Worksheet 19 (March 17)

$T(\vec{v}) = \lambda \cdot \vec{v}$

e-value λ.
e-vector $\vec{v}$
linear trans. $T: V \to V$

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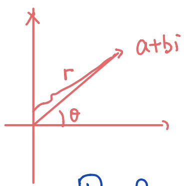
basis $\mathcal{A}$ — basis $\mathcal{B}$

$a+bi = r \cdot e^{i\theta}$.

$r = \sqrt{a^2+b^2}$, $\theta = \arctan\frac{b}{a}$.

$e^{i\theta} = \cos\theta + i\sin\theta = \frac{b}{\sqrt{a^2+b^2}} + i \cdot \frac{a}{\sqrt{a^2+b^2}}$



Real

Complex

1 Review

$A \cdot [\vec{v}]_{\mathcal{A}} = \lambda \cdot [\vec{v}]_{\mathcal{A}}$ $\quad A = {}_{\mathcal{A}}[T]_{\mathcal{A}}$

${}_{\mathcal{B}}[T]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{A}} \cdot {}_{\mathcal{A}}[T]_{\mathcal{A}} \cdot (P_{\mathcal{B}\leftarrow\mathcal{A}})^{-1}$ $\quad B = {}_{\mathcal{B}}[T]_{\mathcal{B}}$

$T: V \to V$.

e-value λ. $\to$ e-value λ

e-vector $\vec{\alpha} = [\vec{v}]_{\mathcal{A}}$ $\to$ e-vector $\vec{\beta} = [\vec{v}]_{\mathcal{B}}$

$[\vec{v}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{A}} \cdot [\vec{v}]_{\mathcal{A}}$.

${}_{\mathcal{A}}[T]_{\mathcal{A}} = P_{\mathcal{A}\leftarrow\mathcal{B}} \cdot {}_{\mathcal{B}}[T]_{\mathcal{B}} \cdot (P_{\mathcal{A}\leftarrow\mathcal{B}})^{-1}$

$[\vec{v}]_{\mathcal{A}} = P_{\mathcal{A}\leftarrow\mathcal{B}} \cdot [\vec{v}]_{\mathcal{B}}$.

DEFINITIONS

- eigenvalue and eigenvector of linear transformation;

  λ $\quad \vec{v} \in V$.

  s.t. $T(\vec{v}) = \lambda \cdot \vec{v}$

- complex number, conjugate, absolute value;

  $a+bi$ $(i^2 = -1)$ $\quad \overline{(a+bi)} = a-bi$ $\quad r = \sqrt{a^2+b^2}$

  $= r \cdot e^{i\theta}$, $r = \sqrt{a^2+b^2}$, $\theta = \arctan\frac{b}{a}$ $\quad$ "reflection across the real axis"!

- complex eigenvalue, complex eigenvector.

  $\lambda \in \mathbb{C}$ $\quad \vec{v} \in \mathbb{C}^n$

  s.t. $A \cdot \vec{v} = \lambda \cdot \vec{v}$

  $A \cdot \overline{(\vec{v})} = \bar{\lambda} \cdot \overline{(\vec{v})}$

Fact: If A real matrix.

① If $\vec{v}$ is an e-vector of a complex e-value $\lambda \in \mathbb{C}$, then $\vec{v}$ must be a complex vector.

② If λ is e-value of A, then so is $\bar{\lambda}$.
If $\vec{v}$ is e-vector of A with e-value λ, then $\overline{(\vec{v})}$ is e-vector of A with e-value $\bar{\lambda}$.

$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a}{a^2+b^2} + \frac{bi}{a^2+b^2}$

$\frac{1}{i} = \frac{-i}{i\cdot(-i)}$

2 Problems

Example 1. Find the complex eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Then find a basis of $\mathbb{R}^2$ under which the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a scaling followed by rotation.

Take $P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ then $P^{-1}AP = \begin{pmatrix} 1+2i & \\ & 1-2i \end{pmatrix}$.

eigenbasis $\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$ of $\mathbb{C}^2$

$\to$ Find e-values.

$\chi_A(\lambda) = \det(A - \lambda I)$

$= \lambda^2 - 2\lambda + 5 = 0$

$\lambda = 1 \pm 2i$

$\to$ For $\lambda = 1+2i$.

$E_{1+2i} = \mathrm{Nul}(A - \lambda I) = \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix}$

$i \cdot$ first col. $+ 1 \cdot$ second col. $= \vec{0}$

basis of E_{1+2i}: $\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$.

Similarly, for $\lambda = 1-2i$

basis of E_{1-2i}: $\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$

[either by direct computation

by complex conjugation].

e-value $\lambda = 1+2i = \sqrt{5} \cdot \left(\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} i\right)$

absolute value $\to r$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$.

$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$\vec{x} \mapsto A \cdot \vec{x}$

$\begin{pmatrix} i \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{\alpha}} + i \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{\beta}}$

real part $\quad$ imag. part.

Basis $\mathcal{B} = \{\vec{\alpha}, \vec{\beta}\}$, $\quad P = (\vec{\alpha}\ \vec{\beta}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Under this basis, A becomes

$P^{-1}AP = \underbrace{\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}}_{\text{Scaling by } \lambda = r \cdot e^{i\theta}} = \underbrace{\begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}}_{\text{scaling by } r} \cdot \underbrace{\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}}_{\text{rotation by } \theta}$

$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

"Scaling by the complex number $\lambda = 1+2i$ on $\mathbb{R}^2$"

diagonalize A in $\mathbb{C}$. using the complex e-vectors

express A as scaling ∘ rotation using real vectors (real & imag. parts of the complex e-vectors)

Example 2. Find an example or disprove existence:
(a) a real 3×3 matrix A whose only real eigenvalue is 1 with algebraic multiplicity 2.

X

complex e-values come in pairs

Fact. ① If λ is e-value of A, then it's also e-value of A'

② If A is invertible, λ is e-value of A, then λ^{-1} is e-value of A^{-1}.

③ If $a+bi$ is e-value of $\underset{\text{real}}{A}$, then so is $a-bi$.

④ $\det(A) =$ product of all e-values with multiplicities.

(b) a real 3×3 invertible matrix B with 2 and 3 being two of its eigenvalues, and 4 being an eigenvalue of its inverse.

$\frac{1}{4}$ is e-value of B. e.g. $B = \begin{pmatrix} 2 & & \\ & 3 & \\ & & 1/4 \end{pmatrix}$

(c) a non-diagonal 2×2 matrix C with eigenvalues 2 and 4, and determinant 6.

X $\det = 2 \times 4 = 8 \neq 6$.

(d) a 3×3 matrix F with eigenvalues 2 and 3 such that F^2 is not diagonalizable.

$F = P^{-1} D P$. So F can't diagonalizable.
$F^2 = P^{-1} D^2 P$

Try $F = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $F^2 = \ldots$ Check F^2 is not diagonalizable.

Example 3. Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ given by $T(f(x)) = f'(x) + f(x)$. Find all eigenvalues and eigenvectors of T.

Idea Pick a basis of $\mathbb{P}_2$ and then do computations in $\mathbb{R}^3$.
The final result will not depend on the basis.

→ Take basis $\mathcal{B} = \{1, x, x^2\}$ of $\mathbb{P}_2$.

then $T(1) = 1' + 1 = 1$

$T(x) = x' + x = 1 + x$

$T(x^2) = (x^2)' + x^2 = 2x + x^2$

So $\underset{= A}{{}_{\mathcal{B}}[T]_{\mathcal{B}}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ different

→ Find e-values & e-vectors of A in $\mathbb{R}^3$.

e-value $\lambda = 1$ same

e-vectors $E_1 = \mathrm{Nul}(A - \lambda I)$
$= \mathrm{Nul}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

$E_1 = \left\{ \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} \middle| s \in \mathbb{R} \right\}$ different

So all e-vectors of A are

$E_1 - \{\vec{0}\} = \left\{ \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} \middle| s \neq 0 \right\}$.

→ Go back to $\mathbb{P}_2$. For T,

e-value 1. Same

e-vectors vectors whose $\mathcal{B}$-coor.
Same are in $E_1 - \{\vec{0}\}$.

$\{ s \cdot 1 + 0 \cdot x + 0 \cdot x^2 \mid s \neq 0 \} \subset \mathbb{P}_2$