Worksheet 14 (March 5)

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1 Review

METHODS AND IDEAS

It is more "invariant", independent of choices of bases $\mathcal{A}, \mathcal{B}$ of V, W.

Theorem 1. (rank-nullity)
Let $T : V \longrightarrow W$ be a linear transformation, we will always have

$\mathcal{A}$ $\mathcal{B}$

$$\dim \ker(T) + \dim \operatorname{Im}(T) = \dim V.$$

A matrix

$\dim \underbrace{\operatorname{Nul}(A)}_{\#\text{free var.}} + \underbrace{\operatorname{rank}(A)}_{\#\text{pivot columns}} = \#\text{ Cols of } A.$

Theorem 2. (coordinates under base change)
Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of vector space V, then the coordinates of the same vector $\mathbf{x} \in V$ under $\mathcal{B}$ and $\mathcal{C}$ are related by

"x in C" = "B in C" · "x in B"

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}},$$

$\vec{x}$ $V \xrightarrow[\simeq]{\mathcal{B}} \mathbb{R}^n \ni [\vec{x}]_{\mathcal{B}}$

$V \xrightarrow[\simeq]{\mathcal{C}} \mathbb{R}^n \ni [\vec{x}]_{\mathcal{C}}$

where the base change matrix is given by

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \left([\mathbf{b}_1]_{\mathcal{C}} \quad \cdots \quad [\mathbf{b}_n]_{\mathcal{C}}\right).$$

Theorem 3. (linear transformation under base change)
Let $\mathcal{A}$ and $\mathcal{B}$ be two bases of vector space V, $\mathcal{C}$ and $\mathcal{D}$ be two bases of vector space W, and $T : V \longrightarrow W$ a linear transformation. Then the matrices of the same linear transformation T under different bases are related by

$$_{\mathcal{D}}[T]_{\mathcal{B}} = P_{\mathcal{D}\leftarrow\mathcal{C}}\ {}_{\mathcal{C}}[T]_{\mathcal{A}} \cdot P_{\mathcal{A}\leftarrow\mathcal{B}}.$$

In particular, if $V = W$ and $\mathcal{A}$ and $\mathcal{B}$ are two bases of V,

$$_{\mathcal{B}}[T]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{A}}\ {}_{\mathcal{A}}[T]_{\mathcal{A}} \cdot P_{\mathcal{A}\leftarrow\mathcal{B}}.$$

$T: V \longrightarrow V$ $\mathcal{A},\mathcal{B}$ $\mathcal{A},\mathcal{B}$

codomain domain

these two matrices are inverse of each other.

Sketch of proof

Want: $_D[T]_B \cdot \vec{x} = P_{D\leftarrow C}\, {}_C[T]_A \cdot P_{A\leftarrow B}\vec{x}$

→ Let $\vec{v} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n$

so $[\vec{v}]_B = \vec{x} \in \mathbb{R}^n$

$\Uparrow$ V.

→ $\underbrace{{}_D[T]_B \cdot [\vec{v}]_B}_{[T(\vec{v})]_D} = \underbrace{P_{D\leftarrow C} \cdot \underbrace{{}_C[T]_A \cdot \underbrace{P_{A\leftarrow B}\cdot[\vec{v}]_B}_{[\vec{v}]_A}}_{[T(\vec{v})]_C}}_{[T(\vec{v})]_D}$

$\mathcal{A} = \{\vec{a}_1, \cdots, \vec{a}_n\}$ $\mathcal{B}$ $\mathcal{C} = \mathcal{D}$ $\{\vec{c}_1, \cdots, \vec{c}_m\}$.

$T: V^n \longrightarrow W^m$ $\ni T(\vec{a}_1), \cdots, T(\vec{a}_n)$

$${}_C[T]_A = \begin{pmatrix} [T(\vec{a}_1)]_C & \cdots & [T(\vec{a}_n)]_C \end{pmatrix}_{m\times n}$$

Idea of defining ${}_C[T]_A$:

$$\begin{array}{ccc} \vec{v} \in V & \xrightarrow{T} & W \ni T(\vec{v}) \\ \mathcal{A}\downarrow\simeq & & \mathcal{C}\downarrow\simeq \\ [\vec{v}]_{\mathcal{A}} \in \mathbb{R}^n & \xrightarrow{{}_C[T]_A} & \mathbb{R}^m \ni [T(\vec{v})]_{\mathcal{C}} \end{array}$$

$[T(\vec{v})]_{\mathcal{C}} = {}_{\mathcal{C}}[T]_{\mathcal{A}} \cdot [\vec{v}]_{\mathcal{A}}.$

2 Problems

Example 1. True or false.

(T) Base change matrices $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are always invertible. $(P_{C\leftarrow B})^{-1} = P_{B\leftarrow C}$

(T) If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three bases of the vector space V, then

$$P_{\mathcal{C}\leftarrow\mathcal{A}} = P_{\mathcal{C}\leftarrow\mathcal{B}} \cdot P_{\mathcal{B}\leftarrow\mathcal{A}}.$$

$\parallel$

$([\vec{a}_1]_C \ [\vec{a}_2]_C \cdots [\vec{a}_n]_C)$

$\vec{a}_1 = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n$

$= x_1(y_{11}\vec{c}_1 + y_{12}\vec{c}_2 + \cdots + y_{1n}\vec{c}_n) + x_2(\cdots) + \cdots$

(T) If M and N are two matrices of a linear transformation $T : V \longrightarrow W$ (relative to different bases), then rank M = rank N. $= \dim \operatorname{Im}(T)$

$\dim \operatorname{Nul}(M) = \dim \operatorname{Nul}(N)$

However $\operatorname{Col}(M) \neq \operatorname{Col}(N)$

$\operatorname{Nul}(M) \neq \operatorname{Nul}(N)$

Example 2. *Example 3 from the previous worksheet.* Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1.\mathbf{b}_2, \mathbf{b}_3\}$ be two different bases of $\mathbb{R}^3$, where

$$\mathbf{a}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

$P_{B\leftarrow A} = \left((a_1)_B \ (a_2)_B \ (a_3)_B \right)$

$E = \{\vec{e_1}, \cdots, \vec{e_n}\}$

(a) Find $[\mathbf{a}_i]_{\mathcal{B}}$ for $i = 1, 2, 3$.
(b) If $[\mathbf{x}]_{\mathcal{A}} = (1, 1, 1)^T$, find $[\mathbf{x}]_{\mathcal{B}}$.
(c) If $[\mathbf{y}]_{\mathcal{A}} = [\mathbf{y}]_{\mathcal{B}}$, find $\mathbf{y}$.

General method (of finding base change matrix $P_{B\leftarrow A}$ for two bases A, B of $\mathbb{R}^n$)

$A = \{\vec{a_1}, \cdots, \vec{a_n}\}$, $B = \{\vec{b_1}, \cdots, \vec{b_n}\}$

$$\Big(\underbrace{\vec{b_1} \cdots \vec{b_n}}_{B} \ \Big| \ \underbrace{\vec{a_1} \cdots \vec{a_n}}_{A} \Big)_{n\times 2n} \xrightarrow[\text{red.}]{\text{row}} \Big(I_n \ \Big| \ P_{B\leftarrow A} \Big)$$

$P_{B\leftarrow A} = B^{-1} \cdot A = P_{B\leftarrow E} \cdot P_{E\leftarrow A}$

(a) $[\vec{a_1}]_B$. $\vec{a_1} = c_1 \vec{b_1} + c_2 \vec{b_2} + c_3 \vec{b_3}$

Solve $\left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$.

Solution $[\vec{a_1}]_B = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\vec{a_1} = \vec{b_1} - \vec{b_2}$

$[\vec{a_2}]_B$ Solution $[\vec{a_2}]_B = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\vec{a_2} = \vec{b_1} - \vec{b_3}$

$[\vec{a_3}]_B$ Solution $[\vec{a_3}]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{a_3} = \vec{b_1}$

(b) $[\vec{x}]_B = P_{B\leftarrow A} \cdot [\vec{x}]_A = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}}_{\text{from part (a)}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

(c) $[\vec{y}]_B = P_{B\leftarrow A} \cdot [\vec{y}]_A$. assume $[\vec{y}]_A = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ then $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

$\rightsquigarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} s \\ -s \\ s \end{pmatrix}$, $s \in \mathbb{R}$.

$\rightsquigarrow \vec{y} = s \cdot \vec{a_1} + (-s) \cdot \vec{a_2} + s \cdot \vec{a_3} = \begin{pmatrix} s \\ 0 \\ s \end{pmatrix}$, $s \in \mathbb{R}$.

Example 3. *Example 4 from the previous worksheet.* Consider the linear transformation $T : \mathbb{P}_2 \to M_{2\times 2}(\mathbb{R})$ defined as

$$T(a + bx + cx^2) = \begin{pmatrix} 3a + b & b + c \\ -2b & a + b + c \end{pmatrix}.$$

(a) Consider the basis $\mathcal{B} = \{1+x, x, x^2-x\}$ of $\mathbb{P}_2$ and the basis $\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ of $M_{2\times 2}(\mathbb{R})$. Compute ${}_{\mathcal{E}}[T]_{\mathcal{B}}$.

e.g. $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(b) Consider instead the basis $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ of $M_{2\times 2}(\mathbb{R})$, where

$$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}, C_3 = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}, C_4 = \begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix}.$$

Compute ${}_{\mathcal{C}}[T]_{\mathcal{B}}$.
(c) Is T one-to-one? Onto?

(a). ${}_{\mathcal{E}}[T]_B = \left([T(\vec{b_1})]_{\mathcal{E}} \ [T(\vec{b_2})]_{\mathcal{E}} \ [T(\vec{b_3})]_{\mathcal{E}} \right)_{4\times 3} = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 1 & 0 \\ -2 & -2 & 2 \\ 2 & 1 & 0 \end{pmatrix}$

$T(1+x) = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix} = 4E_{11} + 1\cdot E_{12} + (-2)\cdot E_{21} + 2\cdot E_{22}$. so $[T(1+x)]_{\mathcal{E}} = \begin{pmatrix} 4 \\ 1 \\ -2 \\ 2 \end{pmatrix}$

$a=1, b=1, c=0$

$T(x) = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ " so $[T(x)]_{\mathcal{E}} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix}$.

$a=0, c=0, b=1$

$T(x^2-x) = \begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix}$ so $[T(x^2-x)]_{\mathcal{E}} = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$.

$a=0, b=-1, c=1$

(b) ${}_{C}[T]_B = \left([T(\vec{b_1})]_C \ [T(\vec{b_2})]_C \ [T(\vec{b_3})]_C \right)$

$[T(\vec{b_1})]_C = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. as $T(\vec{b_1}) = C_2 = 0\cdot C_1 + 1\cdot C_2 + 0\cdot C_3 + 0\cdot C_4$

$[T(\vec{b_2})]_C = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. $[T(\vec{b_3})] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

$${}_{C}[T]_{B} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) pivot in each column $\rightsquigarrow$ injective (one-to-one)

not in each row $\rightsquigarrow$ not surjective (not onto).

Rmk. There always exists bases $\mathcal{B}$ & C s.t. ${}_{C}[T]_{B}$ in RREF.