Worksheet 7 (Feb. 5)

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$T: \mathbb{R}^n \to \mathbb{R}^m$ map
s.t. ① $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
② $T(c\cdot\vec{u}) = c\cdot T(\vec{u})$.
$\rightsquigarrow$ $T(\vec{x}) = \underbrace{A}_{\text{matrix of } T}\cdot\vec{x}$. $A = \begin{pmatrix} T(\vec{e_1}) & \cdots & T(\vec{e_n}) \end{pmatrix}$, $\vec{e_1} = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$, $\vec{e_n} = \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$

1 Review

DEFINITIONS

- *Recall:* matrix of a linear transformation;
- range and kernel of a linear transformation;

$\ker T = \{\vec{x} \in \underset{\mathbb{R}^n \text{ domain}}{\mathbb{R}^n} \mid T(\vec{x}) = \vec{0}\}$.

$\hookrightarrow$ $\text{Range } T = \underbrace{\text{Im}}_{\text{image}}\, T = \{\vec{b} \in \mathbb{R}^m \mid \exists\, \vec{x} \in \mathbb{R}^n \text{ s.t. } \vec{b} = T(\vec{x})\} \subset \mathbb{R}^m$ Codomain

($\exists$: there exists)

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation

- one-to-one (injective) linear transformation, onto (surjective) linear transformation, ~~bijective~~ (injective and surjective) linear transformation. $\Leftrightarrow$ $\ker T = \{\vec{0}\}$.

T injective $\Leftrightarrow$ $\forall\, \vec{b} \in \mathbb{R}^m$ codomain, $\exists$ at most 1 $\vec{x} \in \mathbb{R}^n$ domain s.t. $T(\vec{x}) = \vec{b}$.

"no two different vectors in $\mathbb{R}^n$ are sent to the same vector in $\mathbb{R}^m$."

T surjective $\Leftrightarrow$ $\forall\, \vec{b} \in \mathbb{R}^m$ codomain, $\exists$ at least 1 $\vec{x} \in \mathbb{R}^n$ domain s.t. $T(\vec{x}) = \vec{b}$.

($\forall$: for any)

"range of T covers the entire $\mathbb{R}^m$".

$\Leftrightarrow$ $\text{Im}\, T = \mathbb{R}^m$

METHODS AND IDEAS

[For the complete version see P7 of the professor's notes of Lecture 6. Let A be an $m \times n$ matrix.]

$\ker T$ is exactly solution set of $A\vec{x} = \vec{0}$. $\rightsquigarrow$ Equivalence because $A\vec{x} = \vec{0}$ has no nontrivial solution i.e. has only zero sol. $\Updownarrow$ $A\vec{x} = \vec{b}$ ≤ 1 solution ($\forall \vec{b}$)

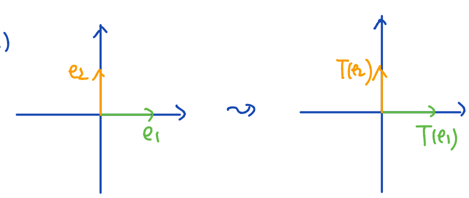
- Further expanded **criterion** for $\geqslant 1$ solution (existence): The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m, T(\mathbf{x}) = A\mathbf{x}$ is surjective $\Leftrightarrow$ the linear system $A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b} \in \mathbb{R}^m$. $\Leftrightarrow$ pivot in all rows $\Leftrightarrow$ columns of coeff. matrix span entire $\mathbb{R}^m$.
- Further expanded **criterion** for $\leqslant 1$ solution (inconsistency or uniqueness): The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m, T(\mathbf{x}) = A\mathbf{x}$ is injective $\Leftrightarrow$ the linear system $A\mathbf{x} = \mathbf{b}$ has $\leqslant 1$ solution for any $\mathbf{b} \in \mathbb{R}^m$. $\Leftrightarrow$ pivot in all columns of coeff. matrix $\Leftrightarrow$ columns are independent.

2 Problems

injective (one-to-one) $\updownarrow$ not opposite surjective (onto)

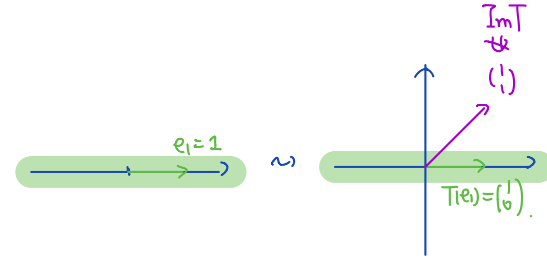
Example 1. Find a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ that is
(a) one-to-one and onto; identity, rotation about the origin, reflection across a line through origin.
(b) one-to-one but not onto;
(c) onto but not one-to-one;
(d) neither one-to-one nor onto.

(a)



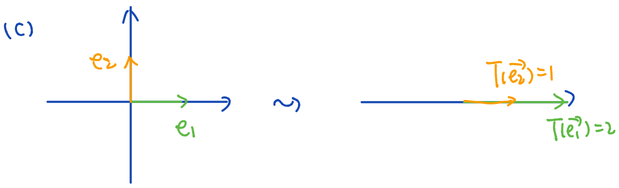
$T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}x\\y\end{pmatrix}$ "identity map" $A = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$
pivots in all columns
pivots in all rows

(b)



$T: \mathbb{R} \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix}x\\0\end{pmatrix}$ $A = \begin{pmatrix}1\\0\end{pmatrix}$ pivot in each column but not in each row.

(c)



$T: \mathbb{R}^2 \to \mathbb{R}$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto 2x+y$

$e_1 \mapsto 2$

$e_2 \mapsto 1$

$A = (2 \quad 1)$.

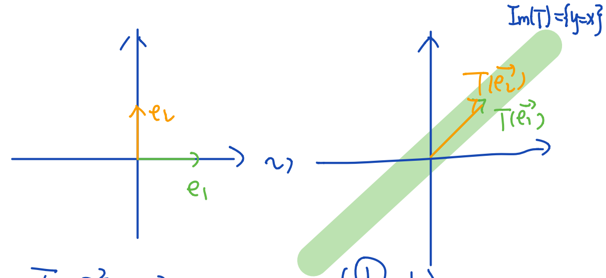
pivot in each row

but not each column

$\Leftrightarrow \operatorname{Ker} A \neq \{\vec{0}\}$

e.g. $T(\vec{e_1} - 2\vec{e_2}) = T(\vec{e_1}) - 2T(\vec{e_2}) = 0$

(d)



$T: \mathbb{R}^2 \to \mathbb{R}^2$

$\vec{e_1} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{e_2} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x+y \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

pivot not in each row

not in each column.

Example 2. True or false.

$A = (T(\vec{e_1}) \cdots T(\vec{e_n}))$

(T) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix. $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

(F) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if it maps each $\mathbf{x} \in \mathbb{R}^n$ to a unique vector in $\mathbb{R}^m$. if this were violated, then $\vec{x} \to 1, 2$

(T) If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfies $T(\mathbf{e}_1 + \mathbf{e}_n) = \mathbf{0}$, then T is not injective. Method ① Let A be the matrix of T. $A \cdot (\vec{e_1} + \vec{e_n}) = \vec{0}$. $\rightsquigarrow$ $A\vec{x} = \vec{0}$ has a nontrivial solution, $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ $\Leftrightarrow$ T is not injective.

Method ② $\operatorname{Ker} T \ni \vec{e_1} + \vec{e_n}$ so $\operatorname{Ker} T \neq \{\vec{0}\}$

$T(\vec{e_1} + \vec{e_n}) = \vec{0}$

$T(\vec{0}) = \vec{0}$

(T) A linear transformation is onto if and only if its matrix has a pivot in each row. $\Leftrightarrow$ Span $\{\vec{a_1}, \cdots, \vec{a_n}\} = \mathbb{R}^m$.

(F) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is surjective if $n > m$.

(F) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is surjective only if $n > m$.

only if $n \geq m$.

Example

$f: \{1,2,3\} \to \{1,2\}$

① $\to$ 1

② $\to$ 2

③ $\to$ 2

f is not injective

one-to-one:

$\forall \vec{b} \in \mathbb{R}^m$, it is mapped from a unique (≤ 1) vector of the domain.

Counterexample,

$T: \mathbb{R}^3 \to \mathbb{R}^2$

with matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

But it's correct to say that if $n<m$, then $T: \mathbb{R}^n \to \mathbb{R}^m$ cannot be surjective.

$T: \mathbb{R}^n \to \mathbb{R}^m$

Injectivity (one-to-one)

① $\ker T = \{\vec{0}\}$.

$A\vec{x} = \vec{0}$ has only the trivial solutions.

② $\forall \vec{b} \in \mathbb{R}^m$ the codomain. $\exists \leq 1$ $\vec{x} \in \mathbb{R}^n$ the domain s.t. $T(\vec{x}) = \vec{b}$.

$A\vec{x} = \vec{b}$ has at most 1 solution for any $\vec{b}$.

Surjectivity (onto)

① $\operatorname{Im} T = \mathbb{R}^m$

$A\vec{x} = \vec{b}$ has at least one solution for any $\vec{b}$

② $\forall \vec{b} \in \mathbb{R}^m$ the codomain. $\exists \geq 1$ $\vec{x} \in \mathbb{R}^n$ s.t. $T(\vec{x}) = \vec{b}$.

Example 3. *'Find values of c' cliché.* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation associated to the matrix

$$A = \begin{pmatrix} 3 & 1 & 3 \\ c & 2 & 6 \\ 1 & 0 & -1 \end{pmatrix}.$$

(1) When is T injective? (2) When is T surjective?