Worksheet 6 (Feb. 3)

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Clarification: Codomain $\neq$ range/image
not only for linear transformations, but for any functions/maps as well.
Example $f: \{1,2\} \longrightarrow \{1,2,3\}$, $f(1)=1$, $f(2)=2$.
Codomain $f = \{1,2,3\}$
$\cup$
range $f = \operatorname{Im} f = \{1,2\}$
"the subset of elements that can be written as $f(\cdots)$".

1 Review

DEFINITIONS

$A\vec{x}=\vec{b}$. Columns of A denoted by $\vec{a_1}, \cdots, \vec{a_n}$.

- Trichotomy of linear systems (vector language);

Solution set:
- no solution : $\vec{b} \notin \operatorname{span}\{\vec{a_1}, \cdots, \vec{a_n}\}$
- unique solution : $\vec{b} \in \operatorname{span}\{\vec{a_1}, \cdots, \vec{a_n}\}$, and $\vec{a_1}, \cdots, \vec{a_n}$ L.I.
- >1 solutions (~~thus~~ ∞ solutions) : $\vec{b} \in \operatorname{span}\{\vec{a_1}, \cdots, \vec{a_n}\}$, and $\vec{a_1}, \cdots, \vec{a_n}$ not L.I.

- linear transformation, domain, codomain;

essential object of linear alg.

$T: \mathbb{R}^n \to \mathbb{R}^m$ a map satisfying
(1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
(2) $T(c\cdot\vec{u}) = c\cdot T(\vec{u})$
$\rightsquigarrow$ Corollary
Cor 1° $T(\vec{0}) = \vec{0}$
By (2). $T(2\cdot\vec{0}) = 2\cdot T(\vec{0})$
$\overset{\shortparallel}{T(\vec{0})} \longrightarrow T(\vec{0}) = \vec{0}$.

- standard basis of $\mathbb{R}^n$, matrix of a linear transformation.

$\vec{e_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \vec{e_n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$\vec{e_1}, \cdots, \vec{e_n}$ are the columns of $n\times n$ identity matrix $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n\times n}$

2° "T preserves linear combination"
$T(a_1\vec{u_1}+a_2\vec{u_2}+\cdots+a_k\vec{u_k})$
$= a_1T(\vec{u_1})+a_2T(\vec{u_2})+\cdots+a_kT(\vec{u_k})$.
[can be proved by applying (1) & (2) k times].

METHODS AND IDEAS

- A map $f: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation of vectors $\Leftrightarrow$ it is a matrix multiplication, i.e. we can find an $m \times n$ matrix such that $f(\mathbf{x}) = A\mathbf{x}$, for any $\mathbf{x} \in \mathbb{R}^n$. The columns of A are exactly $f(\mathbf{e}_1), f(\mathbf{e}_2), \cdots, f(\mathbf{e}_n)$. A is called the matrix of f.

↑ how to compute A

Strong property

- Thus in particular, the linear transformation f is completely determined by the **images** $f(\mathbf{e}_1), f(\mathbf{e}_2), \cdots, f(\mathbf{e}_n)$ of the standard basis of $\mathbb{R}^n$!!!

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{e_1}+x_2\vec{e_2}+\cdots+x_n\vec{e_n}$

$\overset{A}{[f(\vec{e_1}) \cdots f(\vec{e_n})]} \overset{\vec{x}}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}$
$\shortparallel$
So by Cor 2°. $f(\vec{x}) = f(x_1\vec{e_1}+\cdots+x_n\vec{e_n}) = x_1f(\vec{e_1})+\cdots+x_nf(\vec{e_n})$

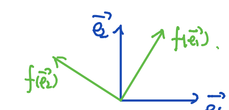
2 Problems

Example 1. Determine if the following maps are linear transformations.
(1) $f: \mathbb{R}^2 \to \mathbb{R}^2$ the rotation of vectors counterclockwise by $\pi/3$ about the origin.

Yes. Checked in Tuesday's lecture.

Matrix of this transformation f is

$$A = \begin{array}{c} \begin{array}{cc} f(\vec{e_1}) & f(\vec{e_2}) \end{array} \\ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \end{array}, \quad \theta = \frac{\pi}{3}$$



(2) $g: \mathbb{R}^2 \to \mathbb{R}^2$ sending any vector $\mathbf{x}$ to $\mathbf{x} + \mathbf{e}_1$.

In coordinates, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. then $g(\vec{x}) = \begin{pmatrix} x_1+1 \\ x_2 \end{pmatrix}$.

Method ① By Cor 1. $\vec{g}(\vec{0}) = \vec{0}$. but here $g(\vec{0}) = \begin{pmatrix} 0+1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
So it's not linear transformation.

Method ② Systematic way $g\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1+1 \\ x_2 \end{pmatrix}$ bad.

$g\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1+2x_2+x_3 \\ x_4 \\ x_1+x_4 \end{pmatrix}$ ✓

g is linear transformation only if all entries in the coordinate expression are linear expressions in terms of x_1 and x_2 (and $x_3, x_4, \cdots$). In particular, it doesn't contain any constant.

(3) $h : \mathbb{R}^3 \to \mathbb{R}^2$ with

$$h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2+y^2 \\ z \end{pmatrix}.$$

No.

Method ① because there're quadratic terms.

Method ② Counterexample. $h(\vec{u}+\vec{v}) = h(\vec{u})+h(\vec{v})$.

take $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$h(\vec{u}+\vec{v}) = \begin{pmatrix} 2^2+0^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \neq$ $h(\vec{u})+h(\vec{v}) = \begin{pmatrix} 1^2+0^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1^2+0^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

(4) $k : \mathbb{R} \to \mathbb{R}$ with

$$g(x) = |x|.$$

No.

$|x|$ is not "linear".

Counterexample: $\vec{u} = (1)$, $\vec{v} = (-1)$.

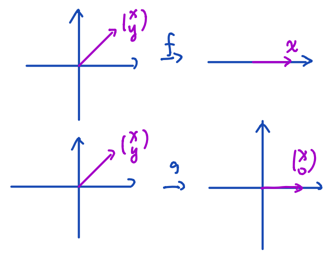
$$\begin{matrix} (0) & \neq & (2) \\ \| & & \| \\ g(\vec{0}) & & (1)+(1) \\ \| & & \| \\ g(\vec{u}+\vec{v}) & \neq & g(\vec{u})+g(\vec{v}) \end{matrix}$$

Example 2. Compare the two linear transformations

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad g : \mathbb{R}^2 \to \mathbb{R}^2$$

different codomains

$$f\begin{pmatrix} x \\ y \end{pmatrix} = x, \qquad g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$



Why are they different linear transformations?

For f, codomain $f = \mathbb{R}$, $\text{Im} f = \mathbb{R}$.

For g, codomain $g = \mathbb{R}^2$, $\text{Im} g =$ "x-axis" $\subset \mathbb{R}^2$.

Example 3. Assume that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation. Consider the following vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}.$$

(1) Assume further that $T(\mathbf{e}_1) = \mathbf{u}$ and $T(\mathbf{e}_2) = \mathbf{v}$. Find the matrix of T.
(2) Assume instead that $T(\mathbf{a}) = \mathbf{u}$ and $T(\mathbf{b}) = \mathbf{v}$. Find the matrix of T. *Hint: why is T also determined completely here?*

(1). The matrix of T: $A = \begin{pmatrix} T(\vec{e_1}) & T(\vec{e_2}) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 3 \\ -1 & 3 \end{pmatrix}$

"Goldilock"
When $m=n$, independent $\iff$ span $= \mathbb{R}^m$.

(2) Intuition. T is determined by $T(\vec{a}), T(\vec{b})$. because $\text{Span}\{\vec{a}, \vec{b}\} = \mathbb{R}^2$

$\forall \vec{x} \in \mathbb{R}^2$, can be written as $\vec{x} = x_1\vec{a} + x_2\vec{b}$.

$T(\vec{x}) = x_1 T(\vec{a}) + x_2 T(\vec{b})$.

Computation. $A = \begin{pmatrix} T(\vec{e_1}) & T(\vec{e_2}) \end{pmatrix}$.

$\vec{e_1} = ?\vec{a} + ?\vec{b}$
$\vec{e_2} = ?\vec{a} + ?\vec{b}$

$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\vec{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{cases} \vec{a} = (-1)\vec{e_1} + 2\cdot\vec{e_2} \\ \vec{b} = \vec{e_1} + \vec{e_2} \end{cases}$$

$$\rightsquigarrow \begin{cases} T(\vec{a}) = -T(\vec{e_1}) + 2T(\vec{e_2}) \\ T(\vec{b}) = T(\vec{e_1}) + T(\vec{e_2}) \end{cases} \rightsquigarrow \begin{cases} -T(\vec{e_1}) + 2T(\vec{e_2}) = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} & ① \\ T(\vec{e_1}) + T(\vec{e_2}) = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} & ② \end{cases}$$

① + ② $3T(\vec{e_2}) = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$, so $T(\vec{e_2}) = \begin{pmatrix} 1 \\ 2 \\ 2/3 \end{pmatrix}$

2·② − ① $3T(\vec{e_1}) = \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}$, so $T(\vec{e_1}) = \begin{pmatrix} 1 \\ 1 \\ 7/3 \end{pmatrix}$

i.e. $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 7/3 & 2/3 \end{pmatrix}$.

$$T(\vec{e_1}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad T(\vec{e_2}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$\left.\begin{aligned} -x + 2a &= 1 \\ -y + 2b &= 3 \\ -z + 2c &= -1 \end{aligned}\right\} ① \qquad ② \left\{\begin{aligned} x + a &= 2 \\ y + b &= 3 \\ z + c &= 3 \end{aligned}\right.$$