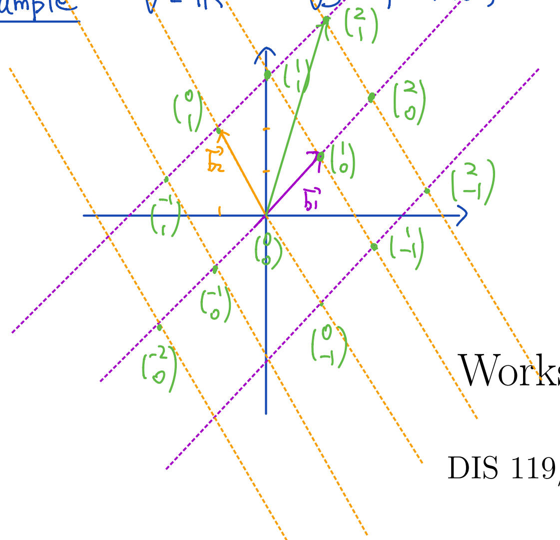
Example $V=\mathbb{R}^2$ $\mathcal{B}=\{\vec{b_1},\vec{b_2}\}$. $\vec{b_1}=\begin{pmatrix}1\\1\end{pmatrix}$, $\vec{b_2}=\begin{pmatrix}-1\\2\end{pmatrix}$.



green vectors are $\mathcal{B}$-coordinate vectors.

Worksheet 12 (Feb. 26)

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Erratum: In Example 5(d) of Worksheet 11, the map T is not actually linear transformation, so please disregard part (d). You may want to think about why it is not linear.

DEFINITIONS

- kernel, image, injectivity, surjectivity;

$\subset V$ $\subset W$ $\mathrm{Ker}\,T=\{\vec{0}\}$ $\mathrm{Im}\,T=W$

for $T: V\to W$ linear transformation.

- isomorphism of vector spaces;

$\exists$ inverse

bijective linear transformation

one-to-one & onto

$T: V \xrightarrow{\simeq} W$

$v \longmapsto T(v)$

Rmk $\exists$ isomorphism between V and W, it means they can be regarded as identical.

- basis, coordinate vector;

$\mathcal{B}=\{\vec{b_1},\cdots,\vec{b_n}\}$ of V is set of vectors satisfying ① linearly independent ② $\mathrm{span}\{\vec{b_1},\cdots,\vec{b_n}\}=V$

Upshot Any $\vec{v}\in V$ can be written uniquely as a linear combination of $\vec{b_1},\cdots,\vec{b_n}$: $\vec{v}=c_1\cdot\vec{b_1}+c_2\cdot\vec{b_2}+\cdots+c_n\cdot\vec{b_n}$

$[\vec{v}]_{\mathcal{B}}=\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}$

Fixing a basis, we get an "isomorphism"

$V \xrightarrow{\simeq} \mathbb{R}^n$

$\vec{v} \longmapsto [\vec{v}]_{\mathcal{B}}$

$c_1\vec{b_1}+\cdots+c_n\vec{b_n}$ $\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}$

$A\vec{x}=\vec{b}$

METHODS AND IDEAS

Theorem 1. *Any finite-dimensional vector space V is isomorphic to the Euclidean space of the same dimension.*

Remark 1. After choosing a basis for V, any linear property of vectors (e.g. linear combination, linear independence) in the abstract vector space V can be checked through the coordinates under the basis.

Similarly, after choosing bases for both V and W, any linear property of linear transformation $T: V\to W$ (e.g. injectivity, surjectivity, isomorphism, kernel, image) can be checked through the standard matrix under the bases.

1 Problems

Example 1. Consider the linear transformation

$$T:\ \mathbb{P}_2 \longrightarrow \mathbb{R}$$
$$f(x)\longmapsto f(1)-f(2).$$

$\mathbb{P}_2=\{a+bx+cx^2 \mid a,b,c\in\mathbb{R}\}$. $\ni -1$

$T(x)=-1$

$-T(x)=1$

$(-b)\cdot T(x)=b,\quad \forall b.$

$\Leftrightarrow T((-b)\cdot x)=b,\quad \forall b.$

(a) $T(x^2)=x^2|_{x=1}-x^2|_{x=2}=1-4=-3$. $T(x)=x|_{x=1}-x|_{x=2}=-1$. $T(3)=3|_{x=1}-3|_{x=2}=0$.

(b). $T(f(x))=f(1)-f(2)=(a+b+c)-(a+2b+4c)=-b-3c=0$. $\boxed{b+3c=0}$, a arbitrary.

(c). Method 1 Take a basis $\mathcal{B}=\{1,x,x^2\}$. This gives isomorphism (correspondence)

$\mathbb{P}_2 \xrightarrow{\simeq} \mathbb{R}^3$

$\vec{v} \longmapsto [\vec{v}]_{\mathcal{B}}$

$a+bx+cx^2 \longmapsto \begin{pmatrix}a\\b\\c\end{pmatrix}$.

(d). T inj? ✗ T surj? ✓

Method 2

① Observation: T is surjective.

② Apply nullity-rank thm.

$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim \operatorname{Domain} T$
[null] [rank] [#columns]
2 1 3

Need 2 linearly independent "vectors" in the kernel

$f(x) = a+bx+cx^2 \in \operatorname{Ker} T \iff \boxed{b+3c=0}$

$f_1(x) = 1$, $(a=1, b=c=0)$

$f_2(x) = x^2 - 3x$. $(a=0, b=-3, c=1)$

$T: \mathbb{P}_2 \to \mathbb{R}$ becomes $T: \mathbb{R}^3 \longrightarrow \mathbb{R}$. So matrix of T is $A = \begin{pmatrix} 0 & -1 & -3 \end{pmatrix}$.

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longmapsto -b-3c$

$[a+bx+cx^2]$

2 free vars

$\operatorname{Ker} T = \left\{ \begin{pmatrix} s \\ -3t \\ t \end{pmatrix} \middle| \; s,t \in \mathbb{R} \right\}$.

basis of $\operatorname{Ker} T$ ($\operatorname{Nul} A$). is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} \right\}$.

Going back to $\mathbb{P}_2$, they're $\{ 1, \; x^2-3x \}$

$1 = 1\cdot 1 + 0\cdot x + 0 \cdot x^2$, $x^2-3x = 0\cdot 1 + (-3)\cdot x + 1 \cdot x^2$

(a) Compute $T(x^2), T(x), T(3)$.
(b) Let $f(x) = a + bx + cx^2$ and $T(f(x)) = 0$, then what do we know about a, b, c?
(c) Find a basis of the kernel of T.
(d) Is T injective? Surjective?

Example 2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis of the vector space V.
(a) Determine $\dim V$ $=3$
(b) Consider the three vectors

$$\mathbf{v}_1 = \mathbf{b}_1 + 2\mathbf{b}_2 + 2\mathbf{b}_3, \mathbf{v}_2 = 2\mathbf{b}_1 + \mathbf{b}_2 + 2\mathbf{b}_3, \mathbf{v}_3 = 2\mathbf{b}_1 + 2\mathbf{b}_2 + \mathbf{b}_3.$$

Are they linearly dependent? Do they span V?
(c) Let $T : V \to V$ be the linear transformation defined by $T(\mathbf{b}_1) = \mathbf{b}_2, T(\mathbf{b}_2) = \mathbf{b}_1, T(\mathbf{b}_3) = T(\mathbf{b}_3)$. Find all $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{v}$. Do they form a subspace of V?

(b) Take the basis $\mathcal{B}$ and consider coordinates.

isomorphism $V \xrightarrow{\simeq} \mathbb{R}^3$
$\vec{v} \longmapsto [\vec{v}]_B$.

$[\vec{v}_1]_B = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $[\vec{v}_2]_B = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $[\vec{v}_3]_B = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$. if there're 3 pivots then $\vec{v}_1 \sim \vec{v}_3$ L.I. ✓

$\det M \neq 0$

(c) $T: V \to V$ becomes $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

matrix A of T is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $A \cdot \vec{v} = \vec{v} = I \cdot \vec{v}$.

Want $T(\vec{v}) = \vec{v}$. $\rightsquigarrow A\vec{v} = \vec{v} \iff A\vec{v} - I\vec{v} = \vec{0}$
$\iff (A-I)\cdot\vec{v} = \vec{0} \iff (A-I)\cdot\vec{v} = \vec{0}$

$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\left\{ \begin{pmatrix} s \\ s \\ t \end{pmatrix} \middle| \; s,t \in \mathbb{R} \right\}$.

Going back to V.

$\{ \vec{v} \in V \mid T(\vec{v}) = \vec{v} \} = \boxed{\{ s\vec{b}_1 + s\vec{b}_2 + t\vec{b}_3 : s,t \in \mathbb{R} \}}$.

It is indeed a subspace because the set it corresponds to in $\mathbb{R}^3$ is a subspace.

Example 3. Consider the following three polynomials in $\mathbb{P}^2$

$$\mathbf{b}_1 = 3 + 4x + 5x^2, \quad \mathbf{b}_2 = 2 + cx + 4x^2, \quad \mathbf{b}_3 = 1 + 2x + cx^2.$$

(a) Find their coordinate vectors under the basis $\{1, x, x^2\}$. Your answer may depend on c.
(b) For what values of c is $\mathbb{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ a basis of $\mathbb{P}^2$?
(c) Suppose that $\mathcal{B}$ is indeed a basis, and that the polynomial $7x$ has coordinate $(1, -2, 1)^T$ relative to it. Find c.