Worksheet 25 (April 12)

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$\vec{u_1} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\vec{v_1} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\vec{u_1}\cdot\vec{v_1}^T = \begin{pmatrix} x \\ y \end{pmatrix}\cdot(a\ b\ c) = \begin{pmatrix} xa & xb & xc \\ ya & yb & yc \end{pmatrix}$

1 Preview

METHODS AND IDEAS

square roots of non-zero eigenvalues of $A^T\cdot A$.

Theorem 1. *Let A be an $m \times n$ matrix, $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0$ be the singular values of A, and*

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

be the singular value decomposition of A. Then $\operatorname{rank} A = r$, *and for any $i = 1, 2, \cdots, r$,*

$$A_i = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

is the best approximation of A by $\operatorname{rank} = i$ *matrices.* ["orthogonal projection" of A to the subspace of "rank $\leq i$" matrices.]

2 Problems

$(A\cdot A^T)_{m\times m}$ $A_{m\times n}$.

Example 1. True or false. $(A^T)^T\cdot A^T = A\cdot A^T$

square roots of non-zero eigenvalues of $(A^T\cdot A)_{n\times n}$

(T) The singular values of A^T are the same as the singular values of A for any A. square roots of non-zero. of $(A^T)^T A^T$

(T) If a square matrix A is symmetric, so is A^2. $(A^2)^T = (A\cdot A)^T = A^T\cdot A^T = A\cdot A = A^2$

(T) If a square matrix A is symmetric, so is A^{-1}. $(A^{-1})^T\cdot A = I$? $(A^{-1})^T\cdot A = (A^T\cdot A^{-1})^T = (A\cdot A^{-1})^T = I^T = I$

(T) If $(1, 2)^T$ and $(2, 1)^T$ are both eigenvectors of a symmetric 2×2 matrix A, so is $(2, 2)^T$. $\chi_A(\lambda)$. deg = 2. 2 e-values counted with mult.

↳ they have the same eigenvalue
↳ the eigenspace of this e-value is 2-dim.
↳ $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

Prop. For a symmetric matrix A, eigenvectors of distinct eigenvalues are orthogonal.

3 Some Calculus

DERIVATIVES

$f(x)$	$f'(x)$
e^x	e^x
$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$	$n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$
$\ln x$	$1/x$
$\sin x, \cos x$	$\cos x$, $-\sin x$

Useful properties of derivatives:

- $(f \pm g)' = f' \pm g'$;
- $(f \cdot g)' = f' \cdot g + f \cdot g'$;
- $(f/g)' = (f' \cdot g - f \cdot g')/g^2$;
- Denote by h the composition of f and g, i.e. $h(x) = f(g(x))$, then $h' = f'(g) \cdot g'$.

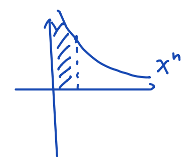
INTEGRALS

$f(x)$	indefinite $\int f(x)dx$	definite $\int_0^x f(s)ds$
e^x	$e^x + C$	$e^x - 1$
$x^n \ (n \neq -1)$	$\frac{1}{n+1} x^{n+1} + C$	$\frac{1}{n+1} x^{n+1}$
x^{-1}	$\ln x + C$	doesn't exist
$\sin x, \cos x$	$-\cos x + C, \sin x + C$	$-\cos x - 1$, $\sin x$

[At $x=0$. $\int_0^0 f(s)ds = 0$]

← for $n<-1$. the expression $\int_0^x f(s)ds$. does not make sense

but $\int_1^x s^{-1} ds = \ln x - \ln 1 = \ln x$.



EXERCISES

Example 2. Find the derivatives of the functions below:

$$x^3 e^2, \tan x, \ln x^2, x \sin \frac{1}{x}.$$

composition of $\sin x$ & $\frac{1}{x}$

$$\left(x \cdot \sin\frac{1}{x}\right)' = x' \cdot \sin\frac{1}{x} + x \cdot \left(\sin\frac{1}{x}\right)'$$

$$= \sin\frac{1}{x} + x \cdot \cos\frac{1}{x} \cdot \left(\frac{1}{x}\right)'$$

$$= \sin\frac{1}{x} + x \cdot \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right)$$

$$= \sin\frac{1}{x} - \frac{1}{x} \cos\frac{1}{x}.$$

Example 3. Find all functions $f = f(x)$ such that $f(x) = f'(x)$.

→ $f(x) = 0$, $f(x) = e^x$, other examples?

→ $f(x) = C \cdot e^x$, then $f'(x) = f(x)$

$\Leftrightarrow f(x) \cdot e^{-x} = C$

$\Leftrightarrow (f(x) \cdot e^{-x})' = 0$

$\Leftrightarrow -e^{-x} \cdot f(x) + e^{-x} \cdot f'(x) = 0$

$\Leftrightarrow e^{-x} (f'(x) - f(x)) = 0$

$\Leftrightarrow f'(x) = f(x)$

$e^{x^2} \cdot (f'(x) + 2x \cdot f(x)) = 0$

$\Updownarrow$

$(e^{x^2} \cdot f(x))'$

$e^{x^2} \cdot f(x) = C.$

$f(x) = C \cdot e^{-x^2}$