Worksheet 5 (Feb. 1)

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1 Review

Recall from last time

- linear independence, how to check;

  $\vec{v}_1, \cdots, \vec{v}_k$ linearly dependent

  $\Leftrightarrow \exists c_1, \cdots, c_k \in \mathbb{R}$, s.t. $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$. $\Leftrightarrow$ [matrix] $A = (\vec{v}_1 \cdots \vec{v}_k)$ does not have pivots in all columns $\Leftrightarrow$ [linear system] $A\vec{x} = \vec{0}$ has nontrivial solutions.

- matrix-vector product.

  [see example below] $A \cdot \vec{v}$ is a vector which is a linear combination of columns of A

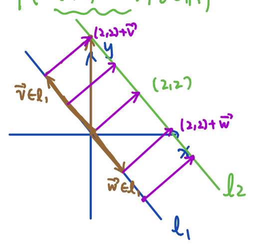
Example

l_1: $\{ (-t, t) \mid t \in \mathbb{R} \}$

straight line

l_2: ($= l_1 +$ the vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$).

$\{ (-t+2, t+2) \mid t \in \mathbb{R} \}$



Theorem 1. (Solution of inhomogeneous linear system)
Let $\mathbf{x} = \mathbf{x}_{prt}$ be one particular solution of $A\mathbf{x} = \mathbf{b}$, then any solution of $A\mathbf{x} = \mathbf{b}$ can be written as

$$\underset{A\vec{x}=\vec{b}}{\mathbf{x}} = \underset{A\vec{x}=\vec{b}}{\mathbf{x}_{prt}} + \underset{A\vec{x}=\vec{0}}{\mathbf{x}_{hmg}},$$

for some solution $\mathbf{x}_{hmg}$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$, and vice versa.

$A\vec{x} = \vec{0}$

$\updownarrow$ relation between the two solution sets

$A\vec{x} = \vec{b}$

Geometrically, they are "parallel".

2 Problems

Example 1. True or false.

(T) The columns of any 4×5 matrix are linearly dependent.

(F) The columns of a matrix A are linearly dependent if the equation $A\mathbf{x} = \mathbf{0}$ is consistent.

(T) If A is a 2×5 matrix with two pivot positions, $A\mathbf{x} = \mathbf{b}$ is consistent for any $b \in \mathbb{R}^2$.

(T) Two vectors are linearly dependent if and only if geometrically they lie on the same line through the origin.

$\vec{u}, \vec{v}$ linearly dependent

$\Downarrow$

$\exists$ c_1, c_2 not both zero

s.t. $c_1\vec{u} + c_2\vec{v} = \vec{0}$.

Assume $c_1 \neq 0$

then $\vec{u} = -\frac{c_2}{c_1}\vec{v}$.

So on the same line

(T) The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent if $\mathbf{v}_3 = \mathbf{0}$.

(T) If the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent for any $\mathbf{v}_4$.

→ The choice $\begin{pmatrix}1\\0\\0\end{pmatrix}$ of the particular solution is arbitrary. For example, we can also take $\vec{x}_{prt} = \begin{pmatrix}3\\2\\-3\end{pmatrix}$ $(t=-1)$, then in general, a solution of $A\vec{x}=\vec{b}$, by Thm1, is

$$\begin{pmatrix}3\\2\\-3\end{pmatrix} + \begin{pmatrix}-2t\\-2t\\3t\end{pmatrix} = \begin{pmatrix}3-2t\\2-2t\\3t-3\end{pmatrix}$$

i.e. the solution set of $A\vec{x}=\vec{b}$ should be $\{(3-2t, 2-2t, 3t-3) \mid t\in\mathbb{R}\}$.

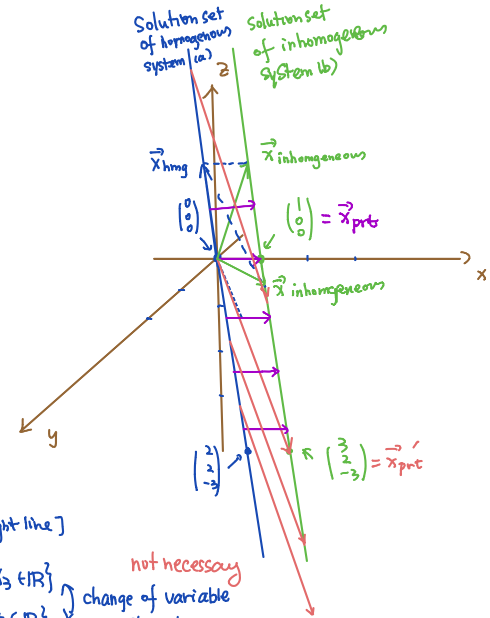
$\downarrow$ $t=s+1$

the same set can be written as $\{(1-2s, -2s, 3s) \mid s\in\mathbb{R}\}$

→ By Thm1. as we know solution set of (a) $\{(-2t, -2t, 3t) \mid t\in\mathbb{R}\}$. we can write out solution set of (b) in terms of one particular solution of (b).

For example, $\vec{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ solves (b), i.e. $\vec{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ is a particular solution of (b). So all solutions of (b) can be written as

$$\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{\vec{x}_{prt}} + \underbrace{\begin{pmatrix}-2t\\-2t\\3t\end{pmatrix}}_{\vec{x}_{hmg}} = \underbrace{\begin{pmatrix}-2t+1\\-2t\\3t\end{pmatrix}}_{\text{general solution } \vec{x} \text{ of } A\vec{x}=\vec{b}}, \text{ for some } t\in\mathbb{R}.$$



Example 2. Consider

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and solve the two linear systems Graph the solution sets

$$(a) A\mathbf{x} = \mathbf{0}. \qquad (b) A\mathbf{x} = \mathbf{b}.$$

(a) $\begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 2/3 & 0 \end{bmatrix}$

Solution set [straight line]

$\{(-\frac{2}{3}x_3, -\frac{2}{3}x_3, x_3) \mid x_3\in\mathbb{R}\}$

$= \{(-2t, -2t, 3t) \mid t\in\mathbb{R}\}$ change of variable $x_3 = 3t$

(b) $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 2/3 & 0 \end{bmatrix}$

Solution set [straight line]

$\{(-\frac{2}{3}x_3+1, -\frac{2}{3}x_3, x_3) \mid x_3\in\mathbb{R}\}$

$= \{(-2t+1, -2t, 3t) \mid t\in\mathbb{R}\}$

Example 3. Compute the matrix-vector multiplication

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}.$$

Remark 1. *With a view toward future lectures:* Regarding vector as a $n \times 1$ matrix, then we actually defined the product of an $m \times n$ matrix and an $n \times 1$ matrix as an $m \times 1$ matrix. (In our example above, $m = 2$ and $n = 3$.) Can we generalize the definition to multiplication of an $m \times n$ matrix with an $n \times p$ matrix? The answer should be $m \times p$.