Worksheet 2 (Jan. 25)

DIS 119/120 GSI Xiaohan Yan

Ex (About free variable).

$$\begin{cases} x_1 + x_2 = 0 \\ x_3 = 1 \end{cases}$$

$$\begin{bmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

constant column

pivots

only non-pivot coefficient column ⇝ free variable.

$\{(-s, \underline{s}, 1) \mid s \in \mathbb{R}\}$

we can take any value here.

1 Review

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

DEFINITIONS

info/numbers/rows/columns

- matrix, augmented matrix ($m \times n{+}1$), coefficient matrix ($m \times n$);

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

n variables, m equations

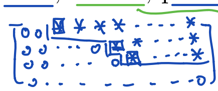
- three types of row reductions: scaling, interchange, replacement;

$① \to ① \times 2$ $\quad ① \leftrightarrow ②$ $\quad ③ \to ③ + (-1)②$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Coefficient columns that are not pivotal give rise to free variables.

- REF, RREF, pivot column, pivot position;



① pivots = 1

② entries above pivots are 0.

heading nonzero entries in the rows

Columns containing pivot positions

- existence and uniqueness of solutions.

≥1 Solutions =1 Solution

	0 Solution	1 solution	≥2 solutions
now	inconsistent	unique, existent	exist

METHODS AND IDEAS

- To solve linear systems, we transform them into simpler systems. This can be done by writing the coefficients and constants of a system into a matrix (called the **augmented matrix**), and then doing row reductions to transform the matrix into its **row Echelon form**, or even its **row reduced Echelon form**. [For the algorithm see P12 of the professor's lecture notes 2.]

- *(Criterion of existence and uniqueness)* Solutions of a linear system **exist** if and only if the last column (the constant column) of REF of the augmented matrix is not pivot. The solution **uniquely exists** if and only if there are no free variables, i.e. all except the last column of REF are pivot.

- Note that we need only **REF** but not RREF to determine existence and uniqueness.

$$\begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[\text{(Start with bottom rows)}]{\text{backward}} \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[\text{①→①-3·③}]{\text{①→①-②}}$$

②→②-③

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

2 Problems

Example 1. *We will do this together.* Solve the linear systems below, by writing out the augmented matrix and applying appropriate row reductions.

Keep coefficients small (and integral) when doing row reductions.

(a) $\begin{cases} x_1 + x_2 + 3x_3 = 3 \\ 2x_1 + 2x_2 - x_3 = -1 \\ x_1 + 3x_2 + 5x_3 = -5 \end{cases}$, (b) $\begin{cases} x_1 + x_2 + 3x_3 = 3 \\ 2x_1 + 2x_2 - x_3 = -1 \end{cases}$.

RREF

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution $\{(-s, s, 1) \mid s \in \mathbb{R}\}$

Rmk Take any value for free vars, and solve other vars in terms of them.

Convention of not writing zero at bottomleft corner

Write out the augmented

$$\begin{bmatrix} 1 & 1 & 3 & 3 \\ 2 & 2 & -1 & -1 \\ 1 & 3 & 5 & -5 \end{bmatrix} \xrightarrow[\text{②→②-2·①}]{\text{Do row reductions}} \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 0 & -7 & -7 \\ 1 & 3 & 5 & -5 \end{bmatrix} \xrightarrow{\text{③→③-①}} \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 0 & -7 & -7 \\ 0 & 2 & 2 & -8 \end{bmatrix}$$

$$\xrightarrow[\text{③→③/2}]{\text{②→②/(-7)}} \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \xrightarrow{\text{②}\leftrightarrow\text{③}} \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

REF RREF

$\{(5, -5, 1)\}$ Solution is unique

Example 2. True or false.

(F) A linear system is inconsistent (no solution) if it has at least ~~two different solutions~~.

(T) A linear system with more than 1 solution has infinitely many solutions.

(F) Whenever a linear system has free variables, the solution set contains at least two solutions. ↘ 0 or ≥ 2

(T) Every elementary row operation is reversible.

(F) Two matrices are row equivalent (A → B through row operations) if they have the same number of rows.

(F) The row echelon form (REF) of a matrix is unique.

RREF is unique

① Trichotomy theorem
0 | 1 | ∞

② ≥ 2 solutions { consistent, free variables exist }

Can take any value in $\mathbb{R}$ for free vars, which will give rise to infinitely many solution

③ [geometric]
⇒ solutions of linear system
⇔ pts on intersection of planes.
⇒ If intersection of planes contains ≥ 2 pts, then it contains infinitely many pts.
pt, line, plane, ... "flat objects"

Example 3. What do we know of the consistency of a linear system
(a) whose augmented matrix is 3×5 with the 5th column being pivot? inconsistent
(b) whose coefficient matrix is 3×5 with 2 pivot columns? not sure
(c) whose coefficient matrix is 3×5 with 3 pivot columns? consistent

Example 4. Find all intersection points of the following three planes in $\mathbb{R}^3$

$$x_1 + x_2 + 3x_3 = 3, 2x_1 + 2x_2 - x_3 = -1, x_1 + 3x_2 + 5x_3 = -5.$$

Hint: Recall Example 1.

Remark 1. What else can the intersection of three planes in $\mathbb{R}^3$ look like? If this is too hard, what can the intersection of two lines in $\mathbb{R}^2$ look like?