Worksheet 24 (April 9)

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1 Review

Symmetric matrix $A = P \cdot D \cdot P^T$

$\downarrow$ generalization

any matrix $A = P \cdot \underset{\sim}{D} \cdot Q^T$

diagonal matrix of singular values

DEFINITIONS

Symmetric $\Updownarrow$

- symmetric matrices, orthogonally diagonalizable;

  $A = A^T$ $\exists$ orthogonal P and diagonal D s.t. $A = P \cdot D \cdot P^{-1}$ $= P \cdot D \cdot P^T$

- singular value (of non-square matrices). $A_{m\times n}$

  are square roots of the nonzero eigenvalues of $A^T \cdot A$.

  $n\times n$ symmetric

  all e-values are non-negative.

METHODS AND IDEAS

Theorem 1. (Properties of Symmetric Matrices)
Let A be a symmetric $n \times n$ matrix, then:
(a) all eigenvalues of A are real;
(b) eigenspaces of different eigenvalues of A are orthogonal;
(c) there is a basis of $\mathbb{R}^n$ consisting of orthonormal eigenvectors of A (which means A is orthogonally diagonalizable).

In fact the converse of (c) is also true: if a square matrix is orthogonally diagonalizable, it must be symmetric, too.

Method 1. (Algorithm of Singular Value Decomposition)
Assume that we are given an $m \times n$ matrix A with $m > n$.

$\boxed{A}_{m\times n}$ $\boxed{A^TA}_{n\times n}$

1 *Find eigenvalues of the symmetric matrix A^TA, which are all real and non-negative. Denote them by $\lambda_1, \cdots, \lambda_n$. Assume that the first r of them are strictly positive and the rest are zero.*

$D_{r\times r} = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{pmatrix}.$

2 *Find orthonormal eigenvectors of A^TA corresponding to the strictly positive eigenvalues $\lambda_1, \cdots, \lambda_r$. Denote them by $\mathbf{v}_1, \cdots, \mathbf{v}_r$.*

$Q_{n\times r} = \begin{pmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_r} \end{pmatrix}.$

3 *Let $\sigma_i = \sqrt{\lambda_i}$ for $i = 1, \cdots, r$. These are the **singular values** of A.*

$P_{m\times r} = \begin{pmatrix} \vec{u_1} & \cdots & \vec{u_n} \end{pmatrix}$

4 *Let $\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$ for $i = 1, \cdots, r$. Then,*

$$\boxed{A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T.}$$

$$\boxed{A_{m\times n} = P_{m\times r} \cdot D_{r\times r} \cdot Q^T_{r\times n}}$$

*This is the **singular value decomposition** (SVD) of A.*

Note that in this case rank $A = r$. Moreover, $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ is an orthonormal basis of Row(A), while $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ is an orthonormal basis of Col(A).

2 Problems

Example 1. Diagonalize the matrix below with orthogonal matrices.

$$S = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

e-value λ. $\det(S-\lambda I)=0$.

Want λ s.t. $S-\lambda I$ is not invertible.

$\lambda=1$ $S-I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$\dim E_1 = 2$

$\lambda=4$ $S-4I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

Step 1 Find eigenvalues.

$\chi_S(\lambda) = \cdots$

$\lambda_1 = 1$ $\lambda_2 = 4$

Step 2. Find basis for each eigenspace

For $\lambda_1=1$, basis of $E_1 = Nul(S-I)$

$\left\{ \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{\vec{u_1}}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{u_2}} \right\}$

For $\lambda_2=4$. basis of $E_4 = Nul(S-4I)$

$\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\vec{u_3}} \right\}$.

Step 3 Apply G-S to make the eigenvectors orthonormal.

Note It is sufficies to do G-S within each eigenspace.

$\vec{w_1} = \vec{u_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. $\xrightarrow{\text{unit}}$ $\begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$.

$\vec{w_2} = \vec{u_2} - \frac{\vec{u_2}\cdot\vec{w_1}}{\vec{w_1}\cdot\vec{w_1}} \cdot \vec{w_1} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$. $\xrightarrow{\text{unit}}$ $\begin{pmatrix} -\sqrt{6}/6 \\ -\sqrt{6}/6 \\ \sqrt{6}/3 \end{pmatrix}$

$\vec{w_3} = \vec{u_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\xrightarrow{\text{unit}}$ $\begin{pmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{pmatrix}$.

Example 2. Let us still consider the matrix S from the previous example. Is there a vector $\mathbf{x} \in \mathbb{R}^3$ such that $\mathbf{x}^T S\mathbf{x} < 0$?

Conclusion. $S = P\cdot D\cdot P^{-1} = P\cdot D\cdot P^T$

where $P = \begin{pmatrix} -\sqrt{2}/2 & -\sqrt{6}/6 & \sqrt{3}/3 \\ \sqrt{2}/2 & -\sqrt{6}/6 & \sqrt{3}/3 \\ 0 & \sqrt{6}/3 & \sqrt{3}/3 \end{pmatrix}$.

and $D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 4 \end{pmatrix}$.

Idea. $\vec{x} = C_1\cdot\vec{w_1} + C_2\cdot\vec{w_2} + C_3\cdot\vec{w_3}$

$S\vec{x} = C_1\cdot S\vec{w_1} + C_2\cdot S\vec{w_2} + C_3\cdot S\vec{w_3}$

$= C_1\vec{w_1} + C_2\cdot\vec{w_2} + 4C_3\cdot\vec{w_3}$

$\vec{x}^T S\vec{x} = \vec{x}\cdot(S\vec{x}) = (C_1\vec{w_1} + C_2\vec{w_2} + C_3\vec{w_3})\cdot(C_1\vec{w_1} + C_2\cdot\vec{w_2} + 4C_3\cdot\vec{w_3})$

$= 1\cdot C_1^2 + 1\cdot C_2^2 + 4\cdot C_3^2 \geq 0$

Reason All e-values of S are positive.

Example 3. Find SVD of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix}.$$