Worksheet 10 (Feb. 12)

DIS 119/120 GSI Xiaohan Yan

Note: There is no discussion session next Monday.

DEFINITIONS

- Subspace, span as an example of subspace;
- null space, column space;
- basis of a subspace, dimension of a subspace.

METHODS AND IDEAS

Remark 1. An important property of **determinant**:

$$\det(AB) = \det(A) \cdot \det(B).$$

Theorem 1. *Below is the algorithm for finding the* ***bases of the null and column space*** *of a matrix A:*

- *row reduce A to REF or RREF;*
- *identify the pivot positions, go back to A, then the columns vectors of A where these pivots reside form a basis of $\mathrm{Col}(A)$;*
- *write out the solution set of $A\mathbf{x} = \mathbf{0}$;*
- *separate the free variables in the solution set, then the "vector coefficients" of these free variables form a basis of $\mathrm{Nul}(A)$.*

Remark 2. Since the dimension of a space is the number of vectors in any basis of it,

$$\dim \mathrm{Col}(A) = \sharp \text{ pivots in } A, \qquad \dim \mathrm{Nul}(A) = \sharp \text{ free variables in } A.$$

1 Problems

Example 1. Which ones of the following subsets of $\mathbb{R}^n$ are subspaces?

(a) $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x = y \right\} \subset \mathbb{R}^2$

(b) $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x, y > 0 \right\} \subset \mathbb{R}^2$

(c) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} | x + y + z = 1 \right\} \subset \mathbb{R}^3$

(d) $\mathbb{R}^3 \subset \mathbb{R}^3$

(e) $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x^2 + y^2 = 1 \right\} \subset \mathbb{R}^2$

(f) $\{\mathbf{0}\} \subset \mathbb{R}^5$.

Example 2. Consider the matrix

$$A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix}.$$

(a) Find a basis and the dimension for $\mathrm{Nul}(A)$. (b) Find a basis and the dimension for $\mathrm{Col}(A)$.

Example 3. Let $P \subset \mathbb{R}^3$ be the plane $x - 2y + 3z = 0$ passing through the origin.
(a) Show that P is a subspace of $\mathbb{R}^3$. (**Hint:** You may use any theorem from the text.)
(b) Find a basis of P.

Example 4. True or false.

( ) $\det(A+B) = \det(A) + \det(B)$.

( ) $\det(A^{-1}) = -\det(A)$.

( ) $\det(3A) = 3\det(A)$

( ) If two rows of A are identical, then $\det(A) = 0$.

( ) If two columns of A are identical, then $\det(A) = 0$.

( ) $\det(A)$ is the product of diagonal entries of A.