Worksheet 17 (March 12)

DIS 119/120 GSI Xiaohan Yan

1 Review

DEFINITIONS

Motivation

$T: V \to V$. $\mathcal{B}, \mathcal{C}$ two bases of V.

$$\boxed{{}_{\mathcal{B}}[T]_{\mathcal{B}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} \cdot {}_{\mathcal{C}}[T]_{\mathcal{C}} \cdot P_{\mathcal{C}\leftarrow\mathcal{B}}}$$

- similar matrices, eigenvalues and eigenvectors of similar matrices;

Def. A, B similar if $\exists$ invertible P s.t. $A = P^{-1}BP$.

$PAP^{-1} = B$

$(P^{-1})^{-1} A \cdot P^{-1} = B$

→ e-values of A, B are the same.

geo. and alg. mult.

→ Correspondence between e-vectors of A & B.

$$\left\{\begin{matrix}\text{e-vectors} \\ \text{of } A\end{matrix}\right\} \underset{P^{-1}\cdot}{\overset{P\cdot}{\rightleftarrows}} \left\{\begin{matrix}\text{e-vectors} \\ \text{of } B\end{matrix}\right\}.$$

$\vec{v} \mapsto P\vec{v}$

$P^{-1}\vec{u} \leftarrow \vec{u}$

This is not a subspace of $\mathbb{R}^n$, but a union of subspaces.

- diagonalization.

A is called diagonalizable if $\exists$ D diagonal, P invertible s.t. $\boxed{D = P^{-1}AP}$

METHODS AND IDEAS

Theorem 1. (Diagonalizability)
An $n \times n$ matrix A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors (which thus form a basis of $\mathbb{R}^n$) $\Leftrightarrow$ the sum of geometric multiplicities of eigenvalues of A is n $\Leftrightarrow$ for all eigenvalues of A, the algebraic multiplicity is equal to the geometric multiplicity.

Remark 1. (Algorithm for diagonalization)
(1) Given matrix A, we first find its characteristic polynomial and solve for eigenvalues.
(2) For each eigenvalue λ of A, we find a basis of the eigenspace $E_\lambda = \text{Nul}(A - \lambda I)$.
(3) If $\dim E_\lambda$ is equal to the algebraic multiplicity of λ for all λ, A is diagonalizable. Take P as the matrix whose columns are the basis vectors we found in those E_λ in the previous step, then $D = P^{-1}AP$ gives a diagonalization of A, and D is the diagonal matrix whose diagonal entries are eigenvalues of the columns of P.

$$\begin{pmatrix} 2 & 1 & & & & \\ & 2 & 1 & & 0 & \\ & & 2 & & & \\ & & & 2 & & \\ & 0 & & & 4 & \\ & & & & & -1 \end{pmatrix}$$

"Jordan block" → alg. mult. = 4. geo. mult. = 4 − 2 = 2

Whenever "1" appears above the diagonal in Jordan form, the matrix is not diagonalizable

2 Problems

Jordan form.

Example 1. Find an example or disprove existence:
(a) Diagonalizable 3×3 matrix M that is not invertible.

$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ diag. ✓ invert ✗

$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ diag ✓. invert ✓

only $\lambda = 1$.
alg. mult. = 3
geo. mult. = 2

$M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ diag ✗ invert ✓

$M_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ diag ✗ invert ✗

Rmk 1° Diagonalizable :

$\Leftrightarrow$ $\forall \lambda$, geo. mult. = alg. mult.

$\Leftrightarrow$ $\sum_\lambda$ geo. mult. $= n$.

good way to find examples of diag. matrices. → $\boxed{\text{Cor}}$ If M has n different e-values, then M is diagonalizable. [geo. mult. = alg. mult. = 1]

2° Invertible : all e-values are non-zero.

(b) Diagonalizable 3×3 matrix M with 2 distinct eigenvalues.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Example 2. Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix},$$

i.e. find invertible matrix P and diagonal matrix D such that $D = P^{-1}AP$.

e-values $\lambda = -1$ alg.mult. $=2$ basis of E_{-1} $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ geo.mult. $=2$

$\lambda = 5$ alg.mult. $=1$. basis of E_5 $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ geo.mult. $=1$

A is indeed diagonalizable.

Take $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ Columns of P are L.I. eigenvectors of A.

then $\boxed{D = P^{-1}AP}$ where $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

Example 3. *What is described in this example is entirely hypothetical.* YiFang and FengCha are two boba shops in Berkeley. Denote by $Y(t)$ and $F(t)$ the numbers of customers of these two shops on day t. An economist who newly learned some linear algebra formulated the following recursive relation between $Y(t)$ and $F(t)$:

$$\begin{pmatrix} Y(n+1) \\ F(n+1) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} Y(n) \\ F(n) \end{pmatrix}$$

Assume that this model is correct, and that $Y(0) = 34$ and $F(0) = 32$, i.e. on day 0 they have 34 customers and 32 customers respectively. Can you help the economist to compute $Y(5)$ and $F(5)$?

Observation $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. then $M^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$.

Want $\begin{pmatrix} Y(5) \\ F(5) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} Y(4) \\ F(4) \end{pmatrix} = \cdots = \boxed{\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}^5 \cdot \begin{pmatrix} 34 \\ 32 \end{pmatrix}}$

→ $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$. Find e-values & e-vectors of A.

$\chi_A(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 8$

$\lambda = 2, 4$ two e-values. each has alg.mult. $=1$.

$\lambda = 2$ e-space $E_2 = \mathrm{Nul}(A - 2 \cdot I) = \mathrm{Nul}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

so basis of E_2 is $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \vec{\alpha}$

$\lambda = 4$ e-space $E_4 = \mathrm{Nul}(A - 4I) = \mathrm{Nul}\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$.

so basis of E_4 is $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} = \vec{\beta}$

Method 1 → Diagonalize A.

$D = P^{-1}AP$, where $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$

→ Compute $A^5 = (PDP^{-1})^5 = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})$

$= P \cdot D^5 \cdot P^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2^5 & 0 \\ 0 & 4^5 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$.

Compute $A^5 \cdot \begin{pmatrix} 34 \\ 32 \end{pmatrix}$.

Method 2. $\to$ $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ linear transformation.

$\vec{v} \longmapsto \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \cdot \vec{v}$

Take basis $\mathcal{B} = \{\vec{\alpha}, \vec{\beta}\}$ of $\mathbb{R}^2$.

then ${}_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.

$T(\vec{\alpha}) = 2 \cdot \vec{\alpha}$, $T(\vec{\beta}) = 4 \cdot \vec{\beta}$.

In general, $T(c_1\vec{\alpha} + c_2\vec{\beta}) = c_1 T(\vec{\alpha}) + c_2 T(\vec{\beta})$

$= 2c_1 \cdot \vec{\alpha} + 4c_2 \cdot \vec{\beta}$.

$T^2(c_1\vec{\alpha} + c_2\vec{\beta}) = T(T(c_1\vec{\alpha} + c_2\vec{\beta}))$

$= T(2c_1\vec{\alpha} + 4c_2\vec{\beta})$

$= 2^2 \cdot c_1 \cdot \vec{\alpha} + 4^2 \cdot c_2 \cdot \vec{\beta}$.

$$\boxed{T^5(c_1\vec{\alpha} + c_2\vec{\beta}) = 2^5 \cdot c_1\vec{\alpha} + 4^5 \cdot c_2\vec{\beta}}$$

$\to$ Want $\boxed{\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}^5 \begin{pmatrix} 34 \\ 32 \end{pmatrix}} = T^5 \begin{pmatrix} 34 \\ 32 \end{pmatrix}$.

Since $\begin{pmatrix} 34 \\ 32 \end{pmatrix} = 33 \cdot \vec{\alpha} + (-1) \cdot \vec{\beta}$.

We have $\boxed{} = 2^5 \cdot 33 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4^5 \cdot (-1) \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \cdots$

then

$\left[T^5 \begin{pmatrix} 34 \\ 32 \end{pmatrix}\right]_{\mathcal{B}}$

$= {}_{\mathcal{B}}[T^5]_{\mathcal{B}} \cdot \left[\begin{pmatrix} 34 \\ 32 \end{pmatrix}\right]_{\mathcal{B}}$.

$= {}_{\mathcal{B}}[T]_{\mathcal{B}}^5 \cdot \left[\begin{pmatrix} 34 \\ 32 \end{pmatrix}\right]_{\mathcal{B}}$.

But $\left[\begin{pmatrix} 34 \\ 32 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 33 \\ -1 \end{pmatrix}$.

So $\left[T^5 \begin{pmatrix} 34 \\ 32 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 2^5 & \\ & 4^5 \end{pmatrix} \cdot \begin{pmatrix} 33 \\ -1 \end{pmatrix}$.

$T^5 \begin{pmatrix} 34 \\ 32 \end{pmatrix} = 2^5 \cdot 33 \cdot \vec{\alpha} + 4^5 \cdot (-1) \cdot \vec{\beta}$