Worksheet 27 (April 16)

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1 Review

Theorem 1. *The solution set of the second-order linear homogeneous ODE*

$$a \cdot y''(t) + b \cdot y'(t) + c \cdot y(t) = 0$$

$a\lambda^2 + b\lambda + c = 0$ λ_1, λ_2

is a 2-dimensional vector space. More precisely, let λ_1 and λ_2 be the two roots of the quadratic equation $a\lambda^2 + b\lambda + c = 0$, then the general solution is given by

- ***if** λ_1 **and** λ_2 **are two distinct real numbers**$(b^2 - 4ac > 0)$,*

$$y(t) = c_1 \cdot e^{\lambda_1 t} + c_2 \cdot e^{\lambda_2 t}, \qquad c_1, c_2 \in \mathbb{R};$$

- ***if** λ_1 **and** λ_2 **are the same real number**$(b^2 - 4ac = 0)$,*

$$y(t) = c_1 \cdot e^{\lambda_1 t} + c_2 \cdot te^{\lambda_2 t}, \qquad c_1, c_2 \in \mathbb{R};$$

- ***if** λ_1 **and** λ_2 **are two complex numbers**$(b^2 - 4ac < 0)$, denote $\lambda_1 = u + iv$ and $\lambda_2 = u - iv$,*

$$y(t) = c_1 \cdot e^{ut} \cos vt + c_2 \cdot e^{ut} \sin vt, \qquad c_1, c_2 \in \mathbb{R}.$$

$e^{ut}\cos vt + i e^{ut}\sin vt = e^{ut}\cdot(\cos vt + i\sin vt) = e^{ut}\cdot e^{ivt} = e^{(u+iv)t} = f_1(t)$, $e^{ut}\cos vt - i e^{ut}\sin vt = \cdots = e^{(u-iv)t} = f_2(t)$

Remark 1. Intuitively, the trigonometric functions come from the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$. Following the pattern of the first two cases, the two basis functions in the third case should have been $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. However, by the Euler formula,

$$e^{\lambda_1 t} = e^{ut+ivt} = e^{ut} \cdot e^{ivt} = e^{ut}(\cos vt + i \sin vt), \quad e^{\lambda_2 t} = \cdots = e^{ut}(\cos vt - i \sin vt),$$

$\frac{f_1(t)+f_2(t)}{2} = e^{ut}\cos vt$

$\frac{f_1(t)-f_2(t)}{2i} = e^{ut}\sin vt$

so we might as well take the real and imaginary parts to be the basis functions.

2 Problems

Example 1. Solve the equations
(a) $2y''(t) - 5y'(t) + 3y(t) = 0$;
(b) $2y''(t) - 8y'(t) + 8y(t) = 0$;
(c) $y''(t) - 4y'(t) + 5y(t) = 0$.

(a) $2\lambda^2 - 5\lambda + 3 = 0$ $\lambda_1 = \frac{3}{2}$, $\lambda_2 = 1$

$y = c_1 \cdot e^{3t/2} + c_2 \cdot e^t$

(b) $2\lambda^2 - 8\lambda + 8 = 0$ $\lambda_1 = 2 = \lambda_2$

$y = c_1 \cdot e^{2t} + c_2 \cdot te^{2t}$

$\frac{4 \pm \sqrt{16 - 4\cdot1\cdot5}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$

(c) $\lambda^2 - 4\lambda + 5 = 0$ $\lambda_1 = 2+i$, $\lambda_2 = 2-i$

$y = c_1 \cdot e^{2t}\cos t + c_2 \cdot e^{2t}\sin t$

$2y'' - 5y' + 3y = 0.$ $\lambda_1 = \frac{3}{2}$, $\lambda_2 = 1.$

Example 2. Consider the equation in (a) of the previous example. Denote by V its solution set. Prove that the linear transformation

$y = c_1 \cdot e^{3t/2} + c_2 \cdot e^{t}$

$$\begin{aligned} T: \quad V &\longrightarrow \mathbb{R}^2 \\ y(t) &\longmapsto \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} \end{aligned}$$

$y' = \frac{3}{2} \cdot e^{3t/2}$

is an isomorphism. $e^{3t/2}$ e^{t} $\begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Hint: You already know a basis of V. $\mathcal{B} = \{ \vec{b_1} = e^{3t/2}, \vec{b_2} = e^{t} \}$

basis of $\mathbb{R}^2$ $E = \{\vec{e_1}, \vec{e_2}\}$

Matrix $\quad {}_E[T]_B = ([T(\vec{b_1})]_E \quad [T(\vec{b_2})]_E)$

$= \begin{pmatrix} 1 & 1 \\ \frac{3}{2} & 1 \end{pmatrix}$

This is an invertible matrix, so T is an isomorphism.

Example 3. Suppose that for $c \in \mathbb{R}$,

$$y''(t) + 2c \cdot y'(t) + c^2 \cdot y(t) = 0,$$

has a solution with $y(0) = 0, y(1) = 1, y(2) = 4$. Find the value of c or disprove existence.

$\underbrace{y'' - 4y' + 4y = 0.}_{e^{2t}}$

$e^{2t}(2^2 - 4 \cdot 2 + 4) = 0$

$\lambda^2 + 2c \cdot \lambda + c^2 = 0$ $\quad \lambda_1 = \lambda_2 = -c$ "always in 2^{nd} case"

$y = c_1 \cdot e^{-ct} + c_2 \cdot t e^{-ct}$

$$\begin{cases} y(0) = c_1 = 0 \\ y(1) = c_1 \cdot e^{-c} + c_2 \cdot e^{-c} = 1 \\ y(2) = c_1 \cdot e^{-2c} + 2c_2 \cdot e^{-2c} = 4 \end{cases}$$

$c_1 = 0$

$$\begin{cases} c_2 \cdot e^{-c} = 1 \\ 2c_2 \cdot e^{-2c} = 4 \end{cases}$$

$2 \cdot e^{-c} = 4$

$e^{-c} = 2$

$\boxed{c = -\ln 2}$

$c_2 = \frac{1}{2}$

$y'' - 2y' + y = 0$

$\to \lambda^2 - 2\lambda + 1 = 0.$ $\lambda = 1$

$\to y_1(t) = e^{t}$ is a solution

because $y'' - 2y' + y$

$= e^{t} - 2e^{t} + e^{t} = 0$

$\to \frac{d}{dt}\left(\frac{dy}{dt}\right) - 2\frac{dy}{dt} + y = 0$

$\left(\frac{d^2}{dt^2} - 2\frac{d}{dt} + 1\right) \cdot y = 0$

$\to e^{t} \in \mathrm{Ker}\left(\frac{d}{dt} - 1\right)$

$\frac{d}{dt}$ linear transformation

$\frac{d^2}{dt^2} - 2\frac{d}{dt} + 1 = \left(\frac{d}{dt} - 1\right) \cdot \left(\frac{d}{dt} - 1\right)$

$= \frac{d}{dt} \cdot \frac{d}{dt} - \frac{d}{dt} - \frac{d}{dt} +$

$\left(\frac{d}{dt} - 1\right)(e^{t}) = e^{t} - e^{t} = 0$

$\underline{\left(\frac{d}{dt}-1\right)}\boxed{\left(\frac{d}{dt}-1\right)y}=0$

$\Uparrow$

$\mathrm{Ker}\left(\frac{d}{dt}-1\right)$

$e^t \qquad te^t$

$\left(\frac{d}{dt}-1\right)y = e^t \qquad \boxed{\frac{dy}{dt}-y=e^t} \rightsquigarrow y=te^t$

$\underbrace{e^{-t}(y'-y)}=1$

$(e^{-t}\cdot y)'=1$

$e^{-t}\cdot y = t+c$

$y = \underline{t\cdot e^t + c\cdot e^t}$