Worksheet 29 (April 23)

DIS 119/120 GSI Xiaohan Yan

1 Review

DEFINITIONS

For $\vec{x}'(t) = A\cdot\vec{x}(t)$

- fundamental solution set, fundamental solution matrix.

$\{\vec{f}_1(t), \vec{f}_2(t), \cdots, \vec{f}_n(t)\}$ a basis of the solution space

$$X(t) = \begin{pmatrix} \vec{f}_1(t) & \vec{f}_2(t) & \cdots & \vec{f}_n(t) \end{pmatrix} \rightsquigarrow y(t) = X(t)\cdot\vec{c} = c_1\cdot\vec{f}_1(t) + \cdots + c_n\cdot\vec{f}_n(t).$$

$$\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

METHODS AND IDEAS

Theorem 1. *Consider the 1st-order linear ODE system*

$$\mathbf{x}' = A\mathbf{x},$$

with A being a diagonalizable $n\times n$ matrix. Assume $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is an eigenbasis of A, and denote by $\{\lambda_1, \cdots, \lambda_n\}$ the corresponding eigenvalues. Then

$$\{e^{\lambda_1 t}\mathbf{v}_1, \cdots, e^{\lambda_n t}\mathbf{v}_n\}$$

$$X(t) = \begin{pmatrix} e^{\lambda_1 t}\vec{v}_1 & \cdots & e^{\lambda_n t}\vec{v}_n \end{pmatrix}$$

is a fundamental solution set. In other words, the general solution is

$$\mathbf{x}(t) = c_1\cdot e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n\cdot e^{\lambda_n t}\mathbf{v}_n.$$

Remark 1. When A is diagonalizable over $\mathbb{C}$, we have a similar theorem. More precisely, let's assume $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ are the two complex eigenvalues, and $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$ and $\mathbf{v}_2 = \mathbf{a} - i\mathbf{b}$ be the complex eigenvectors (complex eigenvalues and eiigenvectors come in pairs). Then,

$$\{e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b}, e^{\alpha t}\cos\beta t\mathbf{b} + e^{\alpha t}\sin\beta t\mathbf{a}, e^{\lambda_3 t}\mathbf{v}_3, \cdots, e^{\lambda_n t}\mathbf{v}_n\}$$

is a fundamental solution set. Note that for the first two terms, we are simply taking the real and imagnary parts of $e^{\lambda_1 t}\mathbf{v}_1$:

Solution of $\vec{x}' = A\cdot\vec{x}$

$e^{\lambda_1 t}\cdot\vec{v}_1 = e^{(\alpha+i\beta)t}\cdot(\vec{a}+i\vec{b})$

$$e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b} = \mathrm{Re}(e^{\lambda_1 t}\mathbf{v}_1) = \mathrm{Re}(e^{(\alpha+i\beta)t}(\mathbf{a}+i\mathbf{b})),$$

$$e^{\alpha t}\cos\beta t\mathbf{b} + e^{\alpha t}\sin\beta t\mathbf{a} = \mathrm{Im}(e^{\lambda_1 t}\mathbf{v}_1) = \mathrm{Im}(e^{(\alpha+i\beta)t}(\mathbf{a}+i\mathbf{b})).$$

If $\vec{f}(t) + i\vec{g}(t)$ is a solution of $\vec{x}'(t) = A\cdot\vec{x}(t)$

$\vec{f}'(t) + i\vec{g}'(t) = A\cdot\vec{f}(t) + iA\cdot\vec{g}(t)$

2 Problems

Example 1. Consider the 1st order homogeneous linear system of ODE

$$\mathbf{x}'(t) = \underbrace{\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}}_{A}\mathbf{x}(t).$$

(a) Find a fundamental solution matrix of the system.

(b) Solve the initial value problem $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(c) Sketch the trajectory of $\mathbf{x}(t)$ in (b).

(a) Eigenvalues

$\chi_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$

$\lambda_1 = 2, \quad \lambda_2 = 3$

λ_1: Eigenspace $E_2 = Nul\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$. basis $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

λ_2: $E_3 = Nul\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}$ basis $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$X(t) = \left(e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad e^{3t}\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} e^{2t} & e^{3t} \\ e^{2t} & 2e^{3t} \end{pmatrix}$

(b). $\vec{x}(t) = C_1 \cdot \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + C_2 \cdot \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$.

$\vec{x}(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

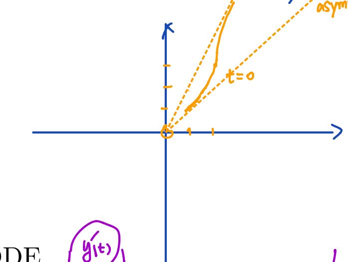
$C_1 = C_2 = 1$

$\vec{x}(t) = \begin{pmatrix} e^{2t} + e^{3t} \\ e^{2t} + 2e^{3t} \end{pmatrix}$

asymptotic $y=2x$ $\quad t \to \infty \quad \sim \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$

asymptotic $y=x$ $\quad t \to -\infty \quad \sim \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$

(c)



$t \to \infty, \quad \dfrac{e^{2t} + 2e^{3t}}{e^{2t} + e^{3t}} = \dfrac{\cancel{e^{-t}} + 2}{\cancel{e^{-t}} + 1} \to 2$

Example 2. Find the general solution of the linear ODE

$$y'''(t) + y(t) = 0.$$

$\begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \\ \vdots \end{pmatrix}$ $\quad \begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \end{pmatrix}$

Idea $\quad \vec{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \end{pmatrix}$.

$\vec{x}'(t) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ * & * & \cdots & * \end{pmatrix}\vec{x}(t)$

↑ from original ODE

$\vec{x}'(t) = \begin{pmatrix} y'(t) \\ y''(t) \\ y'''(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}}_{A}\begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \end{pmatrix}$

$\lambda^3 + 1 = 0$

$y(t) = C_1 \cdot e^{-t} + C_2 \cdot ? + C_3 \cdot ?$ $\quad C_1, C_2, C_3 \in \mathbb{R}$

Eigenvalues

$\chi_A(\lambda) = -\lambda^3 - 1 = 0$. $\quad \lambda_1 = -1, \quad \lambda_2 = \dfrac{1+i\sqrt{3}}{2}, \quad \lambda_3 = \dfrac{1-i\sqrt{3}}{2}$

$\lambda^3 + 1 = 0$

$\parallel$

$(\lambda+1)(\lambda^2 - \lambda + 1)$

$\lambda = r \cdot e^{i\theta}$

$\parallel$

$r\cos\theta + ir\sin\theta$

$r^3 \cdot e^{i3\theta} = -1 \rightarrow \begin{cases} r^3 = 1 \\ 3\theta = \pi, 3\pi, 5\pi, 7\pi, \ldots \end{cases}$

$\rightarrow \begin{cases} r = 1 \\ \theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}. \end{cases}$

$\dfrac{1+i\sqrt{3}}{2} \quad -1 \quad \dfrac{1-i\sqrt{3}}{2}$

$X(t) = \begin{array}{c} y(t) \\ y'(t) \\ y''(t) \end{array}\left(\begin{array}{c|c|c} e^{-t} & & \\ -e^{-t} & e^{\alpha t}\cos\beta t \cdot \vec{a} - e^{\alpha t}\sin\beta t \cdot \vec{b} & ? \\ e^{-t} & & \end{array} \right)$

$\begin{cases} \lambda_1 = -1. & E_{-1} = Nul\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} & \text{basis } \vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \\ \lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2} & E_{\lambda_2} = Nul\begin{pmatrix} -\lambda_2 & 1 & 0 \\ 0 & -\lambda_2 & 1 \\ -1 & 0 & -\lambda_2 \end{pmatrix}. & \text{basis } \vec{v_2} = \begin{pmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \end{pmatrix}. \\ \lambda_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2}. & & \text{basis } \vec{v_2} = \begin{pmatrix} 1 \\ \overline{\lambda_2} \\ \overline{\lambda_2}^2 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_3 \\ \lambda_3^2 \end{pmatrix} \end{cases}$

$(-1)\cdot 1 + 0 \cdot \lambda_2 + (-\lambda_2)\lambda_2^2 = -1 - \lambda_2^3 = 0$