Worksheet 16 (March 10)

DIS 119/120 GSI Xiaohan Yan

1 Review

Rmk. $\chi_A(\lambda)$ does not tell anything about the geo. mult.

Example If char. poly. is $\chi_A(\lambda) = (5-\lambda)^2 \cdot (1+\lambda)(2+\lambda)\lambda^3$ then

eigenvalues	alg. mult.
5	2
−1	1
−2	1
0	3

DEFINITIONS

- eigenvector, eigenvalue; defined for square matrix $A_{n\times n}$.

  ↳ $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$ s.t. $A\cdot\vec{v} = \lambda \vec{v}$ (*), for some $\lambda \in \mathbb{R}$.

  λ: eigenvalue; $\vec{v}$: eigenvector

- characteristic polynomial;

  (*) ⇒ $A\cdot\vec{v} = \lambda\cdot I\cdot\vec{v}$ ⇝ $(A-\lambda I)\cdot\vec{x} = \vec{0}$ ⇝ (Invertible matrix thm) $\det(A-\lambda I) = 0$

  ⇒ $(A-\lambda I)\cdot\vec{v} = \vec{0}$

  the linear system has nontrivial solution.

  $\boxed{\chi_A(\lambda) = \det(A-\lambda I)}$ deg-n polynomial of λ.

  Rmk. λ is e-value of A ⟺ λ is a root of the polynomial $\chi_A(\lambda)$.

- eigenspace, algebraic multiplicity, geometric multiplicity.

  ↳ eigenspace of λ is defined as $E_\lambda = Nul(A-\lambda I)$.

  Rmk. $E_\lambda - \{\vec{0}\}$ is the set of eigenvectors of λ.

  alg. mult. of λ = # of times λ appears as a root of $\chi_A(\lambda)$.

  geo mult. of λ is $\dim Nul(A-\lambda I)$. "how many L.I. e-vectors there are of λ".

METHODS AND IDEAS

Theorem 1. (Fundamental Theorem of Algebra)
A polynomial of degree n has exactly n complex roots, and thus at most n real roots.

Remark 1. An $n \times n$ matrix has at most n **real eigenvalues** (and exactly n **complex eigenvalues**), counted with algebraic multiplicity.

Theorem 2. *Eigenvectors of different eigenvalues are linearly independent.*

Remark 2. As a corollary, one can find n linearly independent eigenvectors for an $n \times n$ matrix A in the following two cases:

Call A "diagonalizable"

⇕

for any e-value λ of A, geo. mult. of λ = alg. mult. of λ.

- A has n distinct eigenvalues; ↪ $\forall \lambda$, geo. = alg. = 1
- A has less than n eigenvalues $\lambda_1, \ldots, \lambda_k (k < n)$, but all of the eigenvalues have full geometric multiplicities, i.e.

$$\dim E_{\lambda_1} + \dim E_{\lambda_2} + \cdots + \dim E_{\lambda_k} = n.$$

for any e-value λ. $\boxed{1 \leq \dim Nul(A-\lambda I) \leq \text{alg. mult. of } \lambda}$

Note that in general, the **geometric multiplicity** is smaller than or equal to the **algebraic multiplicity**, so the sum above might be strictly smaller than n.

2 Problems

Example 1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

(a) Find all eigenvalues of A, and determine their algebraic multiplicities.
(b) For each eigenvalue, find a basis for its eigenspace, and determine its geometric multiplicity.
(c) Note that A is 3×3. Are there 3 linear independent eigenvectors of A?

Factorization $\chi_A(\lambda) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5$.

→ All possible roots of $\chi_A(\lambda)$ are factors of $\frac{5}{-1} = -5$. $\pm 1, \pm 5$.

→ Check if $(\lambda - 1)$ is a factor of $\chi_A(\lambda)$.
Plugging $\lambda = 1$. $\chi_A(1) = -1 + 3 + 9 + 5 \neq 0$.
So $(\lambda - 1)$ is not a factor.

→ Check if $(\lambda+1)$ is a factor. $\chi_A(-1) = 0$.
So $\chi_A(\lambda) = (\lambda+1)(-\lambda^2 + 4\lambda + 5)$
$= -(\lambda+1)(\lambda-5)$.

(a) $\chi_A(\lambda) = \det(A - \lambda I)$.

$= \det \begin{pmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{pmatrix}$.

$= (1-\lambda)^3 + 8 + 8 - 4(1-\lambda) \times 3$

$= -\lambda^3 + 3\lambda^2 + 9\lambda + 5$

$= -(\lambda+1)^2(\lambda-5)$

eigenvalues of A are

	$\lambda = -1$	$\lambda = 5$
alg. mult.	2	1

(b)
For $\lambda = -1$. the eigenspace
$E_{-1} = Nul(A + I)$
$= Nul \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$

basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$, $\dim E_{-1} = 2$.

For $\lambda = 5$. the eigenspace
$E_5 = Nul(A - 5I) = Nul \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}$

basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, $\dim E_5 = 1$.

geometric multiplicities

Rmk In this case, for all e-values, geo mult. = alg. mult.
reason.

(c) Yes.
The three vectors are linearly independent e-vectors of A. Here we used Thm 2.

Example 2. Find the values of c and d such that $(1, 1, 1)^T$ is an eigenvector of of the matrix

$$A = \begin{pmatrix} 2 & 1 & c \\ -2 & 4 & -2 \\ 0 & b & 7 \end{pmatrix}.$$

$\begin{pmatrix} 2 & 1 & c \\ -2 & 4 & -2 \\ 0 & b & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\begin{pmatrix} 3+c \\ 0 \\ b+7 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ \lambda \end{pmatrix}$. $\rightsquigarrow$ $\begin{cases} \lambda = 0 \\ c = -3 \\ b = -7 \end{cases}$

Note. Only geo. mult. can be read from the picture.
The alg. mult. can only be found by solving $\chi_A(\lambda) = 0$.

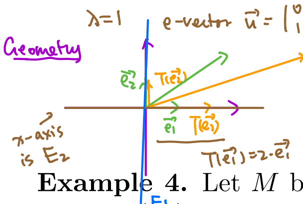
Example 3. Find all real eigenvalues and an eigenvector for each real eigenvalue of the following linear transformations of $\mathbb{R}^2$.
(a) Scale the x-direction by 2. (b) Rotation counterclockwise by $\pi/2$. (c) Reflection across $y = x$. (d) Sheer transformation sending $\mathbf{e}_1$ to $\mathbf{e}_1$ but $\mathbf{e}_2$ to $\mathbf{e}_1 + \mathbf{e}_2$.

(a) Standard matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

$\lambda = 2$ e-vector $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{e_1}$

$\lambda = 1$ e-vector $\vec{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{e_2}$

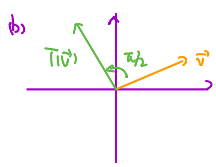
Geometry



Rmk 1° Geometry of eigenvector $\vec{v}$:
the effect of the linear trans. on $\vec{v}$ is just scaling (i.e. $T(\vec{v})$ is in the same/opposite direction of $\vec{v}$).

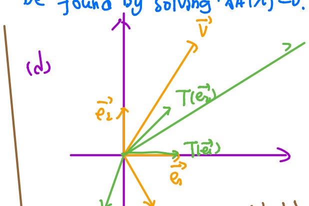
2° There are at most n e-values in $\mathbb{R}^n$.

(b)



no (real) e-value
no e-vector.
Rmk e-vectors of a complex e-value are complex vectors.
(at least 1 non-real entry).

(d)

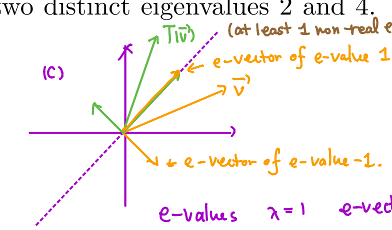


Standard matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Geometry direction of $\vec{v}$ is preserved iff $\vec{v}$ does not have y-component, i.e. $\vec{v}$ is on x-axis.
only e-value 1
an evector $\vec{v} = \vec{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Rmk $\chi_A(\lambda) = (1-\lambda)^2$
So $\lambda = 1$ has alg. mult = 2
but $\dim E_1 = 1$, so its geo. mult. = 1.

Example 4. Let M be a 2×2 matrix with two distinct eigenvalues 2 and 4. Find $\det(M)$.

(c)



e-values $\lambda = 1$ e-vector $\begin{pmatrix} 100 \\ 100 \end{pmatrix}$
$\lambda = -1$. e-vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Rmk E-values of reflections are ± 1.