$\mathbb{R}^n$

linear

addition

scalar multiple

~~distance~~
~~angles~~
~~inner products~~
~~multiplication of vectors??~~

Worksheet 3 (Jan. 27)

DIS 119/120 GSI Xiaohan Yan

$0 = \vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

professor

me

1 Review

DEFINITIONS

n-dimensional Euclidean space $= \{(x_1, \cdots, x_n) \mid x_1, \cdots, x_n \in \mathbb{R}\}$.

- vector, zero vector, $\mathbb{R}^n$, addition, scalar multiplication;

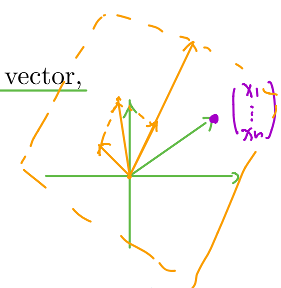
  vector $= (x_1, \cdots, x_n)$; $\mathbb{R}^n \supseteq$ n-dimensional vector

  scalar × vector → vector

  Only legitimate operations for vectors.

- three things are identified: the numerical vector, the geometric vector, and the endpoint of the geometric vector;

  $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.



- linear combination, span;

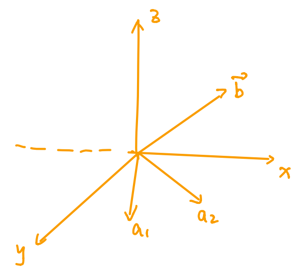
  $\vec{u}$ is lin.comb. of $\vec{v_1}, \cdots, \vec{v_k}$ means $\vec{u} = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_k\vec{v_k}$

  span: the set of all linear combinations

  Rmk. It looks like a "plane" passing through the origin.
  " line "
  " 3-dim plane"

linear system ⟷ matrix ⟷ geometry (vectors)

METHODS AND IDEAS



- Expanded **criterion for existence**: the constant column is a linear combination (i.e. in the **span**) of the variable coefficient columns ⇔ solutions exist for a linear system ⇔ the last column in REF is not pivotal.

Remark 1. One implication of the expanded criterion above is that, if the coefficient matrix has pivot in each **row** already, which means the column vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ span the entire $\mathbb{R}^m$, the associated linear system is always consistent, no matter what the constant vector $\mathbf{b}$ is. In particular, this is **impossible** when $n < m$.

2 Problems

Example 1. *We will do this together.* Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, where

$$\mathbf{b} = \begin{bmatrix} 9 \\ 1 \\ 10 \end{bmatrix}, \mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}.$$

$Ax + By + Cz = 0$

$x + y - z = 0$

$\begin{cases} 2+1=3 \\ 4+1=5 \end{cases}$

$x_1\vec{a_1} + x_2\vec{a_2} = \vec{b}$

By the expanded criterion above, it is equivalent to determine if the system below is consistent

$$\begin{cases} 2x_1+4x_2=9 \\ x_1+x_2=1 \\ 3x_1+5x_2=10 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 4 & 9 \\ 1 & 1 & 1 \\ 3 & 5 & 10 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 9 \\ 3 & 5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 7 \\ 0 & 2 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 7/2 \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

$\vec{a_1}$ $\vec{a_2}$ $\vec{b}$

REF

this is not pivot position.

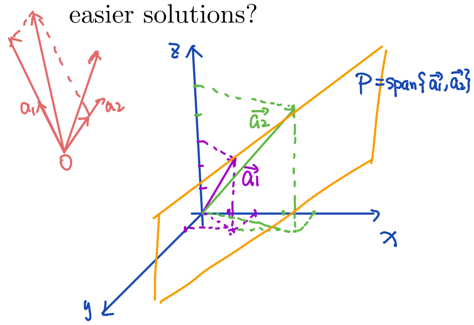
$$\begin{cases} x_1 = 1 - 7/2 = -5/2 \\ x_2 = 7/2 \end{cases}$$

unique solution

$\therefore$ the system is consistent. $\vec{b} = \frac{-5}{2}\vec{a_1} + \frac{7}{2}\vec{a_2}$

Solutions of the system

After you have solved this, think: what does it mean geometrically? Are there easier solutions?



$\vec{b}$ is on P ?

P : $x+y-z=0$

It is easy to check that $\vec{b}$ is on P.

coefficient

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $m=2, n=3$

Example 2. True or false.

(T) The 3-dimensional zero vector $\mathbf{0}$ is a linear combination of the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ as in Example 1. $\vec{0}$ is in any span.

(F) The columns of an $m \times n$ matrix A span the entire $\mathbb{R}^m$ if and only if $m = n$.

Rmk 1 $\iff$ $x_1\vec{a_1} + \cdots + x_n\vec{a_n} = \vec{b}$ consistent for any $\vec{b}$.

(F) $\mathbf{b} \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ if and only if the augmented matrix $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_n & \mathbf{b}\end{bmatrix}$ has the last column as one of its pivot columns. exactly the opposite.

Example 3. Find the values of h such that $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{v} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}.$$

Method ① Want $\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix}$ to be consistent

Do row reductions

Method ② $x_1\vec{u_1} + x_2\vec{u_2} = \vec{v}$

$$\begin{cases} x_1 \cdot 0 + x_2 \cdot 1 = -5 \\ x_1 \cdot (-2) + x_2 \cdot 8 = -3 \end{cases}$$

$x_1 + x_2 \cdot (-3) = h$

Example 4. *With a view toward future lectures.* Graph the following spans in $\mathbb{R}^2$:

$$(1)\,\text{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}; \qquad (2)\,\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\};$$

$$(3)\,\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}; \qquad (4)\,\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}.$$

How is the 4th case different, and why?