Worksheet 15 (March 8)

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In $\mathbb{R}^n$, for $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$ we naturally know $P_{\mathcal{E}\leftarrow\mathcal{B}} = (\vec{b}_1 \cdots \vec{b}_n)$.
standard basis $\mathcal{E} = \{\vec{e}_1, \cdots, \vec{e}_n\}$

$(P_{C\leftarrow B} \mid P_{C\leftarrow A}) \to (I \mid P_{B\leftarrow A})$

1 Problems

Example 1. Find an example or disprove existence of examples:

Such example does not exist.

A linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and two bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^2$ such that

$$\det({}_{\mathcal{B}}[T]_{\mathcal{B}}) \neq \det({}_{\mathcal{C}}[T]_{\mathcal{C}})$$

$P_{B\leftarrow A} = P_{B\leftarrow C} P_{C\leftarrow A} = P_{C\leftarrow B}^{-1} \cdot P_{C\leftarrow A}$

$\det(A\cdot B) = \det(A)\cdot\det(B)$

${}_{\mathcal{B}}[T]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}} \cdot {}_{\mathcal{C}}[T]_{\mathcal{C}} \cdot P_{\mathcal{C}\leftarrow\mathcal{B}}$, $(P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1} = P_{\mathcal{C}\leftarrow\mathcal{B}}$

RREF method works (most commonly) in Euclidean spaces $\mathbb{R}^n$. It also works when we need $P_{B\leftarrow A}$ but we know $P_{C\leftarrow A}$ and $P_{C\leftarrow B}$.

$\det({}_{\mathcal{B}}[T]_{\mathcal{B}}) = \det(P_{\mathcal{B}\leftarrow\mathcal{C}}) \cdot \det({}_{\mathcal{C}}[T]_{\mathcal{C}}) \cdot \det(P_{\mathcal{C}\leftarrow\mathcal{B}})$
$= \underbrace{\det(P_{\mathcal{B}\leftarrow\mathcal{C}}) \cdot \det(P_{\mathcal{C}\leftarrow\mathcal{B}})}_{=1} \cdot \det({}_{\mathcal{C}}[T]_{\mathcal{C}}) = \det({}_{\mathcal{C}}[T]_{\mathcal{C}})$.

Example 2. *Regular computations.* Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(f(x)) = f'(x) + f(x).$$

(a) Let $\mathcal{E} = \{1, x, x^2\}$ be the canonical basis of $\mathbb{P}_2$. Find ${}_{\mathcal{E}}[T]_{\mathcal{E}}$. ($\vec{e}_1, \vec{e}_2, \vec{e}_3$)
(b) Let $\mathcal{B} = \{1 + x, x + x^2, 1 + x^2\}$ be another basis of $\mathbb{P}_2$. Find the base change matrix $P_{\mathcal{B}\leftarrow\mathcal{E}}$. ($\vec{b}_1, \vec{b}_2, \vec{b}_3$)

(a) ${}_{\mathcal{E}}[T]_{\mathcal{E}} = \left([T(\vec{e}_1)]_{\mathcal{E}}\ [T(\vec{e}_2)]_{\mathcal{E}}\ [T(\vec{e}_3)]_{\mathcal{E}}\right)$

$[T(1)]_{\mathcal{E}} = [0 + 1]_{\mathcal{E}} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $0+1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$[T(x)]_{\mathcal{E}} = [1 + x]_{\mathcal{E}} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $1+x = 1\cdot 1 + 1\cdot x + 0\cdot x^2$

$[T(x^2)]_{\mathcal{E}} = [2x + x^2]_{\mathcal{E}} = \begin{pmatrix}0\\2\\1\end{pmatrix}$

So ${}_{\mathcal{E}}[T]_{\mathcal{E}} = \begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix}$.

(b) Method ①: $P_{\mathcal{B}\leftarrow\mathcal{E}} = \left([\vec{e}_1]_{\mathcal{B}}\ [\vec{e}_2]_{\mathcal{B}}\ [\vec{e}_3]_{\mathcal{B}}\right)$, $[\vec{e}_1]_{\mathcal{B}} = \begin{pmatrix}1/2\\-1/2\\1/2\end{pmatrix}$

Want: Write $\vec{e}_1$ as linear comb. of $\vec{b}_1, \vec{b}_2, \vec{b}_3$ and take the coefficients.

$\vec{e}_1 = \_\vec{b}_1 + \_\vec{b}_2 + \_\vec{b}_3$ i.e. $1 = c_1(1+x) + c_2(x+x^2) + c_3(1+x^2)$
i.e. $1 = (c_1+c_3)\cdot 1 + (c_1+c_2)\cdot x + (c_2+c_3)\cdot x^2$

All coefficients should match, i.e. $\begin{cases}c_1+c_3=1\\c_1+c_2=0\\c_2+c_3=0\end{cases} \leadsto \begin{cases}c_1=1/2\\c_2=-1/2\\c_3=1/2\end{cases}$

Method ②: Observation: it's easier to write $\vec{b}_1, \vec{b}_2, \vec{b}_3$ as lin. comb. of $\vec{e}_1, \vec{e}_2, \vec{e}_3$.
$\vec{b}_1 = 1 + x = 1\cdot 1 + 1\cdot x + 0\cdot x^2$
so $P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}$, $P_{\mathcal{B}\leftarrow\mathcal{E}} = (P_{\mathcal{E}\leftarrow\mathcal{B}})^{-1} = \dots$

Example 3. Consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by reflectioin across $y = x$ followed by rotation counter-clockwise by $\pi/2$.
(a) Find the matrix ${}_{\mathcal{E}}[T]_{\mathcal{E}}$ of T under the standard basis $\mathcal{E} = \{\mathbf{e}_1.\mathbf{e}_2\}$.
(b) Find a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ of $\mathbb{R}^2$, such that ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ is a diagonal matrix.
Hint: Geometrically, what do we know of $T(\mathbf{b}_1)$ and $T(\mathbf{b}_2)$ given that ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ is a diagonal matrix?

(a) ${}_{\mathcal{E}}[T]_{\mathcal{E}} = \begin{pmatrix}-1&0\\0&1\end{pmatrix}$. $T(\vec{e}_1) = -\vec{e}_1$, $T(\vec{e}_2) = \vec{e}_2$.

(b). Take $\mathcal{B} = \mathcal{E}$.

Rmk ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ is diagonal means that T sends every basis vector $\vec{b}_i$ to a multiple of $\vec{b}_i$.

Example 4. *Pauli matrices. For fun only.* The algebra $\mathbb{H}$ of **quaternions** is extensively used in both maths and physics. As a set, it is defined as

$$\mathbb{H} = \{x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k} | x, y, z, w \in \mathbb{R}\}.$$

In other words, a quaternion is an expression of the form $x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k}$. When $y = z = w = 0$, it is just a real number x. $\mathbb{H}$ can be endowed with an $\mathbb{R}$-vector space structure, by the natural "coefficient-wise addition and scalar multiplication", i.e.

$$(x+y\mathbf{i}+z\mathbf{j}+w\mathbf{k})+(x'+y'\mathbf{i}+z'\mathbf{j}+w'\mathbf{k}) := (x+x')+(y+y')\mathbf{i}+(z+z')\mathbf{j}+(w+w')\mathbf{k}$$

and for any real number c,

$$c \cdot (x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k}) := cx + cy\mathbf{i} + cz\mathbf{j} + cw\mathbf{k}.$$

$\mathbb{H}$ is said to be an algebra over $\mathbb{R}$ because a multiplication (of any two elements of $\mathbb{H}$) is defined. Note that this multiplication structure does not appear in general vector spaces like Euclidean spaces. More precisely, we define

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = -1,$$

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j}.$$

The multiplication is defined to satisfy the distribution law, so the above equalities entirely determine the multiplication of any two quaternions. The multiplication is known to be associative, but not commutative.
(a) The vector space $\mathbb{H}$ has a natural basis $\mathcal{B} = \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Let $T : \mathbb{H} \to \mathbb{H}$ be the linear transformation of multiplying $\mathbf{i}$ from the left

$$T(x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k}) = \mathbf{i} \times (x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k}).$$

Find the matrix $\sigma_{\mathbf{i}} = {}_{\mathcal{B}}[T]_{\mathcal{B}}$ of T under the basis $\mathcal{B}$.
(b) Similarly, find $\sigma_{\mathbf{j}}$ and $\sigma_{\mathbf{k}}$, the matrices of multiplying $\mathbf{j}$ and $\mathbf{k}$ from the left, respectively.
(c) Check that $\sigma_{\mathbf{i}}^2 = \sigma_{\mathbf{j}}^2 = \sigma_{\mathbf{k}}^2 = -I_4$. Can you explain why?
The three matrices $\sigma_{\mathbf{i}}, \sigma_{\mathbf{j}}, \sigma_{\mathbf{k}}$ are (variations of) the famous **Pauli matrices**, which represent the angular momenta in the three spatial directions of spin-1/2 particles like hydrogen.