Worksheet 22 (March 31)

dot product

matrix multiplication

$\vec{x}\cdot\vec{y} = x^T\cdot y$

$1\times n \quad n\times 1$

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Col. of U: $\vec{u_1}, \cdots, \vec{u_n}$

then $U^T\cdot U = (\vec{u_i}\cdot\vec{u_j})_{i,j}$

Let A, B be $n\times m$ matrices.

then $\underset{m\times n}{A^T}\cdot\underset{n\times m}{B} = \begin{pmatrix} & & \\ i\text{-th} & \square & \\ & j\text{-th} & \end{pmatrix}_{m\times m}$

$\square = \vec{\alpha_i}\cdot\vec{\beta_j} = \alpha_i^T\cdot\beta_j$

where $\vec{\alpha_i}$ is the i-th col. of A, and $\vec{\beta_j}$ is the j-th col. of B.

1 Review

$U^TU = UU^T = I_n$ (or $U^T = U^{-1}$).

DEFINITIONS

- orthogonal matrix;

$\Updownarrow$?

square + orthonormal columns $\Longleftrightarrow$ square + orthonormal rows

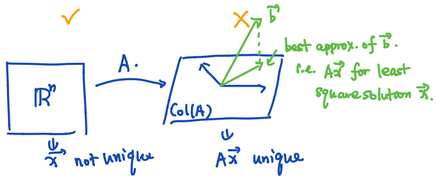
- least square solution of a linear system, existence and uniqueness?

$A\vec{x} = \vec{b}$. ✓ ✗

Solution $\vec{x}$ of the linear system

$A^TA\vec{x} = A^T\vec{b}$

always consistent



METHODS AND IDEAS

Theorem 1. (Orthogonal Projection)
Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ be an ***orthonormal basis*** *of the subspace $W \subset \mathbb{R}^n$. Then the orthogonal projection $\mathrm{Proj}_W\,\mathbf{v}$ of a vector $\mathbf{v} \in \mathbb{R}^n$ is*

$$\mathrm{Proj}_W\,\mathbf{v} = UU^T\mathbf{v}, \quad = (\vec{v}\cdot\vec{u_1})\vec{u_1} + (\vec{v}\cdot\vec{u_2})\vec{u_2} + \cdots + (\vec{v}\cdot\vec{u_k})\vec{u_k}.$$

where U is the matrix whose columns are $\mathbf{u}_1, \cdots, \mathbf{u}_n$. In other words, the standard matrix (i.e. the matrix under the standard basis) of the linear transformation Proj_W is UU^T.

orthogonal proj. to W. it's a linear transformation

Remark 1. The kernel and image of Proj_W are $W^\perp$ and W respectively.

Remark 2. Proj_W can only have two eigenvalues 0 and 1. Moreover, $E_1 = W$ and $E_0 = W^\perp$.

$\mathrm{Proj}_W\vec{v} = \vec{0}$

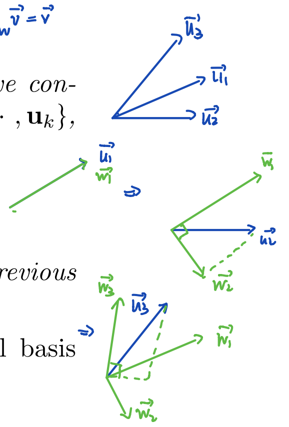
$\mathrm{Proj}_W\vec{v} = \vec{v}$

Method 1. (Gram-Schmidt Process)
Given a (not necessarily orthogonal) basis $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ of $W \subset \mathbb{R}^n$, we construct inductively an orthogonal basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ of W out of $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$, by taking

$$\mathbf{w}_1 = \mathbf{u}_1, \mathbf{w}_{i+1} = \mathbf{u}_{i+1} - \left(\frac{\mathbf{u}_{i+1}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 + \cdots + \frac{\mathbf{u}_{i+1}\cdot\mathbf{w}_i}{\mathbf{w}_i\cdot\mathbf{w}_i}\mathbf{w}_i\right).$$

In other words, we remove from u_{i+1} its "orthogonal projection to the previous vectors" to obtain w_{i+1}.

Remark 3. If you want an orthonormal basis, divide the orthogonal basis vectors by their norms.



$$\vec{x} = \underbrace{\mathrm{Proj}_W\vec{x}}_{\in W} + \underbrace{\mathrm{Proj}_{W^\perp}\vec{x}}_{\in W^\perp}$$

2 Problems

$\sum_{\lambda}$ geo. mult. of λ $= n$.

$\Updownarrow$

geo. mult. = alg. mult
for all e-values

$\Updownarrow$

Example 1. True or false.

(T) The dot product $\mathbf{x} \cdot \mathbf{y}$ is the only entry of the 1×1 matrix $\mathbf{x}^T\mathbf{y}$.

(T) The standard matrix of an orthogonal projection is always diagonalizable.

(T) If the linear system $A\mathbf{x} = \mathbf{b}$ is consistent, the set of its least square solutions is exactly its solution set.

$\vec{x}$ s.t. $A\vec{x}$ is the best approx. of $\vec{b}$ in Col(A).

e-values of ortho. proj. :

0 $\quad E_0 = W^{\perp}$ $\quad$ geo. mult. = $\dim W^{\perp}$

⊕ 1. $\quad E_1 = W$ $\quad$ geo. mult. = $\dim W$.

$\dim W^{\perp} + \dim W = n$

Example 2. Consider

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 1 \\ 4 & -1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Find the least square solutions of $A\mathbf{x} = \mathbf{b}$.

(b) Find the best approximation (orthogonal projection) of $\mathbf{b}$ ~~to~~ in Col(A).

$//$ $A\vec{x}$ for any least square sol. $\vec{x}$.

(a) $\quad A^TA\vec{x} = A^T\vec{b}$.

$$A^TA = \begin{pmatrix} 30 & -2 \\ -2 & 4 \end{pmatrix}$$

$$A^T\vec{b} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}.$$

Solve $\begin{pmatrix} 30 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10/29 \\ 5/29 \end{pmatrix}$$

(b) $\quad A\vec{x}$ for $\vec{x}$ least square solution.

So $\quad \text{Proj}_{\text{Col}(A)}\vec{b} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 10/29 \\ 5/29 \end{pmatrix} = \cdots$

Example 3. Find an orthonormal basis of the subspace $W = \{(x, y, z, w) | x + y + z + w = 0\}$ of $\mathbb{R}^4$.

$x + y + z + w = 0$.

Nul$(1 \ 1 \ 1 \ 1)$

$$\rightarrow \text{basis} \quad \left\{ \overset{\vec{u}_1}{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \overset{\vec{u}_2}{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \overset{\vec{u}_3}{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$$

$$\rightarrow \vec{w}_1 = \vec{u}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \longrightarrow \frac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

innerproduct $\vec{u}_2 \cdot \vec{w}_1$

$$\vec{w}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \cdot \vec{w}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}. \longrightarrow \frac{\sqrt{6}}{3}\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}.$$

$$\vec{w}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle}\vec{w}_1 - \frac{\langle \vec{u}_3, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle}\vec{w}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix}. \rightarrow \frac{\sqrt{3}}{2}\begin{pmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix}.$$

not $\vec{u}_1, \vec{u}_2$