Worksheet 23 (April 2)

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1 Review

METHODS AND IDEAS

Theorem 1. *The row rank* (dim Row(A)) *is equal to the column rank* (dim Col(A)) *for any matrix. In fact we have*

$$\dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A) = n \text{ and } \dim \overset{\text{pivot}}{\mathrm{Col}(A)} + \dim \overset{\text{free var.}}{\mathrm{Nul}(A)} = n,$$

where the former comes from the orthogonality, and the latter comes from the Rank-Nullity theorem.

Remark 1. $\subset \mathbb{R}^n \supset$ $\quad$ $\subset \mathbb{R}^m \supset$

$$\mathrm{Row}(A) = \mathrm{Nul}(A)^{\perp}, \mathrm{Col}(A) = \mathrm{Nul}(A^T)^{\perp}.$$

Consider A^T. $\mathrm{Row}(A^T) = \mathrm{Nul}(A^T)^{\perp}$, $\mathrm{Row}(A^T) = \mathrm{Col}(A)$

for $A_{m\times n}$.

Another idea of proof

Remark 2. Another way to see the theorem is by row reduction. In fact, both the row rank and the column rank are preserved by row reductions. (Why?) So we may reduce the theorem to the case of RREF. But in RREF both ranks are equal to the number of pivots.

→ norms, angles

Theorem 2. *Otrhgonal matrices preserve the inner product. In other words, given an orthogonal matrix U, we have*

$$U\mathbf{x} \cdot U\mathbf{y} = \mathbf{x} \cdot \mathbf{y}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

$$U\vec{x} \cdot U\vec{y} = \underset{1\times n}{(U\vec{x})^T} \underset{n\times 1}{U\vec{y}} = \vec{x}^T U^T U \vec{y} = \vec{x}^T\vec{y} = \vec{x}\cdot\vec{y}$$

Remark 3. In particular, this gives $||U\mathbf{x}|| = ||\mathbf{x}||$ if we take $\mathbf{x} = \mathbf{y}$.

2 Problems

$$\underset{\det(U)^2}{\det(U^T)\cdot\det(U)} = \det(U^T U) = 1 \qquad U^T U = I_n \quad \rightsquigarrow \det(U) = \pm 1.$$

Example 1. True or false.

Example $U = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. $\det(U) = 1$

$U = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\det(U) = -1$

(F) Let U be an orthogonal matrix, then $\det(U) = 1$.

(T) Let U be an orthogonal matrix and $\mathbf{x}$ a vector such that $U\mathbf{x}$ and $\mathbf{x}$ are linearly dependent, then $U\mathbf{x} = \pm\mathbf{x}$.

Eigenvalues of orthogonal matrix can only be 1 or -1.

① $\vec{x} = \vec{0}$ then $U\vec{x} = \vec{0}$, so $U\vec{x} = \vec{x}$

② $\vec{x} \neq \vec{0}$ $U\vec{x}$ is a multiple of $\vec{x}$

($\vec{x}$ is an e-vector of U)

$\|U\vec{x}\| = \|\vec{x}\|$ $\quad U\vec{x} = \pm\vec{x}$

(F) If U is diagonal and orthogonal, then U must be an identity matrix.

Counterexample $U = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

(T) The Gram-Schmidt process produces from a linearly independent set $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ an orthogonal set $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ with the property that for each k,

$$\operatorname{span}\{\mathbf{w}_1, \cdots, \mathbf{w}_i\} = \operatorname{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_i\}, \forall i = 1, 2, \cdots, k.$$

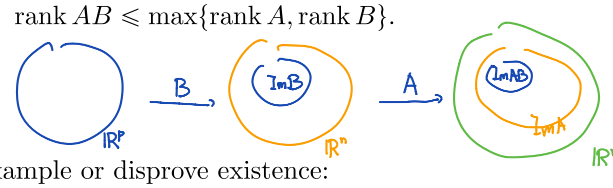
(T) For any two matrices $\underset{m\times n}{A}$ and $\underset{n\times p}{B}$ such that $\underset{m\times p}{AB}$ is well-defined,

$$\operatorname{rank} AB \leqslant \max\{\operatorname{rank} A, \operatorname{rank} B\}.$$

rank AB = dim Im(AB)

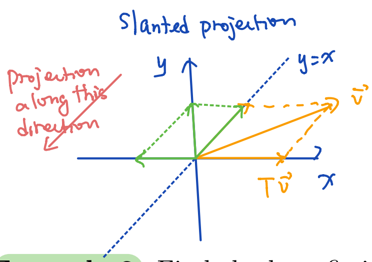
$\leqslant \qquad\qquad \cap$

rank A = dim Im(A)



Example 2. Find an example or disprove existence:
a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $T^2 = T$ but is not an orthogonal projection.

idempotence



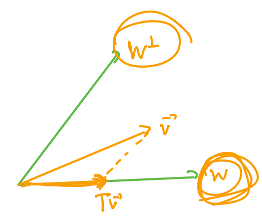
Take basis $\mathcal{B} = \{\vec{b_1} = \binom{1}{0}, \vec{b_2} = \binom{1}{1}\}$ arbitrary

Then $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \xrightarrow{P_1} \mathbb{R}^2$

$\vec{v} \longmapsto [\vec{v}]_{\mathcal{B}} \longmapsto \operatorname{Proj}_{x\text{-axis}} [\vec{v}]_{\mathcal{B}}$

$\binom{x}{y} \longmapsto \binom{x}{0}.$

Standard matrix of T is $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. $T^2 = T$.



$T^2 = T.$

$\Rightarrow T^2\vec{v} = T\vec{v}$

$T(T\vec{v}) = T\vec{v}.$

$\Rightarrow \operatorname{Im} T \subset E_1$

$E_1 \oplus E_0 = \mathbb{R}^n$

Example 3. Find the best fitting model in the form $y = ax^2 + bx + c$ of the data points

$x=-1, y=1$

$$(-1, 1), (0, 0), (1, 1), (2, 1).$$

Find the least square solution of

$$\begin{cases} 1 = a - b + c \\ 0 = c \\ 1 = a + b + c \\ 1 = 4a + 2b + c \end{cases}$$

$A^T A \vec{x} = A^T \vec{b}$

Rmk This method works even though the model is not "a linear model". It works as long as the model depends linearly on the coefficients.

Example 4. Find the values of a, b, c, d, e, f, g such that U is an orthogonal matrix:

$$\begin{pmatrix} 1 & c & e \\ a & \frac{\sqrt{2}}{2} & f \\ b & d & g \end{pmatrix}.$$