Worksheet 31 (April 28)

DIS 119/120 GSI Xiaohan Yan

$\langle c_1\vec{u_1}+c_2\vec{u_2},\vec{v}\rangle = c_1\langle\vec{u_1},\vec{v}\rangle+c_2\langle\vec{u_2},\vec{v}\rangle$

$\langle \vec{u}, c_1\vec{v_1}+c_2\vec{v_2}\rangle = c_1\langle\vec{u},\vec{v_1}\rangle+c_2\langle\vec{u},\vec{v_2}\rangle$

equivalent when $\langle\vec{u},\vec{v}\rangle=\langle\vec{v},\vec{u}\rangle$

1 Review

DEFINITIONS

Example. the "dot product" on $\mathbb{R}^n$ is an inner product.

- inner product: bilinear, commutative, positive definite;

  $\langle\cdot,\cdot\rangle : \underset{\vec{u}}{V}\times \underset{\vec{v}}{V} \longrightarrow \mathbb{R}$, $\langle\vec{u},\vec{v}\rangle$

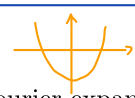
  $\langle\vec{u},\vec{v}\rangle=\langle\vec{v},\vec{u}\rangle$

  ① $\forall \vec{u}\in V$. $\langle\vec{u},\vec{u}\rangle\geqslant 0$

  ② If $\langle\vec{u},\vec{u}\rangle=0$, then $\vec{u}=\vec{0}$.

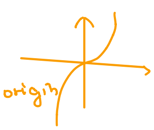
  Rmk. After inner product is defined, we can say: norm, orthogonality, angle, orthogonal projection, ONB.

- piece-wise continuous function, even function, odd function;

  $f(x)=f(-x)$ the graph is symmetric w.r.t the y-axis

  $f(x)=-f(-x)$ the graph is symmetric w.r.t the origin

- Fourier series, Fourier expansion of a function

  infinite sum of $1=\cos 0x$, $\cos nx$, $\sin nx$.

  $f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$.

  $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$ $(n\geqslant 0)$, $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$ $(n>0)$

METHODS AND IDEAS

Idea 1. *The space V_π of piece-wise continuous functions* (over $[-\pi,\pi]$) *is an* ***inner product space****, under the inner product*

$$\langle f,g\rangle := \int_{-\pi}^{\pi} f(x)g(x)dx.$$

This is because $\langle\cdot,\cdot\rangle$ defined as above is indeed bilinear, commutative and positive definite. V_π has the following properties:

- *It is* ***infinite dimensional****.*
- *It has two distinguished subspaces V_π^{even} and V_π^{odd}. The two subspaces are* ***orthogonal****.* "orthogonal complement" to each other
- *V_π^{even} has an orthogonal basis $\{\cos nx\}_{n\geqslant 0}$, and V_π^{odd} has an orthogonal basis $\{\sin nx\}_{n>0}$. Altogether they form $\underbrace{\text{a}}_{\text{orthogonal}}$ basis of V_π.*

Intuition. $\vec{v}=\mathrm{Proj}_W\vec{v}+\mathrm{Proj}_{W^\perp}\vec{v}$

Here. $f(x)=f^{\text{even}}(x)+f^{\text{odd}}(x)$.

$f^{\text{even}}(x)=\frac{f(x)+f(-x)}{2}$. even function

$f^{\text{odd}}(x)=\frac{f(x)-f(-x)}{2}$ odd function.

Theorem 1. *The Fourier expansion of $f(x)$ is given by*

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx,$$

Intuition If $\vec{w_1},\cdots,\vec{w_n}$ is OGB of V.

then $\vec{v}=\frac{\langle\vec{v},\vec{w_1}\rangle}{\langle\vec{w_1},\vec{w_1}\rangle}\vec{w_1}+\cdots+\frac{\langle\vec{v},\vec{w_n}\rangle}{\langle\vec{w_n},\vec{w_n}\rangle}\vec{w_n}$.

$\dfrac{\langle f(x), \sin nx\rangle}{\langle \sin nx, \sin nx\rangle}$ as $\langle \sin nx, \sin nx\rangle = \pi$ $(\forall n>0)$

$\parallel$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$$

Remark 1. The two functions

$$f(x) \quad \text{and} \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

are not completely equal. In fact, if $f(x)$ is piece-wise continuous, they are equal "**almost everywhere**". The discrepancy exists because V_π is infinite-dimensional and the basis $\{1, \cos x, \sin x, \cdots\}$ is not "**complete**".

Reason. Assume for $g(x)$

$g(x)$ = its Fourier expansion

Then if we change the value of $g(x)$ at a certain pt and call the new function $f(x)$, then

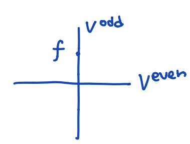
$f(x)$ and $g(x)$ has the same Fourier expansion.

2 Problems

Example 1. Find a linear transformation $T : V_\pi \to V_\pi$ such that V_π^{even} and V_π^{odd} are both eigenspaces of T.

$T: V_\pi \to V_\pi$ $\qquad V^{even} = E_1$

$f(x) \mapsto f(-x)$ $\qquad V^{odd} = E_{-1}$



Example 2. True or false,

(T) If $f(x)$ is an odd piece-wise continuous function, then $a_n = 0$ for all $n \geqslant 0$ in its Fourier expansion.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{odd} \cdot \underbrace{\cos nx}_{even}\, dx = 0$

(T) The Fourier expansion of a piece-wise continuous function is unique. a_n, b_n. explicitly computed.

(T) Let A be a 2×2 symmetric matrix which has two distinct eigenvalues 2 and 3, then the pairing

$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T A \mathbf{v}$$

gives an inner product on $\mathbb{R}^2$.

Rmk In general, if A is symmetric $n \times n$, with all e-values being positive, then $\langle \vec{u}, \vec{v}\rangle = \vec{u}^T A \vec{v}$ is an inner product on $\mathbb{R}^n$.

① bilinear $\langle c_1\vec{u_1} + c_2\vec{u_2}, \vec{v}\rangle = c_1\langle \vec{u_1}, \vec{v}\rangle + c_2\langle \vec{u_2}, \vec{v}\rangle$ Correct because matrix multiplication is linear.

A is symmetric → ② commutative. $\langle \vec{u}, \vec{v}\rangle = \underset{1\times2}{\vec{u}^T} \underset{2\times2}{A} \underset{2\times1}{\vec{v}} = (\vec{u}^T A \vec{v})^T = \vec{v}^T A^T \vec{u} = \vec{v}^T A \vec{u} = \langle \vec{v}, \vec{u}\rangle$.

A has only positive e-values ③ positive definite $\langle \vec{u}, \vec{u}\rangle \geq 0$ 2 , $\langle \vec{u}, \vec{u}\rangle = 0$ if and only if $\vec{u} = \vec{0}$.

Let $\vec{w_1}, \vec{w_2}$ be an orthonormal eigenbasis of $\mathbb{R}^2$

$A\vec{w_1} = 2\vec{w_1}$, $A\vec{w_2} = 3\vec{w_2}$

$\vec{u} = a\vec{w_1} + b\vec{w_2}$

$\langle \vec{u}, \vec{u}\rangle = \vec{u}^T A \vec{u} = \underset{\text{dot prod.}}{\vec{u} \cdot A\vec{u}} = (a\vec{w_1} + b\vec{w_2}) \cdot (2a\vec{w_1} + 3b\vec{w_2})$

$= 2a^2 + 3b^2 \geq 0$

Example 3. Consider the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{P}^2$ defined by

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)dx.$$

(a) Find $\langle 1 + x, x^2 \rangle$.
(b) Let $W = \cancel{\text{sgn}}\overset{\text{span}}{}\{1, x^2\}$ and $u(x) = 2 + 3x + 4x^2$. Find $\text{Proj}_W\, u(x)$ under the above inner product.

(a) $\langle 1+x, x^2 \rangle = \int_{-1}^{1} (1+x)x^2\, dx = \int_{-1}^{1} x^3 + x^2\, dx = \frac{1}{4}x^4 + \frac{1}{3}x^3 \Big|_{-1}^{1} = \frac{2}{3}$

(b) Method ① $\to$ Find an orthogonal basis. [Apply G-S]

$\vec{w_1} = 1$

$\vec{w_2} = x^2 - \frac{\langle 1, x^2 \rangle}{\langle 1, 1 \rangle} \cdot 1 = x^2 - \underbrace{\frac{2/3}{2}}_{\text{scalar}} \cdot \underbrace{1}_{\text{constant function} \in \mathbb{P}^2} = x^2 - \frac{1}{3}$

$\text{Proj}_W\, u(x) = u(x) - \frac{\langle 1, u(x) \rangle}{\langle 1, 1 \rangle} \cdot 1 - \frac{\langle x^2 - \frac{1}{3}, u(x) \rangle}{\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \rangle} \cdot (x^2 - \frac{1}{3}) = \ldots$

Method ②

$W^\perp = \text{span}\{x\}$

$\langle \underbrace{1, x}_{\text{odd}} \rangle = \langle \underbrace{x^2}_{\text{even}}, x \rangle = 0$

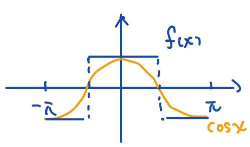
$\text{Proj}_{W^\perp}\, u(x) = \frac{\langle u(x), x \rangle}{\langle x, x \rangle} \cdot x$

$\text{Proj}_W\, u(x) = u(x) - \text{Proj}_{W^\perp}\, u(x).$

Example 4. Find the Fourier expansion of

$$f(x) = \text{sgn}(\cos x)$$

on $[-\pi, \pi]$. Here sgn means taking the sign. In other words, $f(x) = 1$ when $\cos x > 0$, $f(x) = -1$ when $\cos x < 0$, and $f(x) = 0$ when $\cos x = 0$.



$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

where $\begin{cases} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx \end{cases}$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$

For $n>0$ $a_n = \frac{-1}{\pi}\int_{-\pi}^{-\pi/2} \cos nx\, dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx\, dx$

$+ -\frac{1}{\pi}\int_{\pi/2}^{\pi} \cos nx\, dx$

$= -\frac{1}{n\pi} \sin nx \Big|_{-\pi}^{-\pi/2} + \frac{1}{n\pi} \sin nx \Big|_{-\pi/2}^{\pi/2} - \frac{1}{n\pi} \sin nx \Big|_{\pi/2}^{\pi}$

$= -\frac{1}{n\pi}\left[\sin\left(-\frac{n\pi}{2}\right) - \cancel{\sin(-n\pi)}\right] + \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right)\right]$

$- \frac{1}{n\pi}\left[\cancel{\sin(n\pi)} - \sin\left(\frac{n\pi}{2}\right)\right]$

$= \frac{4}{n\pi} \sin\frac{n\pi}{2}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{even}} \underbrace{\sin nx}_{\text{odd}}\, dx = 0$

$\therefore f(x) \sim \frac{4}{\pi}\left(\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \cdots\right)$

Example 5. *This is probably harder than you think.* Find the Fourier expansion of

$$g(x) = e^x$$

on $[-\pi, \pi]$.