Worksheet 26 (April 14)

DIS 119/120 GSI Xiaohan Yan

singular values : sqrts of e-values of A^TA.

reduced SVD $A = \sigma_1 \vec{u_1}\vec{v_1}^T + \sigma_2 \vec{u_2}\vec{v_2}^T + \cdots + \sigma_r \vec{u_r}\vec{v_r}$

$\vec{v_1} \sim \vec{v_r}$ orthonormal eigenvectors of A^TA

$\vec{u_1} \sim \vec{u_r}$ $\vec{u_i} = \frac{1}{\sigma_i} A\vec{v_i}$. also orthonormal.

$$= \underbrace{\begin{pmatrix} \vec{u_1} \cdots \vec{u_r} \end{pmatrix}_{m\times r}}_{\hat{U}} \underbrace{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}_{r\times r}}_{\hat{\Sigma}} \underbrace{\begin{pmatrix} \vec{v_1}^T \\ \vdots \\ \vec{v_r}^T \end{pmatrix}_{r\times n}}_{\hat{V}^T}$$

1 Review

DEFINITIONS

- Reduced SVD, full SVD;

Given $A_{m\times n}$ $(m>n)$

full SVD $= \underbrace{\begin{pmatrix} \vec{u_1} \cdots \vec{u_r} \cdots \vec{u_m} \end{pmatrix}_{m\times m}}_{U} \cdot \underbrace{\begin{pmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ & 0 & & 0 \end{pmatrix}_{m\times n}}_{\Sigma} \cdot \underbrace{\begin{pmatrix} \vec{v_1}^T \\ \vdots \\ \vec{v_r}^T \\ \vdots \\ \vec{v_n}^T \end{pmatrix}_{n\times n}}_{V^T}$.

Complete orthonormal set to orthonormal basis

$\vec{u_1} \sim \vec{u_r} \mapsto \vec{u_1} \sim \vec{u_m} \in \mathbb{R}^m$

$\vec{v_1} \cdots \vec{v_r} \mapsto \vec{v_1} \cdots \vec{v_n} \in \mathbb{R}^n$

(*). $a\cdot y'' + b\cdot y' + c\cdot y = 0$ $(a, b, c \in \mathbb{R}, a \neq 0)$

- second-order, linear, homogeneous, constant-coefficient, ordinary, differential equations;

Non-examples ① ~~Second-order~~ $y''' + y = 3$, $y' + 2y = 0$ ③ ~~homogeneous~~ $y'' + 2y' + 2y = x^2$

② ~~linear~~ $y''\cdot y = 2$, $(y')^2 + e^y = 0$ ④ ~~Constant-coefficient~~ $x\cdot y'' + e^x\cdot y' + (\sin x)\cdot y = 0$

⑤ ~~ordinary~~ $y = y(x)$ $y = y(x_1, x_2)$ $\frac{\partial y}{\partial x_1} = \frac{\partial y}{\partial x_2} + 3$

- linear algebra point of view of linear differential equations.

$T: C^\infty \to C^\infty$

$f(x) \mapsto T(f(x)) = a\cdot f''(x) + b\cdot f'(x) + c\cdot f(x)$

METHODS AND IDEAS

Idea T is linear transformation (because (*) is a linear equation).
Solving (*) $\Leftrightarrow$ finding $\operatorname{Ker} T$

Theorem 1. *For any $m\times n$ matrix A with rank $A = r$, $\exists$ diagonal matrix $\hat{\Sigma}_{r\times r}$, matrices with orthonormal columns $\hat{U}_{m\times r}$ and $\hat{V}_{n\times r}$ such that*

$$A = \hat{U}\hat{\Sigma}\hat{V}^T = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \cdots + \sigma_r\mathbf{u}_r\mathbf{v}_r^T. \qquad \textit{(Reduced SVD)}$$

Moreover, we may complete $\hat{U}$ and $\hat{V}$ into orthogonal matrices $U_{m\times m}$ and $V_{n\times n}$, and $\hat{\Sigma}$ into a "diagonal" but non-square matrix $\Sigma_{m\times n}$, such that

$$A = U\Sigma V^T. \qquad \textit{(Full SVD)}$$

When A is a symmetric matrix, the latter is the orthogonal diagonalization.

Theorem 2. *The solution set S of the 2nd-order homogeneous linear ODE with constant coefficients*

$$ay'' + by' + cy = 0, \quad y(0) = p, \; y'(0) = q \quad \rightsquigarrow \dim \underbrace{S}_{\text{Solution set of } ay''+by'+cy=0} = 2.$$

where $a \neq 0 \in \mathbb{R}, b, c \in \mathbb{R}$, is a 2-dimensional subspace of the vector space of all smooth functions. Moreover, the initial value problems have unique solutions. In other words, for any initial value condition $y(0) = p, y'(0) = q$, there exists a unique function $y = y(x)$ satisfying both the ODE and the initial value conditions.

2 Problems

Example 1. *Example 3 of Worksheet 24.* Find both the reduced and the full SVD of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix}.$$

$\begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix}$ $\begin{pmatrix} -10 & 10 \\ 10 & -10 \end{pmatrix}$

Step 1 $A^T A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 10 \\ 10 & 14 \end{pmatrix}$

eigenvalues $\lambda_1 = 4$, $\lambda_2 = 24$

eigenspace of $\lambda_1 = 4$ has basis $\vec{v_1} = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$

$\lambda_2 = 24$ has basis $\vec{v_2} = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$

Step 2 singular values $\sigma_1 = 2$, $\sigma_2 = 2\sqrt{6}$

$\vec{u_1} = \frac{1}{\sigma_1} A \vec{v_1} = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix}$

$\vec{u_2} = \frac{1}{\sigma_2} A \vec{v_2} = \begin{pmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{pmatrix}$

Step 3

reduced SVD $= \sigma_1 \vec{u_1} \vec{v_1}^T + \sigma_2 \vec{u_2} \vec{v_2}^T = \begin{pmatrix} \sqrt{2}/2 & \sqrt{3}/3 \\ 0 & \sqrt{3}/3 \\ -\sqrt{2}/2 & \sqrt{3}/3 \end{pmatrix} \begin{pmatrix} 2 & \\ & 2\sqrt{6} \end{pmatrix} \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$

full SVD $= \begin{pmatrix} \sqrt{2}/2 & \sqrt{3}/3 & \sqrt{6}/6 \\ 0 & \sqrt{3}/3 & -\sqrt{6}/3 \\ -\sqrt{2}/2 & \sqrt{3}/3 & \sqrt{6}/6 \end{pmatrix} \begin{array}{l} a \\ b \\ c \end{array} \begin{pmatrix} 2 & 0 \\ 0 & 2\sqrt{6} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$

$\frac{\sqrt{2}}{2} \cdot a + 0 \cdot b + \left(-\frac{\sqrt{2}}{2}\right) \cdot c = 0$

$a - c = 0$

$\frac{\sqrt{3}}{3} a + \frac{\sqrt{3}}{3} b + \frac{\sqrt{3}}{3} c = 0$

$a + b + c = 0$

$a - c = 0$

$a + b + c = 0$

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Example 2. Consider the second-order linear ODE

$$y'' + y = 0.$$

(1) Check that $f(x) = \sin x$ and $g(x) = \cos x$ are both solutions of the equation. (2) Solve the initial value problem $y(0) = 2, y'(0) = -1$. (3) Solve the initial value problem $y(\pi) = 2, y'(\pi) = -1$. (4) Find all solutions of the ODE.

(1) $(\sin x)'' + \sin x$ $= (\cos x)' + \sin x$ $= -\sin x + \sin x$ $= 0$

$(\cos x)'' + \cos x$ $= (-\sin x)' + \cos x$ $= -\cos x + \cos x$ $= 0$

$ay'' + by' + cy = 0$

$a\lambda^2 + b\lambda + c = 0$

Idea $\sin x$ & $\cos x$ form a basis of the 2-dim solution set.

(4) $y = c_1 \sin x + c_2 \cos x$

$y' = c_1 \cos x - c_2 \sin x$

(2) $\begin{cases} y(0) = c_1 \cdot \sin 0 + c_2 \cdot \cos 0 = c_2 = 2 \\ y'(0) = c_1 \cdot \cos 0 - c_2 \cdot \sin 0 = c_1 = -1 \end{cases}$

$T: V \longrightarrow \mathbb{R}^2$

$y(x) \mapsto \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}$

$\boxed{y(x) = -\sin x + 2\cos x}$

(3) $\begin{cases} y(\pi) = c_1 \sin \pi + c_2 \cos \pi = -c_2 = 2 \\ y'(\pi) = c_1 \cos \pi - c_2 \sin \pi = -c_1 = -1 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -2 \end{cases}$

$\boxed{y(x) = \sin x - 2\cos x}$