Worksheet 10 (Feb. 12)

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Note: There is no discussion session next Monday.

Def $V \subset \mathbb{R}^n$ is called subspace if satisfies:
① $\vec{0} \in V$.
② whenever $\vec{u}, \vec{v} \in V$, then $a\vec{u} + b\vec{v} \in V$ ($\forall \vec{a}, \vec{b} \in V$).

DEFINITIONS

Intuition: A "plane" (or "line", or higher dim "planes") passing through the origin.

- Subspace, span as an example of subspace;

  $\vec{v_1}, \cdots, \vec{v_k} \in \mathbb{R}^n$, then $V = \text{Span}\{\vec{v_1}, \cdots, \vec{v_k}\} \subset \mathbb{R}^n$ is a subspace

  $\vec{u} + \vec{v}$ $\quad c \cdot \vec{u}$

- null space, column space;

  If A is $m \times n$.

  $Nul(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\} \subset \mathbb{R}^n$ nullspace → subspace of the domain "$Ker(T)$"

  Recall Similar in def of linear transformation $T(a\vec{u} + b\vec{v}) = a \cdot T(\vec{u}) + b \cdot T(\vec{v})$.

  $Col(A) = \text{Span}\{\vec{a_1}, \cdots, \vec{a_n}\} \subset \mathbb{R}^m$ Column space → Subspace of the codomain "$Im(T)$"

  there might be redundancy here. → linear dependence relations. e.g. the free variable columns can be written as linear combinations of pivot columns. [so if we remove free var. columns the span won't change]

- basis of a subspace, dimension of a subspace.

  $\dim V = \ell$, $V \subset \mathbb{R}^n$

  $\vec{u_1}, \cdots, \vec{u_\ell}$ form a basis of $V \subset \mathbb{R}^n$. means ① $\vec{u_1}, \cdots, \vec{u_\ell}$ L.I. ② $\text{Span}\{\vec{u_1}, \cdots, \vec{u_\ell}\} = V$.

  "just enough to span V"

METHODS AND IDEAS

Remark 1. An important property of **determinant**:

$$\det(AB) = \det(A) \cdot \det(B).$$

Theorem 1. *Below is the algorithm for finding the* ***bases of the null and column space of*** *a matrix A:*

- *row reduce A to REF or RREF;*
- *identify the pivot positions, go back to A, then the columns vectors of A where these pivots reside form a basis of* $\text{Col}(A)$*;*
- *write out the solution set of $A\mathbf{x} = \mathbf{0}$;*
- *separate the free variables in the solution set, then the "vector coefficients" of these free variables form a basis of* $\text{Nul}(A)$.

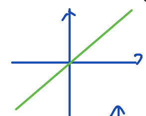
Remark 2. Since the dimension of a space is the number of vectors in any basis of it,

$$\dim \text{Col}(A) = \sharp \text{ pivots in } A, \qquad \dim \text{Nul}(A) = \sharp \text{ free variables in } A.$$

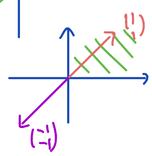
1 Problems

Example 1. Which ones of the following subsets of $\mathbb{R}^n$ are subspaces?

(a) $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x = y \right\} \subset \mathbb{R}^2$ $= \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$



(b) $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x, y > 0 \right\} \subset \mathbb{R}^2$



reason: ① $\vec{0} \notin \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x, y > 0 \right\}$

② for example $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x, y > 0 \right\}$.

but $(-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \notin \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x, y > 0 \right\}$

(c) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} | x + y + z = 1 \right\} \subset \mathbb{R}^3$

(d) $\mathbb{R}^3 \subset \mathbb{R}^3$

(e) $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x^2 + y^2 = 1 \right\} \subset \mathbb{R}^2$

(f) $\{\mathbf{0}\} \subset \mathbb{R}^5$.

Example 2. Consider the matrix

$$A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix}.$$

$$\rightarrow \begin{pmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 1 & -5 & 9/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 1/2 \\ 0 & 1 & -5 & -4 \\ 0 & 1 & -5 & 9/2 \end{pmatrix} \rightarrow \begin{pmatrix} \boxed{1} & 2 & -1 & 1/2 \\ 0 & \boxed{1} & -5 & -4 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

REF

these three are not basis of Col(A).

(a) Find a basis and the dimension for Nul(A). (b) Find a basis and the dimension for Col(A).

(a) $\begin{cases} x_1 + 2x_2 - x_3 + x_4/2 = 0 \\ x_2 - 5x_3 - 4x_4 = 0 \\ x_4 = 0 \end{cases}$ $\begin{cases} x_1 = -2x_2 + x_3 = -9x_3. \\ x_2 = 5x_3 \\ x_3 \text{ free} \\ x_4 = 0 \end{cases}$

$\therefore$ sol. set of $A\vec{x} = \vec{0}$ is $\left\{ \begin{pmatrix} -9s \\ 5s \\ s \\ 0 \end{pmatrix} \middle| s \in \mathbb{R} \right\}$

$= \left\{ \begin{pmatrix} -9 \\ 5 \\ 1 \\ 0 \end{pmatrix} \cdot s + \begin{pmatrix} \\ \\ \end{pmatrix} t \middle| s, t \in \mathbb{R} \right\}$

will appear when there're more free variables

$\therefore$ Basis of Nul(A): $\left\{ \begin{pmatrix} -9 \\ 5 \\ 1 \\ 0 \end{pmatrix} \right\}$.

dim Nul(A) = 1

"Nul(A) is a 1-dimensional subspace of $\mathbb{R}^4$".

(b) Pivot columns are 1^{st}, 2^{nd} and 4^{th} of A.

Basis of Col(A).

$\left\{ \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \right\}$

dim Col(A) = 3

"Col(A) is 3-dimensional subspace of $\mathbb{R}^3$"

So it's just the entire $\mathbb{R}^3$

Example 3. Let $P \subset \mathbb{R}^3$ be the plane $x - 2y + 3z = 0$ passing through the origin.
(a) Show that P is a subspace of $\mathbb{R}^3$. (**Hint:** You may use any theorem from the text.)
(b) Find a basis of P.

Example 4. True or false.

( ) $\det(A+B) = \det(A) + \det(B)$.

( ) $\det(A^{-1}) = -\det(A)$.

( ) $\det(3A) = 3\det(A)$

( ) If two rows of A are identical, then $\det(A) = 0$.

( ) If two columns of A are identical, then $\det(A) = 0$.

( ) $\det(A)$ is the product of diagonal entries of A.