Worksheet 11 (Feb. 24)

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DEFINITIONS

Explain the following definitions without refering to "matrix", "pivot", "column", "linear system", "free variable" or "solution".

- vector space, addition, scalar multiplication;

  Set + addition + Scalar multiplication + 9 axioms

- linear combination, span, linear dependence;
- (linear) subspace, basis of vector space;

  Rmk. Any (finite-dimensional) vector space is isomorphic to a Euclidean space, after choosing a basis.

- linear transformation, domain, codomain, one-to-one, onto.

  $T: V \to W$

  satisfying $T(c_1\vec{v_1} \oplus c_2\vec{v_2}) = c_1 \cdot T(\vec{v_1}) \oplus c_2 \cdot T(\vec{v_2})$

  addition defined on V; addition on W; scalar multiplication on V; scalar multiplication on W

1 Problems

Example 1. Determine which ones of following sets are vector spaces under the given operations. For the vector spaces, find their dimensions. For those that are not vector spaces, explain why.

$\left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \middle| x, z \in \mathbb{R} \right\}$

(a) The subspace (xOz-plane) of $\mathbb{R}^3$ spanned by $\mathbf{e}_1$ and $\mathbf{e}_3$, under the addition and scalar multiplication inherited from $\mathbb{R}^3$.

dim = 2 basis $\vec{e_1}, \vec{e_3}$ $\mathbb{R}^3$ is a vector space. Any subspace of a vector space is also a vector space.

(b) The set of a single element $\{\bullet\}$, with addition defined as $\bullet + \bullet = \bullet$ and scalar multiplication defined as $c\bullet = \bullet, \forall c \in \mathbb{R}$.

dim = 0 $\bullet = \vec{0}$ $\mathbb{R}^0$

(c) The set $\mathbb{R}^{\geqslant 0}$ of all non-negative real numbers, with usual addition of real numbers as addition, and scalar multiplication $c\mathbf{v}$ defined (for real number c and element $\mathbf{v} \in \mathbb{R}^{\geqslant 0}$) as the multiplication of $|c|$ and $\mathbf{v}$ as real numbers.

Violates the distribution law: $1 \cdot \vec{v} + (-1) \cdot \vec{v} \neq (1+(-1)) \cdot \vec{v}$.

(d) The set $\mathbb{C}$ of complex numbers, with usual addition of complex numbers as addition, and usual multiplication by real number as scalar multiplication by

dim = 2 basis $1, i$ $a+bi = a \cdot 1 + b \cdot i \iff \begin{pmatrix} a \\ b \end{pmatrix}$

basis of $M_{2\times 2}(\mathbb{R})$: $E_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $E_{12}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $E_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $E_{22}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$

① $\text{Span}\{E_{11},E_{12},E_{21},E_{22}\}=M_{2\times2}(\mathbb{R})$: $\begin{pmatrix}a&b\\c&d\end{pmatrix} = a\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+b\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix}+c\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix}+d\begin{pmatrix}0&0\\0&1\end{pmatrix}$.

② $E_{11},E_{12},E_{21},E_{22}$ are linearly independent: If $c_{11}E_{11}+c_{12}E_{12}+c_{21}E_{21}+c_{22}E_{22}=\vec{0}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, then $LHS=\begin{pmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}=RHS$. So $c_{11}=c_{12}=c_{21}=c_{22}=0$.

real number.

(e) The set $M_{2\times 2}(\mathbb{R})$ of all 2×2 matrices with real entries, under usual addition of matrices and usual scalar multiplication of matrix by real number.

dim=4

$\begin{pmatrix}a&b\\c&d\end{pmatrix}+\begin{pmatrix}e&f\\g&h\end{pmatrix}=\begin{pmatrix}a+e&b+f\\c+g&d+h\end{pmatrix}$, $x\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}xa&xb\\xc&xd\end{pmatrix}$.

Example 2. Consider the vector space $\mathbb{C}$ of complex numbers.
(a) Find the zero vector.
(b) Prove that 1 and i are linearly independent vectors.
(c) Prove that the conjugation map $T:\underset{\text{domain}}{\mathbb{C}}\to\underset{\text{codomain}}{\mathbb{C}}$ defined as $T(a+bi)=a-bi, \forall a,b\in\mathbb{R}$ is a linear transformation. Is T one-to-one? Onto?

$0\cdot\vec{v}=\vec{0}$

$0\cdot(a+bi)=0a+0\cdot bi=0$.

(a) $\vec{0}+\vec{v}=\vec{v}$, $\forall\vec{v}\in V$. $\vec{0}=0\in\mathbb{C}$ $0+(a+bi)=a+bi$

(b) If $a\cdot 1+b\cdot i=\vec{0}=0\in\mathbb{C}$, for $a,b\in\mathbb{R}$. WTS $a=b=0$

This is because $a\cdot1+b\cdot i=a+bi=0 \Rightarrow a=b=0$

$\begin{cases}a+b=0\\a-b=0\end{cases}$

Prove $1+i$ and $1-i$ are linearly independent. WTS if $a(1+i)+b(1-i)=0$ then $a=b=0$

(c) WTS $T(x\vec{u}+y\vec{v})=xT(\vec{u})+yT(\vec{v})$. Reason. Suppose $\vec{u}=a+bi$, $\vec{v}=c+di$ then

$T(x\vec{u}+y\vec{v})$
$=T(x(a+bi)+y(c+di))$
$=T((xa+yc)+(xb+yd)i)$
$=(xa+yc)-(xb+yd)i$
$=x(a-bi)+y(c-di)$
$=xT(\vec{u})+yT(\vec{v})$

Example 3. Consider the vector space $M_{2\times 2}(\mathbb{R})$ of 2×2 matrices.
(a) Find the zero vector.
(b) Compute $A+2B-C$ and $2\cdot A$ for

$$A=\begin{pmatrix}1&0\\0&0\end{pmatrix}, B=\begin{pmatrix}0&0\\0&1\end{pmatrix}, C=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

(c) Find a basis of $M_{2\times 2}(\mathbb{R})$.
(d) Consider the symmetrization map $T: M_{2\times 2}(\mathbb{R})\to M_{2\times 2}(\mathbb{R})$ defined as

$$T(A)=A+A^T.$$

Prove that T is a linear transformation. Is T injective? Surjective?

Surjective (onto): $\forall\vec{v}\in\mathbb{C}$ codomain $\exists\vec{u}\in\mathbb{C}$ domain s.t. $T(\vec{u})=\vec{v}$

Suppose $\vec{v}=a+bi$, where $a,b\in\mathbb{R}$.
$T(a-bi)=a-(-b)i=a+bi$
So we can just take $\vec{u}=a-bi$.

injective (one-to-one): Goldilock's thm dim Dom = dim Codom then one-to-one $\Leftrightarrow$ onto.

Example 4. Consider the vector space V of all continuous functions over $[0,2]$, with addition and scalar multiplication defined as

$$(f+g)(x)=f(x)+g(x), \qquad \forall f(x),g(x)\in V;$$

$$(c\cdot f)(x)=c\cdot f(x), \qquad \forall c\in\mathbb{R}, f(x)\in S.$$

(a) Prove that $\cos^2 x, \sin^2 x, \cos(2x)$ are linearly dependent.
(b) If a linear transformation $T : V \to \mathbb{R}$ satisfies $T(1) = 2$ and $T(\sin^2 x) = 1$, find $T(\cos(2x))$. Note that the 1 $T(1)$ refers the constant function $f(x) \equiv 1$ on $[0, 2]$.

Example 5. Consider the vector space $\mathbb{P}_2$ of polynomials of degree $\leqslant 2$.
(a) Prove that the subset $W = \{f(x) \in \mathbb{P}_2 | f(1) = 0\}$ is a subspace.
(b) Prove that the subset $S = \{f(x) \in \mathbb{P}_2 | f'(x) = 1\}$ is not a subspace.
(c) Prove that x and $x^2 + x$ are linearly independent.
(d) Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined as

$$T(f(x)) = f'(x) + x^2.$$

Find element $p(x)$ such that $T(p(x)) = p(x)$.