Worksheet 9 (Feb. 10)

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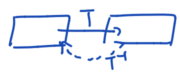
Transpose $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \to A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

1 Review

DEFINITIONS

- Inverse of linear transformation, inverse of matrix;



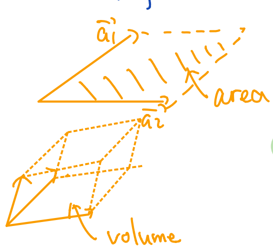
$T: \mathbb{R}^n \to \mathbb{R}^m$ then $T^{-1}: \mathbb{R}^m \to \mathbb{R}^n$ s.t. $\begin{cases} T^{-1}(T(\vec{x})) = \vec{x}, \ \forall \vec{x} \in \mathbb{R}^n \\ T(T^{-1}(\vec{b})) = \vec{b}, \ \forall \vec{b} \in \mathbb{R}^m \end{cases}$

Rmk 1° $\det A \neq 0 \iff A$ invertible

- determinant of matrix.

2° geometric meaning of determinant: n-dim volume of the n-dim parallelogram whose sides are columns of A.

① If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(A) = ad - bc$. ② For A larger, det is defined inductively using expansion w.r.t rows.



METHODS AND IDEAS

" $T^{-1}(\vec{b}) = \vec{x}$ ". (Only possible when $m=n$).

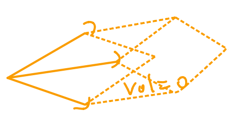
- A transformation is **invertible** if and only if it is **bijective**, i.e. both one-to-one and onto.
- To compute the inverse of a matrix A, apply row reduction

$$\begin{pmatrix} A & I_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} I_n & A^{-1} \end{pmatrix}.$$

i.e., when A is reduced to I_n, what appears on RHS is the A^{-1}.

Theorem 1. [Equivalent conditions for invertibility] [see professor's notes for a more complete version.]
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and A be its standard matrix, then all the following statements are equivalent conditions of invertibility of T:

(linear transformation) $\Leftrightarrow$ *T is invertible* $\Leftrightarrow$ *T is bijective*



(vector) $\Leftrightarrow$ *$T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)$ are linearly independent and span the entire $\mathbb{R}^n$* — columns of A.

(matrix) $\Leftrightarrow$ *A is invertible* $\Leftrightarrow$ *A has n pivots (and thus one in each row and each column)* $\Leftrightarrow$ $\det(A) \neq 0$ $\iff$ RREF(A) = I_n.

(linear system) $\Leftrightarrow$ *$A\mathbf{x} = \mathbf{b}$ has unique solution for any $\mathbf{b} \in \mathbb{R}^n$.*

$\Updownarrow$ Since dim domain = dim codomain

T injective

$\Updownarrow$

T surjective

Method ③ [Only for 3×3]

$$\begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 0 & -2 & 4 \end{vmatrix} \begin{matrix} 1 & -1 \\ 2 & 0 \\ 0 & -2 \end{matrix}$$

$= 1\times0\times4 + (-1)\times1\times0 + 3\times2\times(-2)$
$- 3\times0\times0 - 1\times1\times(-2) - (-1)\times2\times4$

$= -2$

Method ② row operations

- interchange : det is multiplied by (-1).
- scaling by $c \in \mathbb{R}^\times$: det is multiplied by c.
- replacement : det is not changed.

$$d = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 0 & -2 & 4 \end{vmatrix} \xrightarrow[\det\times(-\frac{1}{2})]{③\mapsto③\times(-\frac{1}{2})} \begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 0 & 1 & -2 \end{vmatrix} \xrightarrow[\det\times(-1)]{②\leftrightarrow③} \begin{vmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{vmatrix} \xrightarrow{③\mapsto③-2\cdot①} \begin{vmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -5 \end{vmatrix} \xrightarrow{③\mapsto③-2\cdot②} \begin{vmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} = -1$$

$d\times(-\frac{1}{2})\times(-1) = -1 \rightsquigarrow d = -2$

2 Problems

Example 1. Compute the determinants below

determinant.

(a) (b)

$$\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix}, \qquad \begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 0 & -2 & 4 \end{vmatrix}.$$

(a) $\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 2\times3 - 1\times5 = 1$

choose the simplest row/column (with the most zeros)

(b) Method ①

$$\begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 0 & -2 & 4 \end{vmatrix} = 1\times\begin{vmatrix} 0 & 1 \\ -2 & 4 \end{vmatrix} - (-1)\times\begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} + 3\times\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -2$$

Rmk We can do expansion w.r.t any row/column, but we should add signs $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$. $(-1)^{i+j}$ (i row, j column)

$$\begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 0 & -2 & 4 \end{vmatrix} = -2\times\begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} + 0\times\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} - 1\times\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} = -2$$

Example 2. Consider the 2×2 matrices

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

(a) Find $A^{-1}B$. $= (A^{-1}\vec{b_1} \;\; A^{-1}\vec{b_2})$
(b) Solve the linear systems $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$, for $\mathbf{b}_1$ and $\mathbf{b}_2$ the two column vectors of B.

Formula. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible, i.e. $\det(A) = ad - bc \neq 0$.
then explicit formula for A^{-1}: $A^{-1} = \frac{1}{ad-bc}\cdot\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$B^{-1}A^{-1}$ is the inverse of AB.

Example 3. True or false.

A^{-1} B^{-1}
A, B

Show AB is also invertible. Reason $C = B^{-1}\cdot A^{-1}$

$C\cdot(AB) = B^{-1}A^{-1}\cdot AB = B^{-1}B = I_n$
$(AB)\cdot C = ABB^{-1}A^{-1} = AA^{-1} = I_n$.

(T) The product of two invertible matrices is still invertible.

(T) The composition of two invertible linear transformations is still invertible.

(T) The inverse of an invertible matrix is invertible.

$(A^{-1})^{-1} = A$.
$(A^{-1})\cdot(A^{-1})^{-1} = A^{-1}\cdot A = I_n$
$(A^{-1})^{-1}\cdot A^{-1} = A\cdot A^{-1} = I_n$.

(F) An upper-triangular matrix is invertible.

(T) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a surjective linear transformation and A be its standard matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ has unique solution for any $\mathbf{b} \in \mathbb{R}^n$.

T surjective $\Longleftrightarrow$ T bijective $\Longleftrightarrow$ A invertible $\Longleftrightarrow$ $A\vec{x} = \vec{b}$ always has unique sol.

(T) If $C = AB$ where A is 3×2 and B is 2×3, then C can never be invertible.

$\begin{pmatrix} & & \\ & 0 & \\ & & \end{pmatrix}$ all entries below the diagonal are zero.

Rmk. We don't require entries on the diagonal to be zero.

e.g. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is upper triangular but not invertible

$\in \mathrm{Span}\{\vec{a_1}, \vec{a_2}\}$

$\mathrm{Col}(C) = \mathrm{Span}\{\vec{C_1}, \vec{C_2}, \vec{C_3}\}$
$= \mathrm{Span}\{b_{11}\vec{a_1} + b_{21}\vec{a_2}, b_{12}\vec{a_1} + b_{22}\vec{a_2}, b_{13}\vec{a_1} + b_{23}\vec{a_2}\}$
$= \{x(b_{11}\vec{a_1} + b_{21}\vec{a_2}) + y(b_{12}\vec{a_1} + b_{22}\vec{a_2}) + z(b_{13}\vec{a_1} + b_{23}\vec{a_2}) \mid x,y,z \in \mathbb{R}\}$
$= \{(xb_{11} + yb_{12} + z\cdot b_{13})\vec{a_1} + (xb_{21} + yb_{22} + zb_{23})\vec{a_2} \mid x,y,z \in \mathbb{R}\}$
$\subset \mathrm{Span}\{\vec{a_1}, \vec{a_2}\}$

Fact. $C = A\cdot B$. then all columns of C are linear combinations of columns of A.

For example, $A = \begin{pmatrix} \vec{a_1} & \vec{a_2} \end{pmatrix}_{3\times2}$, $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$.

$\rightsquigarrow C = A\cdot B = \begin{pmatrix} b_{11}\vec{a_1} + b_{21}\vec{a_2} & b_{12}\vec{a_1} + b_{22}\vec{a_2} & b_{13}\vec{a_1} + b_{23}\vec{a_2} \end{pmatrix}_{3\times3}$

$\mathrm{Col}(C) \subset \mathrm{Col}(A)$ so $\dim \mathrm{Col}(C) \leq \dim \mathrm{Col}(A) \leq 2$

$\therefore$ There're at most 2 pivots in C.

$\therefore$ C not invertible.