Worksheet 12 (Feb. 26)

DIS 119/120 GSI Xiaohan Yan

Erratum: In Example 5(d) of Worksheet 11, the map T is not actually linear transformation, so please disregard part (d). You may want to think about why it is not linear.

DEFINITIONS

- kernel, image, injectivity, surjectivity;
- isomorphism of vector spaces;
- basis, coordinate vector;

METHODS AND IDEAS

Theorem 1. *Any finite-dimensional vector space V is isomorphic to the Euclidean space of the same dimension.*

Remark 1. After choosing a basis for V, any linear property of vectors (e.g. linear combination, linear independence) in the abstract vector space V can be checked through the coordinates under the basis.

Similarly, after choosing bases for both V and W, any linear property of linear transformation $T : V \to W$ (e.g. injectivity, surjectivity, isomorphism, kernel, image) can be checked through the standard matrix under the bases.

1 Problems

Example 1. Consider the linear transformation

$$\begin{aligned} T: \quad \mathbb{P}_2 &\longrightarrow \quad \mathbb{R} \\ f(x) &\longmapsto f(1) - f(2). \end{aligned}$$

(a) Compute $T(x^2), T(x), T(3)$.
(b) Let $f(x) = a + bx + cx^2$ and $T(f(x)) = 0$, then what do we know about a, b, c?
(c) Find a basis of the kernel of T.
(d) Is T injective? Surjective?

Example 2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis of the vector space V.
(a) Determine $\dim V$.
(b) Consider the three vectors

$$\mathbf{v}_1 = \mathbf{b}_1 + 2\mathbf{b}_2 + 2\mathbf{b}_3, \mathbf{v}_2 = 2\mathbf{b}_1 + \mathbf{b}_2 + 2\mathbf{b}_3, \mathbf{v}_3 = 2\mathbf{b}_1 + 2\mathbf{b}_2 + \mathbf{b}_3.$$

Are they linearly dependent? Do they span V?
(c) Let $T : V \to V$ be the linear transformation defined by $T(\mathbf{b}_1) = \mathbf{b}_2, T(\mathbf{b}_2) = \mathbf{b}_1, T(\mathbf{b}_3) = T(\mathbf{b}_3)$. Find all $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{v}$. Do they form a subspace of V?

Example 3. Consider the following three polynomials in $\mathbb{P}^2$

$$\mathbf{b}_1 = 3 + 4x + 5x^2, \quad \mathbf{b}_2 = 2 + cx + 4x^2, \quad \mathbf{b}_3 = 1 + 2x + cx^2.$$

(a) Find their coordinate vectors under the basis $\{1, x, x^2\}$. Your answer may depend on c.
(b) For what values of c is $\mathbb{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ a basis of $\mathbb{P}^2$?
(c) Suppose that $\mathcal{B}$ is indeed a basis, and that the polynomial $7x$ has coordinate $(1, -2, 1)^T$ relative to it. Find c.