Worksheet 30 (April 26)

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1 Problems

fund. sol. set : basis of the solution space

fund. sol. matrix : column vectors are the basis vectors

Example 1. True or false.

(T) A column of a fundamental solution matrix of the ODE system $\mathbf{x}' = A\mathbf{x}$ is a solution.

(F) Any two fundamental solution matrices of the ODE sytem $\mathbf{x}' = A\mathbf{x}$ differ by a series of row reductions.

column reductions

(F) The initial value problem $y'''(t) - 2y''(t) + y'(t) + y(t) = 0, y(0) = 0, y'(0) = 1$ has a unique solution.

3rd order equation ⇝ dim Sol. set = 3

2 constraints

Add in another constraint (e.g. $y''(0)=2$) to obtain uniqueness.

Example.

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \vec{x}.$$

$$\to X(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ e^{2t} & 2e^{3t} \end{pmatrix}.$$

$$\to Y(t) = \begin{pmatrix} e^{2t}+e^{3t} & e^{3t} \\ e^{2t}+2e^{3t} & 2e^{3t} \end{pmatrix}$$

Rmk In general, any two fund. sol. mat. differ by

$$\boxed{Y(t) = X(t) \cdot P}$$

"base change matrix"

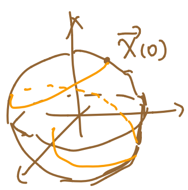
where P is invertible.

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Example 2. Consider the 1st order homogeneous linear system of ODE

$$\mathbf{x}'(t) = \underbrace{\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}}_{A} \mathbf{x}(t).$$

Prove that the norm of any solution $\mathbf{x}(t)$ is constant in t.



$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$

trajectory lies on the same sphere centered at the origin.

Method ①

$$\to \chi_A(\lambda) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ -1 & -\lambda & 1 \\ -1 & -1 & -\lambda \end{pmatrix} = -\lambda^3 - 3\lambda = -\lambda(\lambda+\sqrt{3}i)(\lambda-\sqrt{3}i)$$

eigenvalues $\lambda_1 = 0$, $\lambda_2 = \sqrt{3}i$, $\lambda_3 = -\sqrt{3}i$

For $\lambda_1 = 0$. eigenvector $\vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = \sqrt{3}i$ eigenvector $\vec{v_2} = \begin{pmatrix} 1+\sqrt{3}i \\ -1+\sqrt{3}i \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + i\begin{pmatrix} \sqrt{3} \\ \sqrt{3} \\ 0 \end{pmatrix}$

$\vec{a} + i\vec{b}$

$\alpha + \beta i$, $\alpha = 0$, $\beta = \sqrt{3}$

For $\lambda_3 = -\sqrt{3}i$ eigenvector $\vec{v_3} = \begin{pmatrix} 1-\sqrt{3}i \\ -1-\sqrt{3}i \\ -2 \end{pmatrix}$

Say $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

$$A\cdot\vec{v} = \begin{pmatrix} v_2+v_3 \\ -v_1+v_3 \\ -v_1-v_2 \end{pmatrix}.$$

$$\vec{v}\cdot A\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} v_2+v_3 \\ -v_1+v_3 \\ -v_1-v_2 \end{pmatrix}.$$

$$= v_1v_2 + v_1v_3 - v_2v_1 + v_2v_3 - v_3v_1 - v_3v_2$$

$$= 0$$

Method ②

Idea

$$\vec{x}(t)\cdot\vec{x}(t) = \text{constant}$$

$$\Leftrightarrow (\vec{x}(t)\cdot\vec{x}(t))' = 0$$

$$\vec{x}'(t)\cdot\vec{x}(t) + \vec{x}(t)\cdot\vec{x}'(t)$$

$$2\cdot\vec{x}(t)\cdot\vec{x}'(t)$$

$$2\,\vec{x}(t)\cdot A\vec{x}(t)$$

Suffices to check $\forall \vec{v} \in \mathbb{R}^3$. $\vec{v}\cdot A\vec{v} = 0$

→ General solution

$$\vec{x} = C_1\cdot\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2\cdot\begin{pmatrix} \cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t \\ -\cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t \\ -2\cos\sqrt{3}t \end{pmatrix} + C_3\begin{pmatrix} \sqrt{3}\cos\sqrt{3}t + \sin\sqrt{3}t \\ \sqrt{3}\cos\sqrt{3}t - \sin\sqrt{3}t \\ -2\sin\sqrt{3}t \end{pmatrix}$$

Check $\forall C_1, C_2, C_3$. $\vec{x}(t)\cdot\vec{x}(t) = $ constant (depending on C_1, C_2, C_3 but not on t)