Worksheet 4 (Jan. 29)

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not needed

deg	0	1	1	1	5+1+2
	5	x	y	z	$2x^5yz^2$

"each of the term is a deg=1 monomial"

$x+y+z$ homog.
$x+y+z+5$ inhom.
x^3+xy^2+xyz homog.
x^3+x^2 inhom.

$$A\vec{u} = \begin{pmatrix} a_{11}u_1+a_{12}u_2+\cdots+a_{1n}u_n \\ a_{21}u_1+a_{22}u_2+\cdots+a_{2n}u_n \\ \vdots \\ a_{m1}u_1+a_{m2}u_2+\cdots+a_{mn}u_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

1 Review

DEFINITIONS

① $\begin{cases} a_{11}x_1+\cdots+a_{1n}x_n=b_1 \\ \vdots \\ a_{m1}x_1+\cdots+a_{mn}x_n=b_m \end{cases}$

② $\boxed{A\cdot\vec{x}=\vec{b}}$ where $\vec{x}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$, $\vec{b}=\begin{pmatrix}b_1\\ \vdots\\ b_m\end{pmatrix}$

③ $\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$

④ $x_1\vec{a_1}+x_2\vec{a_2}+\cdots+x_n\vec{a_n}=\vec{b}$, where $\vec{a_1},\cdots,\vec{a_n}$ are column vectors of A

- matrix-vector product;

  $A_{m\times n} = \begin{pmatrix}\vec{a_1} & \cdots & \vec{a_n}\end{pmatrix} \cdot \vec{u} = \begin{pmatrix}u_1\\ \vdots \\ u_n\end{pmatrix} = \begin{pmatrix} u_1\vec{a_1}+u_2\vec{a_2}+\cdots+u_n\vec{a_n}\end{pmatrix}_m$

- 4 equivalent ways of writing a linear system, homogeneous system, inhomogeneous system;

  $A\vec{x}=\vec{b}$, where $\vec{b}\neq\vec{0}$.

  $A\vec{x}=\vec{0}$ (all constants $b_1,\cdots,b_m=0$) $a_{11}x_1+\cdots+a_{1n}x_n=0$

- linear independence.

  That $\vec{v_1}\cdots\vec{v_k}$ are linearly dependent means that $\exists\ c_1,\cdots,c_k\in\mathbb{R}$ (not all zero) s.t. $c_1\vec{v_1}+\cdots+c_k\vec{v_k}=\vec{0}$

METHODS AND IDEAS

- Expanded **criterion for uniqueness**: provided that solutions of a **linear system** $A\mathbf{x} = \mathbf{b}$ exist, its solution is unique $\Leftrightarrow$ the homogenized system $A\mathbf{x} = \mathbf{0}$ has unique solution (the trivial solution $\mathbf{0}$) $\Leftrightarrow$ the column **vectors** of its coefficient matrix A are linearly independent $\Leftrightarrow$ all columns of A have pivot positions (i.e. no free variable) in **REF**. [See the P12 of the professor's notes for a beautiful chart.]

  solution not unique $\Updownarrow$ linearly dependent $\Updownarrow$ $\exists$ free var.

- Solution sets of $A\mathbf{x} = \mathbf{b}$ and of $A\mathbf{x} = \mathbf{0}$ in $\mathbb{R}^m$ are "parallel planes", when both systems are consistent.

Remark 1. In general, if a homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then an inhomogeneous system $A\mathbf{x} = \mathbf{b}$ with the same coefficient matrix can have 0 or 1 solution.

Remark 2. By the relation between the coefficient matrix and its column vectors, for the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ in $\mathbb{R}^m$, they cannot be independent if $n > m$, and they cannot span the entire $\mathbb{R}^m$ if $n < m$. Moreover, if $m = n$, then:

#columns > #rows

$x_1\vec{v_1}+x_2\vec{v_2}+\cdots+x_n\vec{v_n}=\vec{0}$

$$\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \text{ are linearly independent } \Leftrightarrow \text{ span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} = \mathbb{R}^m.$$

span $\quad$ column vectors $\to$ pivot positions $\quad$ independence $\to$ pivot position in each column (no free variable).

2 Problems

linear dependence $\Leftrightarrow$ at least one of the vectors can be written as linear combinations of other vectors.

Example 1. Consider vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ in $\mathbb{R}^3$

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) $\vec{u_1} = -\frac{1}{2}\vec{u_2} + 0 \cdot \vec{u_3}$

$\vec{u_2} = -2\vec{u_1} + 0 \cdot \vec{u_3}$

$\vec{u_3} = \ ?\,\vec{u_1} + \ ?\,\vec{u_2}$ won't exist.

(a) Are they linear independent? (b) Can $\mathbf{u}_3$ be written as a linear combination of the other two?

(a) $A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -2 & 1 \\ 2 & -4 & 1 \end{pmatrix}$ do row reduction, and see if there are free var. $\Big<$ yes. $\rightsquigarrow$ nontrivial sol. exists $\rightsquigarrow$ dependent; no. $\rightsquigarrow$ only trivial sol. $\rightsquigarrow$ independent.

$\longrightarrow \underbrace{\begin{pmatrix} 1 & -2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{REF}} \longrightarrow \underbrace{\begin{pmatrix} \boxed{1} & -2 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}}_{\text{RREF}}.$ free variable.

solution set of $A\vec{x} = \vec{0}$. is $\{(2s, s, 0) \mid s \in \mathbb{R}\}$

which means $2s \cdot \vec{u_1} + s \cdot \vec{u_2} + 0 \cdot \vec{u_3} = \vec{0}$ (for any $s \in \mathbb{R}$).

Take $s = 1$. $2\vec{u_1} + \vec{u_2} = \vec{0}$

$\therefore$ linearly dependent

Example 2. Discuss: about linear dependence. *Note that for determining linear independence it does not matter whether you are picturing these vectors in $\mathbb{R}^3$ or $\mathbb{R}^{100}$, as the extra "directions" are irrelevant here.*

(a) If the set of a single vector $\{\mathbf{v}\}$ is linearly dependent, what do we know of $\mathbf{v}$?

$c \cdot \vec{v} = \vec{0}$, with c nonzero real number. $\rightsquigarrow$ $\vec{v} = \vec{0}$

(b) If the set of two vectors $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent, what do we know of $\mathbf{u}, \mathbf{v}$?

$c_1 \cdot \vec{u} + c_2 \cdot \vec{v} = \vec{0}$, with c_1, c_2 not both zero. WLOG, assume $c_1 \neq 0$, $\vec{u} = -\frac{c_2}{c_1}\vec{v}$.

One of the vectors is a multiple of the other

(c) If the set of three vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, what do we know of $\mathbf{u}, \mathbf{v}, \mathbf{w}$?

$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$ $\rightsquigarrow$ geometrically, the three vectors lie on the same plane (2-dim) passing through the origin.

Note We don't necessarily have any of them being multiple of one of the other two.

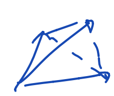
Example $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. $\vec{u} + \vec{v} - \vec{w} = \vec{0}$

Example 3. Find possible values of c such that the following three vectors are linearly dependent

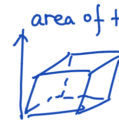
$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ c \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ c \end{bmatrix}.$$

$\begin{bmatrix} 2 & -1 & 3 \\ 5 & c & 3 \\ 3 & 1 & c \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}}$ at least one free var. (not all columns are pivot).

$\vec{V_1}, \vec{V_2}, \vec{V_3}$ independent $\Leftrightarrow$



area of the base



$V(\vec{V_1}, \vec{V_2}, \vec{V_3}) \neq 0$

$\parallel$

$(\vec{V_1} \times \vec{V_2}) \cdot \vec{V_3}$

$=$ area base $\times$ height

$x_1\vec{V_1} + x_2\vec{V_2} \cdots + x_k\vec{V_k}$

$= \underbrace{(\vec{V_1} \cdots \vec{V_k})}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}}_{\vec{x}}$

$\begin{pmatrix} \blacksquare & 0 \\ 0 & \blacksquare \end{pmatrix} \hookrightarrow \begin{pmatrix} \blacksquare & * \\ 0 & \blacksquare \end{pmatrix}$

① row equivalent: $A \leftrightarrow B$ through row red.

② equivalent: solution sets are the same

$\boxed{\text{Prop}}$. $\underline{\vec{v_1}, \cdots, \vec{v_k}}$ are linearly independent.

$\underline{\vec{v_1}, \cdots, \vec{v_k}, \vec{w}}$ are linearly dependent

then $\vec{w}$ is a linear combination of $\vec{v_1} \cdots \vec{v_k}$.

$\vec{u}, \vec{v}$ independent.

$\vec{u}, \vec{v}, \vec{w}$ dependent

$\rightsquigarrow \vec{w} \in \text{Span}\{\vec{u}, \vec{v}\}$

Example $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

$\vec{w} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

① $\vec{u}, \vec{v}$ independent $\quad 1, 1, -1$

② $\vec{u}, \vec{v}, \vec{w}$ dependent : $\underline{\vec{u} + \vec{v} - \vec{w} = 0}$.

$\rightsquigarrow \begin{cases} \vec{u} = \vec{w} - \vec{v} \\ \vec{v} = \vec{w} - \vec{u} \checkmark . \quad (\vec{v} \neq k \cdot \vec{u}) \\ \vec{w} = \vec{u} + \vec{v} \end{cases}$

Pf. $\exists \ \underline{c_1, c_2, \cdots, c_k, d} \in \mathbb{R}$, not all zero.

s.t. $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} + \underline{d\vec{w}} = \vec{0}$.

① If $d \neq 0$, $\quad -\frac{c_1}{d}\vec{v_1} - \frac{c_2}{d}\vec{v_2} - \cdots - \frac{c_k}{d}\vec{v_k} = \vec{w}$

so $\vec{w} \in \text{Span}\{\vec{v_1}, \cdots, \vec{v_k}\}$ we're done.

② If $d = 0$. $\rightsquigarrow \underline{c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} = \vec{0}}$

with $c_1 \sim c_k$ are not all zero

nontrivial solution exists $\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$\rightsquigarrow \vec{v_1}, \cdots, \vec{v_k}$ dependent [Contradiction] ∎

$\{\vec{u}, \vec{v}, \vec{w}\}$ linearly dependent

$\rightsquigarrow \exists$ one of $\vec{u}, \vec{v}, \vec{w}$ that can be written as linear combination of the preceding vectors.

① $\vec{u} = \vec{0}$

② $\vec{v} = k \cdot \vec{u}$

③ $\vec{w} = a\vec{u} + b\vec{v}$.

$\underline{\text{Method 1}}$ $\quad \vec{z} \in \text{Span}\{\vec{x}, \vec{y}\} \Rightarrow c_1\vec{x} + c_2\vec{y} = \vec{z}$ has solution.

$\Rightarrow (\vec{x} \ \vec{y} \ \vec{z})$ is consistent

$\Rightarrow$ REF last column not pivot.

Coeff

$(\vec{x} \ \vec{y} \ \vec{z})$.

$\underline{\text{Method 2}}$. $\quad \vec{z} = a\vec{x} + b\vec{y}$. $\rightsquigarrow a\vec{x} + b\vec{y} - \vec{z} = \vec{0}$

$\rightsquigarrow$ by $\underline{\text{det}}$. dependent