Worksheet 8 (Feb. 8)

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1 Review

$\mathbb{R}^n \xrightarrow{S} \mathbb{R}^m \xrightarrow{T} \mathbb{R}^p$

$\vec{x} \longmapsto S(\vec{x}) \longmapsto T(S(\vec{x}))$

$T\circ S(\vec{x})$

then do this first do this

DEFINITIONS

Rmk. $B\cdot A$ is only defined for B and A satisfying

\# columns of B = \# rows of A.

- composition of linear transformation, matrix product;

Computation. $B_{p\times m} A_{m\times n} = (B\vec{a_1}\ B\vec{a_2} \cdots B\vec{a_n})_{p\times n}$

where $A = (\vec{a_1} \cdots \vec{a_n})$.

- ~~Inverse of linear transformation, inverse of matrix~~;

$T\circ S(\vec{x}) = T(S(\vec{x})) = T(A\cdot\vec{x})$

$= B\cdot(A\cdot\vec{x})$

$= (B\cdot A)\cdot\vec{x}$.

Remark 1. Let A be the matrix of S and B be of T. The matrix of the composition $T\circ S$ is then just BA. In this sense, not any two matrices A and B can be composed, as not any two linear transformations can be composed. They only can when the number of columns of the former is equal to the number of rows of the latter.

$S(\vec{x}) = A_{m\times n}\vec{x}$

$T(\vec{y}) = B_{p\times m}\vec{y}$

2 Problems

Example 1. Determine the injectivity and surjectivity of the composition $T\circ S$ (note that this means to first apply S and then apply T!!!) of linear transformations T and S, where
(1) $S: \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation counterclockwise by $\pi/4$, and $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the projection to y-axis;

(2) $S: \mathbb{R} \to \mathbb{R}^2$ satisfies $S(1) = (2,0)$ and $T: \mathbb{R}^2 \to \mathbb{R}$ is the projection to x-axis;

(3) the matrices of S and T are

$$S = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix}, T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Where injectivity of T fails is not in $\operatorname{Im} S$, so it's not seen by the composition.

Correct version: If $T \circ S$ is injective, then S is injective, but not necessarily T.

Example 2. Find counterexamples to the following statements.

(1) The composition $T \circ S$ of two linear transformations is injective, then both T and S are injective.

$$\mathbb{R} \xrightarrow{S} \mathbb{R}^2 \xrightarrow{T} \mathbb{R}$$

$$e_1 \longrightarrow e_1 \longrightarrow e_1, \qquad e_2 \longrightarrow e_1$$

In matrix, $S = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $T \circ S = (1)$ "identity map"

(2) The composition $T \circ S$ of two linear transformations is ~~injective~~ surjective, then both T and S are ~~injective~~ surjective.

the same example in (1) works: S is not surjective, T is surjective.

Correct version: If $T \circ S$ is injective.

(3) If $n \times n$ matrices satisfy $AB = AC$, then $B = C$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 3. Compute the matrix product

$$\underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \cdot \underset{3\times 2}{\begin{pmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{pmatrix}} = \begin{pmatrix} 20 & 14 \\ 56 & 41 \end{pmatrix}$$

$1\times6+2\times4+3\times2 = 20$, $1\times5+2\times3+3\times1 = 14$

$4\times6+5\times4+6\times2 = 56$, $4\times5+5\times3+6\times1 = 41$

Example 4. *This one looks like a proof problem but it is actually about computation.* Let A and B be two 2×2 matrices and let

$$C = AB - BA.$$

Show that the sum of the two diagonal entries of C, i.e. $C_{11} + C_{22}$, is zero.