Worksheet 20 (March 19)

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inner prod. vector space. ← length, angle, orthogonality, ...

⇑ def. inner product

vector space. ← L.I., span, subspaces, ...

⇑ def. addition & scalar multiple

Set

1 Review

$\mathbb{R}^n$

DEFINITIONS

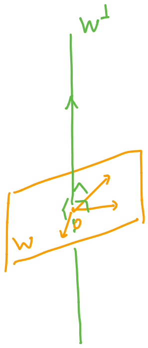
- metric geometry of $\mathbb{R}^n$: inner product, length, angle;

  $\|\vec{x}\| = \sqrt{\vec{x}\cdot\vec{x}}$ $\langle \vec{x}, \vec{y}\rangle = \arccos \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\cdot\|\vec{y}\|}$.

  $\vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$.

  Properties: ① Symmetric $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$.

  ② Bilinear $(a\vec{u}+b\vec{v})\cdot\vec{y} = a(\vec{u}\cdot\vec{y}) + d(\vec{v}\cdot\vec{y})$, $\vec{x}\cdot(c\vec{u}+d\vec{v}) = c(\vec{x}\cdot\vec{u}) + d(\vec{x}\cdot\vec{v})$



- orthogonal set, orthonormal set, orthogonal basis;

  $\{\vec{v_1}, \vec{v_2}, \cdots \vec{v_k}\}$. s.t. $\vec{v_i}\cdot\vec{v_j} = 0$ $(i\neq j)$

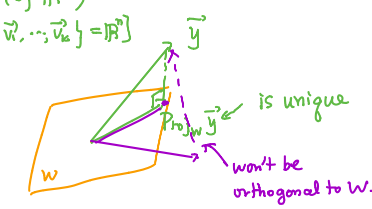
  ↳ ① orthogonal set ② each vector is unit, i.e. $\vec{v_i}\cdot\vec{v_i} = 1$

  ↳ ① orthogonal set ② form a basis (of $\mathbb{R}^n$) [need only $\mathrm{Span}\{\vec{v_1}, \cdots, \vec{v_k}\} = \mathbb{R}^n$]

- orthogonal complement, orthogonal projection.

  of subspace $W \subset \mathbb{R}^n$: $W^\perp = \{\vec{v} \mid \vec{v}\perp\vec{w},\ \forall \vec{w}\in W\}$.

  of a vector $\vec{y}\in\mathbb{R}^n$ to subspace $W\subset\mathbb{R}^n$: $\mathrm{Proj}_W \vec{y}$ " best approx. of $\vec{y}$ in W"



METHODS AND IDEAS

Theorem 1. *If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are orthogonal vectors, then they are linearly independent.*

Theorem 2. (Orthogonal Projection)
*Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be an **orthogonal** basis of subspace $W \subset \mathbb{R}^n$, then for any vector $\mathbf{y} \in \mathbb{R}^n$, its orthogonal projection to W is*

$$\hat{\mathbf{y}} = \mathrm{Proj}_W \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{w}_k}{\mathbf{w}_k\cdot\mathbf{w}_k}\mathbf{w}_k.$$

Remark 1. The orthogonal projection $\hat{\mathbf{y}}$ is the closest to $\mathbf{y}$ among all vectors in W. Moreover, it is the unique vector in W such that $\mathbf{y} - \hat{\mathbf{y}}$ is orthongonal to W. In other words, the decomposition of any vector $\mathbf{y}$ into the sum of W and $W^\perp$ is unique, and it is exactly $\hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}})$.

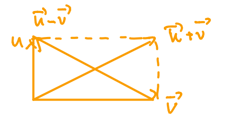
$\vec{y} = \underset{\underset{W}{\cap}}{(\hat{\vec{y}})} + \underset{\underset{W^\perp}{\cap}}{(\vec{y}-\hat{\vec{y}})}$

Remark 2. When W in the theorem is the full subspace $\mathbb{R}^n$, $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ is a basis of $\mathbb{R}^n$ and thus the formula gives an easy way to compute the $\mathcal{W}$-coordinate of $\mathbf{y}$, i.e.

$$[\mathbf{y}]_{\mathcal{W}} = \begin{pmatrix} \frac{\mathbf{y}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1} \\ \vdots \\ \frac{\mathbf{y}\cdot\mathbf{w}_k}{\mathbf{w}_1\cdot\mathbf{w}_k} \end{pmatrix}.$$

$$\vec{u}\cdot\vec{v}=0$$
$$\Updownarrow$$
$$-2\vec{u}\cdot\vec{v} = 2\vec{u}\cdot\vec{v}$$
$$\Updownarrow$$
$$(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$
$$\Updownarrow$$
$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$$
$$\Updownarrow$$

2 Problems



Example 1. True or false.

(T) If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are orthogonal, $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$.

(T) (F) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a set of orthogonal vectors if and only if any two different vectors of it are orthogonal to each other.

(F) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a set of linearly independent vectors if and only if any two different vectors of it are linearly independent to each other.

Same plane but not same direction.

If W is a plane.
① in $\mathbb{R}^3$, then $W^\perp$ is a line
② in $\mathbb{R}^4$, then $W^\perp$ is a plane, because $\dim W^\perp + \dim W = 4$
③ in $\mathbb{R}^{\geq 5}$, $\dim W^\perp \geq 3$

(T) Let $W \subset \mathbb{R}^n$ be a subspace, then its orthogonal complement $W^\perp$ is a subspace of $\mathbb{R}^n$ of complementary dimension. $\dim_{\mathbb{R}} W + \dim_{\mathbb{R}} W^\perp = n.$

(T) Let $W \subset \mathbb{R}^n$ be a subspace, then $(W^\perp)^\perp = W$. ① $W \subset (W^\perp)^\perp$ ② $\dim_{\mathbb{R}} W = \dim_{\mathbb{R}} (W^\perp)^\perp$.

(T) Let $W \subset \mathbb{R}^n$ be a subspace and $W^\perp$ be its orthogonal complement. If $\mathbf{v}$ is in both W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

$\vec{v} \perp \vec{v}$, i.e. $\vec{v}\cdot\vec{v} = 0$

Example 2. Consider the vector

$$\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \in \mathbb{R}^2.$$

(a) Compute $\underset{3}{\mathbf{v} \cdot \mathbf{e}_1}$ and $\underset{4}{\mathbf{v} \cdot \mathbf{e}_2}$.
(b) Suppose $\mathbf{u} \in \mathbb{R}^2$ is a unit vector satisfying $\mathbf{u} \cdot \mathbf{v} = 2$. Find $\mathbf{u}$.
(c) Find the area of the triangle formed by the origin and the endpoints of $\mathbf{u}$ and $\mathbf{v}$.

(b) $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$.

$$\begin{cases} a^2 + b^2 = 1 & ① \\ 3a + 4b = 2. & ② \end{cases}$$

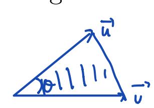
② $a = \dfrac{2-4b}{3}$

Plug in ①. $\dfrac{(4b-2)^2}{9} + b^2 = 1.$

$b = \dfrac{8 \pm 3\sqrt{21}}{25}$

$a =$ " "

(c)



$\boxed{S_\Delta = \frac{1}{2}\|\vec{u}\|\cdot\|\vec{v}\|\cdot\sin\theta}$

but $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$.

$= \dfrac{2}{5}$

So $\sin\theta = \dfrac{\sqrt{21}}{5}$

So $S_\Delta = \frac{1}{2}\cdot 1 \cdot 5 \cdot \dfrac{\sqrt{21}}{5}$

$= \sqrt{21}/2$

Example 3. Let W be the plane in $\mathbb{R}^3$ given by $x + y + z = 0$.
(a) Find the orthogonal projection of $\mathbf{x} = (7, -1, 3)^T$ to W.
(b) Find all $\mathbf{y}$ whose orthogonal projection to W is $(2, 2, -4)^T$.