Worksheet 13 (March 3)

DIS 119/120 GSI Xiaohan Yan

1 Review

Recall. $T: \mathbb{R}^n \to \mathbb{R}^m$

$T(\vec{v}) = A \cdot \vec{v}$.

$A = \left(T(\vec{e_1}) \cdots T(\vec{e_n}) \right) = {}_{\mathcal{E}}[T]_{\mathcal{E}}$

DEFINITIONS

basis $\mathcal{B}$ of V $\vec{b_1}, \cdots, \vec{b_n}$

- coordinate mapping;

$$V \xrightarrow{\vec{v} \mapsto [\vec{v}]_{\mathcal{B}}} \mathbb{R}^n \quad \text{isomorphism}$$
$$c_1\vec{b_1} + \cdots + c_n\vec{b_n} \longleftarrow \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

- matrix of linear transformation relative to bases on domain and codomain;

$T: \underline{V} \to \underline{W}$, $\mathcal{B}$ $\mathcal{C}$ $\quad \mathcal{B} = \{\vec{b_1}, \cdots, \vec{b_n}\}$, $\mathcal{C} = \{\vec{c_1}, \cdots, \vec{c_m}\}$.

$$\begin{array}{ccc} \vec{v} & & T(\vec{v}) \\ V & \xrightarrow{T} & W \\ \mathcal{B}\wr & \circlearrowleft & \wr\mathcal{C} \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ [\vec{v}]_{\mathcal{B}} & \text{given by matrix} & [T(\vec{v})]_{\mathcal{C}} \end{array}$$

$$\boxed{[T(\vec{v})]_{\mathcal{C}} = {}_{\mathcal{B}}[T]_{\mathcal{C}} \cdot \underbrace{[\vec{v}]_{\mathcal{B}}}_{} } \qquad [\vec{v}]_{\mathcal{B}} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$${}_{\mathcal{B}}[T]_{\mathcal{C}} = \left([T(\vec{b_1})]_{\mathcal{C}} \cdots [T(\vec{b_n})]_{\mathcal{C}} \right) \quad m \times n$$

$\vec{v} = v_1\vec{b_1} + v_2\vec{b_2} + \cdots + v_n\vec{b_n}$

$T(\vec{v}) = T(v_1\vec{b_1} + \cdots + v_n\vec{b_n})$

$= v_1T(\vec{b_1}) + \cdots + v_nT(\vec{b_n})$

$[T(\vec{v})]_{\mathcal{C}} = v_1[T(\vec{b_1})]_{\mathcal{C}} + \cdots + v_n[T(\vec{b_n})]_{\mathcal{C}}$

LHS $= \left([T(\vec{b_1})]_{\mathcal{C}} \cdots [T(\vec{b_n})]_{\mathcal{C}} \right) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ RHS

METHODS AND IDEAS

Theorem 1. *Any two vector spaces of the same dimension are isomorphic, since any vector space of dimension n is isomorphic to $\mathbb{R}^n$ under the coordinate mapping under a basis.*

Note that vector spaces of different dimensions can never be isomorphic. This is because isomorphism preserves linear independence and spanning property, and thus always sends a basis to a basis. But bases of vector spaces of different dimensions contain different number of vectors.

2 Problems

Example 1. True or false. In the last three statements, S denotes the vector space of all smooth (infinitely differentiable) functions over $[0, 1]$. In other words, you do not need to worry about differentiability of elements of S

(F) There exists a basis $\mathcal{B}$ of $\mathbb{P}_2$ such that $1 + x$ has coordinate $(1, 1, 1)^T$ while $x + x^2$ has coordinates $(-3, -3, -3)^T$.

$\mathcal{B}$

$T: \mathbb{P}_2 \xrightarrow{\mathcal{B}} \mathbb{R}^3$ iso.

$\vec{v} \longmapsto [\vec{v}]_{\mathcal{B}}$.

$\vec{u}$ $1+x \longmapsto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\vec{w}$ $x^2+x \longmapsto \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix}$.

$3 \cdot T(\vec{u}) + T(\vec{w}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$T(3\vec{u} + \vec{w}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$3\vec{u} + \vec{w} = \vec{0} \in \mathbb{P}_2$

$3 + 4x + x^2 = 0x^2 + 0x + 0$

impossible

memorize the statement

(T) Let V be a 5-dimensional vector space and $\mathbf{v}_1, \mathbf{v}_2$ be two linearly independent vectors in V, then there exists three other vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ in V such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of V.

this statement would be true if $\mathrm{span}\{\vec{v_1}, \cdots, \vec{v_5}\} = W$

(F) Let W be a 2-dimensional vector space and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ be five vectors in W, then we can take two of these vectors to form a basis of W.

In general

→ We can always "complete" a set of L.I. vectors in V to a basis.

→ We can always "remove redundancy" from a set of spanning vectors of V to form a basis.

Idea We can just check this for $\mathbb{R}^5$. (any f.d. vector space is iso. to the Euclidean of the same dim.)

$\vec{v_1}, \vec{v_2}$ two L.I. column vectors

$\left(\vec{v_1}\ \vec{v_2}\ \vec{e_1}\ \vec{e_2}\ \vec{e_3}\ \vec{e_4}\ \vec{e_5} \right)_{5 \times 7}$ column vectors span $\mathbb{R}^5$.

⇒ ∃ 5 pivot columns.

We can determine pivot columns by row reductions.

$\vec{v_1}, \vec{v_2}$ will both be pivotal

$$T: \overset{\mathcal{B}}{V} \longrightarrow \overset{\mathcal{A}}{W}$$

$$[T(\vec{v})]_{\mathcal{A}} = \underbrace{{}_{\mathcal{A}}[T]_{\mathcal{B}}}_{=\left([T(\vec{b_1})]_{\mathcal{A}} \cdots [T(\vec{b_n})]_{\mathcal{A}}\right)} \cdot [\vec{v}]_{\mathcal{B}}$$

Special case $T = id: \overset{\mathcal{B}}{V} \longrightarrow \overset{\mathcal{A}}{V}$.

$$[\vec{v}]_{\mathcal{A}} = \underbrace{{}_{A}[I]_{\mathcal{B}}}_{=\left([\vec{b_1}]_A \cdots [\vec{b_n}]_A\right) = [\mathcal{B}]_{\mathcal{A}}} \cdot [\vec{v}]_B$$

(T) Let $\mathcal{A}, \mathcal{B}$ be two bases of the vector space V, then for any vector $\mathbf{v} \in V$,

$$[\mathbf{v}]_{\mathcal{A}} = [\mathcal{B}]_{\mathcal{A}} \cdot [\mathbf{v}]_{\mathcal{B}},$$

Coordinate change / base change

where $[\mathcal{B}]_{\mathcal{A}}$ is the square matrix whose columns are $\mathcal{A}$-coordinate vectors of the vectors of $\mathcal{B}$.

$$\left([\vec{b_1}]_A \cdots [\vec{b_n}]_A\right)$$

( ) There is no isomorphism $T : \mathbb{P}_2 \to \mathbb{R}^2$.

( ) Let $T : S \to S$ be the linear transformation $T(f(x)) = f'(x)$, then T is surjective.

( ) Let $T : S \to S$ be the linear transformation $T(f(x)) = \int_0^x f(s)ds$, then T is surjective.

( ) Let $T : S \to S$ be the linear transformation $T(f(x)) = f''(x)$, then T has a two-dimensional kernel.

Example 2. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1 . \mathbf{b}_2\}$ be two different bases of $\mathbb{R}^2$ where

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(a) Let $\mathbf{x} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Write $\mathbf{x}$ as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$.

(b) Compute $[\mathbf{x}]_{\mathcal{E}}$ and $[\mathbf{x}]_{\mathcal{B}}$.

(c) Let $B = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$. Check that $[\mathbf{x}]_{\mathcal{E}} = B[\mathbf{x}]_{\mathcal{B}}$. Can you explain the reason behind this?

(d) Now let's generalize this result. Let $\mathcal{A} = \{\mathbf{a}_1 . \mathbf{a}_2\}$ be yet another basis of $\mathbb{R}^2$. Given that

$$[\mathbf{b}_1]_{\mathcal{A}} = \begin{pmatrix} 23 \\ 45 \end{pmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{A}} = \begin{pmatrix} 89 \\ 67 \end{pmatrix},$$

find $[\mathbf{x}]_{\mathcal{A}}$.

(a). $\vec{x} = c_1\vec{b_1} + c_2\vec{b_2}$.

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 5 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

$$\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 5 & 3 & -1 \end{array}\right)$$

$c_1 = -2$, $c_2 = 3$.

So $\vec{x} = -2\vec{b_1} + 3\vec{b_2}$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

$$[\vec{x}]_{\mathcal{B}} = \underbrace{[\mathcal{E}]_{\mathcal{B}}}_{=\left([\vec{e_1}]_{\mathcal{B}}\ [\vec{e_2}]_{\mathcal{B}}\right)} [\vec{x}]_{\mathcal{E}}.$$

(b) $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

$[\vec{x}]_{\mathcal{E}} = \vec{x} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

(c). $[\vec{x}]_{\mathcal{E}} = B \cdot [\vec{x}]_{\mathcal{B}}$

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$[\vec{x}]_{\mathcal{E}} = \left([\vec{b_1}]_{\mathcal{E}}\ [\vec{b_2}]_{\mathcal{E}}\right) \cdot [\vec{x}]_{\mathcal{B}}$$

(d) $[\mathcal{B}]_{\mathcal{A}} = \left([\vec{b_1}]_A\ [\vec{b_2}]_A\right)$.

$$= \begin{pmatrix} 23 & 89 \\ 45 & 67 \end{pmatrix}.$$

$$[\vec{x}]_{\mathcal{A}} = [\mathcal{B}]_{\mathcal{A}} \cdot [\vec{x}]_{\mathcal{B}}$$

$$= \begin{pmatrix} 23 & 89 \\ 45 & 67 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Rmk.

1° If $\mathcal{A}, \mathcal{B}$ two bases of the same vector space V.

then $\boxed{[\mathcal{B}]_{\mathcal{A}} = [\mathcal{A}]_{\mathcal{B}}^{-1}}$

2° If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ three bases, then

$$\boxed{[\mathcal{B}]_{\mathcal{C}} \cdot [\mathcal{A}]_{\mathcal{B}} = [\mathcal{A}]_{\mathcal{C}}}$$

Example 3. *One more question about base change.* Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1.\mathbf{b}_2, \mathbf{b}_3\}$ be two different bases of $\mathbb{R}^3$, where

$$\mathbf{a}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix} \mathbf{b}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(a) Find $[\mathbf{a}_i]_{\mathcal{B}}$ for $i = 1, 2, 3$.
(b) If $[\mathbf{x}]_{\mathcal{A}} = (1, 1, 1)^T$, find $[\mathbf{x}]_{\mathcal{B}}$.
(c) If $[\mathbf{y}]_{\mathcal{A}} = [\mathbf{y}]_{\mathcal{B}}$, find $\mathbf{y}$.

Example 4. Consider the linear transformation $T : \mathbb{P}_2 \to M_{2\times 2}(\mathbb{R})$ defined as

$$T(a + bx + cx^2) = \begin{pmatrix}3a + b & b + c\\ -2b & a + b + c\end{pmatrix}.$$

(a) Consider the basis $\mathcal{B} = \{1+x, x, x^2-x\}$ of $\mathbb{P}_2$ and the basis $\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ of $M_{2\times 2}(\mathbb{R})$. Compute ${}_{\mathcal{B}}[T]_{\mathcal{E}}$.
(b) Consider instead the basis $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ of $M_{2\times 2}(\mathbb{R})$, where

$$C_1 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, C_2 = \begin{pmatrix}4 & 1\\ -2 & 2\end{pmatrix}, C_3 = \begin{pmatrix}1 & 1\\ -2 & 1\end{pmatrix}, C_4 = \begin{pmatrix}-1 & 0\\ 2 & 0\end{pmatrix}.$$

Compute ${}_{\mathcal{B}}[T]_{\mathcal{C}}$.
(c) Is T one-to-one? Onto?