The Key Topics in a Successful Math Curriculum

R. James Milgram
Department of Mathematics
Stanford University
Stanford, CA

Hung-Hsi Wu
Department of Mathematics
University of California, Berkeley
Berkeley, CA

Analysis of the results from TIMSS suggests that the U.S. school mathematics curriculum is a mile wide and an inch deep.[1] It covers too many topics and each topic is treated superficially. By contrast, the structure of mathematics instruction in countries which outperformed the U.S. follows a strikingly different pattern. In all cases, only a few carefully selected focus topics are taught and learned to mastery by students in the early grades. At the fourth grade level, since the students in these countries have not been exposed to as broad a curriculum as U.S. students, it sometimes appears on standardized tests such as TIMSS that they perform at a comparable level to U.S. students, but by grade eight the students in the leading countries are far outperforming our students. In fact, key test items already show serious weaknesses in our fourth grade student performances.[2] This difference becomes even greater by the end of high school, where even our top students do not match up well with the average achievement levels of students in these countries.[3]

It seems reasonable that some effort be devoted to revising our mile-wide-inch-deep curriculum. The following material is a description of the requirements for an intervention program in K - 7 mathematics that the state of California requested of us. It is based on the structure of the programs in the early grades in the high achieving countries where, in fact, remediation is seldom necessary. Thus, the course structure indicated here, an intense focus on six key topics, can also serve as the foundation courses for all students in the early grades - perhaps through grade 7 in this country.

It is worth noting that the NCTM intends to roll out a discussion of *focus topics* early in 2006, with a strong suggestion that these topics become the main part of instruction in grades Pre K - 8. It is too early to predict what the final list of NCTM recommended focus topics will be, but preliminary lists are very similar to the list that we discuss here.

Having said all this, there is more to successfully teaching mathematics than the *mathematical topics* that comprise the curriculum. In the high achieving countries there

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[1] W. Schmidt, C. McKnight, S. Raizen, ***A Splintered Vision***, Kluwer Academic Publishers, 1997

[2] A. Ginsburg, G. Cook, S. Leinwand, J. Noell, E. Pollock, ***Reassessing U.S. International Mathematics Performance: New Results from the 2003 TIMSS and PISA***, American Institutes for Research, 2005

[3] "Thus, the most advanced mathematics students in the United States, about 5 percent of the total age cohort, performed similarly to 10 to 20 percent of the age cohort in most of the other countries." S. Takahira, P. Gonzales, M. Frase, L.H. Salganik, ***Pursuing Excellence: A Study of U.S. Twelfth-Grade Mathematics and Science Achievement in Internation Context,*** U.S. Dept. of Ed., 1998, p.44

is, from the beginning, an intense focus on (1) definitions and precision, and (2) abstract reasoning. In our discussion of the focus topics we constantly ask for definitions and precision in setting things up. This is even more crucial for at risk students than it is for other students, since these are the students who have a greater need for precise and accurate definitions to guide their learning than others. Beyond this, definitions and precision are a critical component of successful mathematics instruction because correct mathematical reasoning is literally impossible without them. We are less insistent on abstract reasoning, given the focus on intervention. But a careful study of how abstraction is built into these top programs would be of benefit to everyone who needs to develop mathematics curricula for our schools. In further work we intend to discuss this issue in detail.

Intervention Program

This article addresses the needs of students in grades 4 to 7 whose mathematical achievements are below grade level. The common approach to the intervention program consists of offering courses which have half the content of the regular courses but use twice the instruction time. The fact that such an approach is not effective not only can be argued on theoretical grounds, but is borne out by ample empirical evidence. Here we propose a completely different solution by offering an **intensive, accelerated** program for these students, with the sole purpose of bringing them up to grade level in the shortest time possible so that they will be ready for algebra in grade eight. The implementation of such a program requires the cooperation and support of schools, school districts, and textbook publishers.

Schools and school districts will have to make a serious commitment of effort and time to such an accelerated program. We suggest *two hours* of mathematics instruction every day, using the special instructional materials to be discussed below. In most cases, we also suggest supplementing the regular hours with after-school programs as well as special mathematics sessions in the summer. We emphasize that, far from recommending *slow* classes for these students with special needs, we are asking for the creation of *more intense and more demanding* classes, to be taught by mathematically well-informed teachers. (In point of fact, the volumes for the intervention program should also be effective as references for regular classes as well as professional development materials.) More needs to be done for these students.

In general terms, two aspects of the proposed instruction stand out:

(1) Diagnostic assessment should be given frequently to determine students' progress. The special instructional materials below will provide assistance on this issue.

(2) There should an abundance of exercises for both in-class practice and homework. No acceleration will be possible if students are not intensely immersed in the doing of mathematics.

The heart of this proposed program is the creation of six volumes of special instructional materials, each volume devoted to one of the following six topics:

Place Value and Basic Number Skills
Fractions and Decimals
Ratios, Rates, Percents, and Proportion
The Core Processes of Mathematics

Functions and Equations
Measurement

The rest of this article is devoted to a detailed description of the content of these individual volumes. Let us first give an overview.

These six volumes will be made available as needed to each and every student in this program, *regardless of grade level*. The main purpose of creating these six volumes is to provide maximum flexibility to the teachers in this program. Depending on the special needs of the students in a given class, the teacher can use the diagnostic tests provided with each volume (see below) to determine the appropriate starting point for the class. For example, an intervention program in grade 5 may start with the chapter on the addition algorithm (second grade level) or the chapter on the long division algorithm instead (fourth grade level). Or, it can happen that three quarters of the students in the class are ready for the long division algorithm but the remaining one quarter of the students are behind and require help with the addition algorithm. In that case, one strategy would be to start the whole class on long division but give separate after-school instruction to the quarter of the class on the addition algorithm. This example may also help to explain why we want all six volumes to be available to students in this program no matter what their grades may be.

Because we are asking that these six volumes replace the textbooks of grades 4 to 7 for students in this program, we call attention to several special features. We ask that:

(1) Emphasis be given to the *clarity* of the exposition and *mathematical reasoning* in the mathematics. Clarity is a *sine qua non* in the present context because one may assume that indecipherable mathematics textbooks in students' past contributed to these students' underachievement. Moreover, the absence of reasoning in mathematics textbooks and mathematics instruction makes learning-by-rote the only way to learn the material. Our obligation to these students demands that we do better.
(2) The grade level of each section and each chapter in these volumes be clearly specified in the Teacher's Edition so that students' progress can be accurately gauged. (*For the sake of definiteness, we have made* **California's Mathematics Content Standards** *as our basic reference in this article, but other states can make suitable modifications.*)
(3) Abundant exercises of varying degrees of difficulty be given at the end of each section so that students will be constantly challenged to improve.
(4) Summative and diagnostic assessment be made an integral part of each section to allow students to determine their level of achievement at each stage.

(5) The expository level be *age appropriate* in the following sense: because these volumes will be used by students in grades 4 to 7, even the sections addressing mathematics standards of grades 1 to 3 should reflect the awareness of the age of the readership. For example, instead of "counting cookies" in teaching the important topic of counting whole numbers, "counting musical CD's" would likely get a better reception.
(6) The exposition be kept to a "no frills" level: multi-color pictures or references to extraneous topics such as rock concerts are distractions. The focus should be on the mathematics instead. Because these six volumes will be used in all four grades (4 to 7), it is imperative that the number of pages be kept to a minimum. Keeping things at a "no frills" level is one way to achieve this goal.

In the remainder of this article, we give a detailed guideline of what we consider to be truly essential in the content of each of these six volumes. Emphases have been placed on topics that are traditionally slighted or misunderstood in standard textbooks. *We believe that this guideline will also serve well as a guideline for the writing of regular textbooks in grades 4 to 7.*

Place Value and Basic Number Skills

Many students misunderstand place value, and without a solid understanding of this topic they will be unable to handle the basic algorithms and develop basic skills with numbers, let alone develop them to automaticity. Consequently, we start with the basic place value standards.

Of course counting starts with grade 1, but because we are addressing students in grade 4 and beyond, counting can be approached from a more sophisticated level. One can begin by explaining why, with the use of only ten symbols 0, 1, 2, ... , 9, counting can proceed beyond the ones place only by creating the tens place (to the left, *by convention*), so that after 9, one starts the counting all over again from 10, 11, 12, etc. Likewise, counting can proceed beyond the tens place (after 99) only by creating the hundreds place (to the left), etc. Observe that each new place has a value 10 times the preceding one. The reasoning here can be given as follows.

Consider how we go from the hundreds place to the thousands place. In the same way that one goes from 99 to 100, one goes to 200 upon reaching 199. Then another 100 later it is 300, and then 400, ... , 900 and therefore (after 999) it has to be 1000. So we see that 1000 is 10 steps from 0, i.e., 0, 100, 200, ... , 1000 if we skip count by 100, and therefore 1000 is 10 times the value of 100. This knowledge about counting also gives a clearer picture of addition because the latter is nothing but "iterated counting", in the sense that $12 + 5$ is the number one arrives at by counting 5 more starting at 12. Perhaps the crucial thing here is that students should clearly understand that place value is an *additiive* representation of the counting numbers. They should know, for example, that 7301 is a shorthand way of writing the number

$$7 \times 1000 + 3 \times 100 + 0 \times 10 + 1.$$

(See the detailed discussion below.)

Counting

In terms of the California Mathematics Content Standards, the key standards here are

Grade 1

1.1 Count, read, and write whole numbers to 100.

1.2 Compare and order whole numbers to 100 by using the symbols for less than, equal to, or greater than (<, =, >).

1.3 Represent equivalent forms of the same number through the use of physical models, diagrams, and number expressions (to 20) (e.g., 8 may be represented as 4 + 4, 5 + 3, 2 + 2 + 2 + 2, 10 - 2, 11 - 3).

1.4 Count and group objects in ones and tens (e.g., three groups of 10 and 4 equals 34, or 30 + 4).

2.1 Know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory.

2.5 Show the meaning of addition (putting together, increasing) and subtraction (taking away, comparing, finding the difference).

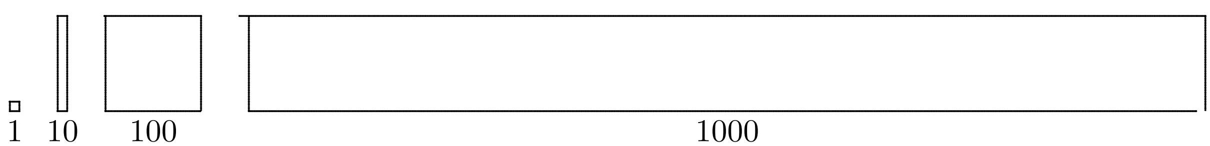
The concept of multiplication *among whole numbers* is a shorthand for counting groups of the same size. 7×5 means the number of objects in 7 groups of objects with 5 in each group. Therefore the *meaning* of 7×5 is $5 + 5 + \cdots 5$ (7 times). This is a point worth emphasizing in the book.

Grade 2

3.1 Use repeated addition, arrays, and counting by multiples to do multiplication.

3.3 Know the multiplication tables of 2s, 5s, and 10s (to "times 10") and commit them to memory.

Students should understand that $10 \times 10 = 100$, because $10 \times 10 = 10 + 10 + \cdots 10$ (10 times), and $10 \times 100 = 1000$ because $10 \times 100 = 100 + 100 + \cdots 100$ (10 times). As remarked above, these are consequences of the way we count in this numeral system. Students should also see an *area model* comparing the relative sizes of 1, 10, 100, and 1000[3]



as well as construct similar models for 10,000 to get a clear idea of the magnitude of these numbers.

It follows from the method of counting that, for example, $3 \times 100 = 100 + 100 + 100 = 300$, that $7 \times 1000 = 1000 + 1000 + 1000 + 1000 + 1000 + 1000 + 1000 = 7000$, etc. Similar facts are also true for the multiplication of the numbers 1, 10, 100, 1,000, and even 10,000 by 1, 2, 3, ..., 9. Students should construct these numbers, particularly for multiples of 1 and 10, and place them on the *number line.*

We strongly recommend the introduction of the number line as early as possible (in high achieving countries, it is done as early as grade 2). For a discussion of some

[3] Illustrations that give 1000 as a $10 \times 10 \times 10$ cube while illustrating 100 by a 10×10 square are not helpful in understanding the relative magnitude of numbers.

instructional issues related to the teaching of the number line at early grades, see the opening paragraphs of the chapter on *Measurement*.

A number such as 37 is said to be bigger than 23, because 37 comes after 23 in our way of counting. On the number line, 37 is to the right of 23. In terms of the number line, "bigger than" is synonymous with "to the right of". Thus 1000 is bigger than 100 in this sense. These facts are connected with the key **grade 2** standard:

1.3 Order and compare whole numbers to 1,000 by using the symbols $<$, $=$, $>$.

With these preliminaries, students should be ready to understand place value. Here are the key place value standards.

Grade 2

1.1 Count, read, and write whole numbers to 1,000 and identify the place value for each digit.

1.2 Use words, models, and expanded forms (e.g., $45 = 4$ tens $+ 5$) to represent numbers (to 1,000).

Grade 3

1.3 Identify the place value for each digit in numbers to 10,000.

1.5 Use expanded notation to represent numbers (e.g., $3,206 = 3,000 + 200 + 6$).

Grade 4

1.1 Read and write whole numbers in the millions.

1.2 Order and compare whole numbers and decimals to two decimal places.

1.3 Round whole numbers through the millions to the nearest ten, hundred, thousand, ten thousand, or hundred thousand.

Remark The **grade 4** standard

1.6 Write tenths and hundredths in decimal and fraction notations and know the fraction and decimal equivalents for halves and fourths (e.g., $\frac{1}{2} = 0.5$ or $.50$; $\frac{7}{4} = 1\frac{3}{4} = 1.75$).

represents a significant increase in sophistication, and should be deferred till fractions are introduced and there is a firm foundation for the discussion of decimals.

A key objective at this point is that students understand that the place value representation of a number such as $10,703$ signifies the full sum

$$1 \times 10,000 + 0 \times 1,000 + 7 \times 100 + 0 \times 10 + 3 \times 1.$$

In other words, each number is the addition of successive products of a single digit number multiplied by a power of 10. This is the so-called *expanded form* of $10,703$. Notice the representation in the expanded form of the first 0 from the left in $10,703$ as $0 \times 1,000$. People often talk of the special role of 0 as a *place-holder*, but this emphasizes the *form* of the written number, as in $10,703$, and not the number itself. Students should clearly understand that $0\times$ a power of 10 is always 0, such as $0 \times 1,000$ above. Same for 0×10, of course. They should understand that when we write the number $10,703$ as

$$1 \times 10,000 + 7 \times 100 + 3 \times 1,$$

where the 0's are suppressed, we are abbreviating. (Often, in instruction, there is a tendency to teach only the abbreviated form from the beginning, and not the full expansion. This can be result in core confusion for at risk students, who will often not distinguish a number such as 2643000012 from 264300012, since they fail to fully comprehend the role of the zeros.) Such abbreviations should be used only when students are already secure in their understanding of the unabbreviated form.

Addition and Subtraction

A main justification of place value lies in the ease of computation with numbers as embodied in the standard algorithms. First the addition and subtraction algorithms.

Grade 1

2.1 Know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory.

2.6 Solve addition and subtraction problems with one- and two-digit numbers (e.g., $5+58 = \_\_$).

2.7 Find the sum of three one-digit numbers.

Grade 2

2.2 Find the sum or difference of two whole numbers up to three digits long.

2.3 Use mental arithmetic to find the sum or difference of two two-digit numbers.

Grade 3

2.1 Find the sum or difference of two whole numbers between 0 and 10,000.

Grade 4

3.1 Demonstrate an understanding of, and the ability to use, standard algorithms for the addition and subtraction of multi-digit numbers.

A main point of the addition algorithm is that when applied to, for instance, $259+671$ is that it replaces the *cumbersome* counting of 671 times starting with 259. Likewise, the subtraction algorithm makes it unnecessary to count backward 259 times from 671 before finding out what $671-259$ is. These simple fact ought to be pointed out to students. The following discussion will concetrate on the addition algorithm; the subtraction algorithm can be dealt with in like manner.

Students first learn to acquire fluency in the use of the addition algorithm with only an informal explanation. Consider the addition of *two* numbers. For example, the column by column addition of $14+23$ in the usual format

$$\begin{array}{rcc} & 1 & 4 \\ + & 2 & 3 \\ \hline & 3 & 7 \end{array}$$

can be explained in terms of money. Think of 14 as represented by 1 ten-dollar bill and 4 one-dollar bills, and 23 as 2 ten-dollar bills and 3 one-dollar bills. Then $14+23$ becomes the counting of the total number of dollars, which can be done by counting the total number of ten-dollar bills ($3=1+2$) and the total number of one-dollar bills ($7=4+3$). This explains the separate additions of the digits in the tens place and those in the ones place.

The careful introduction of place value, however, enables students to revisit the algorithm with greater understanding. They get to see how place value together with the commutative and associative properties of addition lead to a full explanation of the addition algorithm for any two numbers. For the particular case of $14+23$, one can reason as follows:

$$\begin{array}{rcll} 14+23 & = & (10+4)+(20+3) & \\ & = & 10+(4+20)+3 & \text{(associative property)} \\ & = & 10+(20+4)+3 & \text{(commutative property)} \\ & = & (10+20)+(4+3) & \text{(associative property)} \end{array}$$

and the last line is the precise explanation of why the addition of $14+23$ can be carried out column by column. This is the essence of the addition algorithm. Once this is understood (*but not before*), students would be in a position to acquire the skill of carrying, as in

$$\begin{array}{rcc} & 1 & 7 \\ & 2 & 9 \\ + & {\scriptstyle 1} & \\ \hline & 4 & 6 \end{array}$$

This is nothing more than another application of the associative property of addition: We have just seen that $17+29=(10+20)+(7+9)$, but now $7+9=16=10+6$, so

$$\begin{aligned} 17+29 &= (10+20)+(10+6) \\ &= \{(10+20)+10\}+(4+3) \quad \text{(associative property)} \end{aligned}$$

The last line expresses precisely the carrying of the 1 into the tens place.

Discussion of the addition algorithm should include at least the following two points: (a) it simplifies the addition of any two numbers, no matter how large, to a sequence of additions of two one-digit numbers, and (b) the key idea of the algorithm is not the skill of carrying, but the possibility of changing the addition of any two numbers to column-by-column additions of one-digit numbers.

The addition algorithm of more than two numbers is entirely analogous. For example, the addition of any five numbers is reduced to the addition of a sequence of additions of five *one-digit* numbers.

Multiplication

Next one can turn to multiplication. The following is the foundational standard for the multiplication algorithm.

Grade 3

2.2 Memorize to automaticity the multiplication table for numbers between 1 and 10.

Knowing the multiplication table before any discussion of the multiplication of two arbitrary numbers is entirely analogous to knowing the alphabet before any discussion of reading and writing. The next step is for students to develop proficiency with the **grade 3** standard

2.4 Solve simple problems involving multiplication of multi-digit numbers by one-digit numbers ($3,671\times 3=$ __)

Let us first consider a simple example 213×3. As with the addition algorithm, the most important point of the instruction at this stage is to give students a sense of power by letting them see how the multiplication algorithm replaces the tedious process of adding 3 to itself 213 times (recall: this is exactly the *meaning* of 213×3) by the simple process of performing three one-digit multiplications: 2×3, 1×3, 3×3, as in

$$\begin{array}{rccc} & 2 & 1 & 3 \\ \times & & & 3 \\ \hline & 6 & 3 & 9 \end{array}$$

The explanation is simple: it is the distributive property at work, as in

$$\begin{aligned}213 \times 3 &= (200 + 10 + 3) \times 3 \\ &= (200 \times 3) + (10 \times 3) + (3 \times 3) \quad \text{(distributive property)} \\ &= (600) + (30) + (9)\end{aligned}$$

Note that the distributive property is taken up only in the **grade 5** algebra and function standard

1.3 Know and use the distributive property in equations and expressions with variables.

What this means in terms of the present volume is that, as usual, the detailed explanation of the multiplication algorithm can be delayed until students have achieved mastery of the procedures of the algorithm.

At this point, a parallel with the addition algorithm can be drawn: once the basic idea of the algorithm has been explained in a simple case, the more complicated issue of carrying is next. Consider the example in standard 2.4 above, $3,671 \times 3$. The algorithm yields the result as follows:

$$\begin{array}{rccccc} & & 3 & 6 & 7 & 1 \\ & & & & & 3 \\ \times & & {\scriptstyle 2} & {\scriptstyle 2} & & \\ \hline & 1 & 1 & 0 & 1 & 3\end{array}$$

The reasoning is entirely similar to the case of carrying in addition:

$$\begin{aligned}3671 \times 3 &= (3000 + 600 + 70 + 1) \times 3 \\ &= 9000 + 1800 + 210 + 3 \qquad \text{(distributive property)} \\ &= 9000 + 1800 + (200 + 10) + 3 \\ &= 9000 + (1800 + 200) + 10 + 3 \quad \text{(associative property)} \\ &= 9000 + 2000 + 10 + 3 \\ &= 11000 + 10 + 3 \\ &= 10000 + 1000 + 10 + 3\end{aligned}$$

The fourth line explains the carrying of the 2 into the hundreds place, and the fifth line explains the carrying of the 2 into the thousands place. Incidentally, this is the first, but hardly the last illustration of the point made above, that it is important to know the multiplication table: *multiplication between one-digit numbers is the foundation of the algorithm.*

We should call students' attention to the special case of multiplication by 1 and observe that, from both the definition of multiplication and the algorithm, multiplication by 1 does not change the number being multiplied. This is related to the **grade 3** standard

2.6 Understand the special properties of 0 and 1 in multiplication and division.

We will stay with multiplication for the moment and delay the consideration of division to later. When multiplication by 0 is discussed, (see, for example, the discussion of the role of 0 in place value notation that we gave above), it should not be treated as a curiosity but, rather, as a logical consequence of the definition of multiplication (if there are no groups, then there are no elements), and the result is also consistent with the rules for multiplication.

Finally, we come to the general multiplication algorithm for multi-digit numbers. This is contained in the **grade 4** number sense standard

3.2 Demonstrate an understanding of, and the ability to use, standard algorithms for multiplying a multi-digit number by a two-digit number and for dividing a multi-digit number by a one-digit number; use relationships between them to simplify computations and to check results.

Again, delaying the discussion of division, we concentrate on the multi-digit multiplication algorithm. The key idea of the algorithm is to reduce the multiplication between multi-digit numbers to a sequence of multiplications of a multi-digit number by a *one-digit* number. However, this algorithm for even a simple product such as 43×25 *cannot be taught by rote*, because while this product is broken up into the two products of 43 by each of the digits of 25, thus 43×2 $(= 86)$ and 43×5 $(= 215)$, it is the way these two numbers 86 and 215 are added together that mystifies most beginners:

$$\begin{array}{rrrrr} & & & 4 & 3 \\ \times & & & 2 & 5 \\ \hline & & 2 & 1 & 5 \\ + & & 8 & 6 & \\ \hline & 1 & 0 & 7 & 5 \end{array}$$

The obvious question is: why is 86 shifted one place to the left? The explanation comes from place value:

$$\begin{aligned} 43 \times 25 &= 43 \times (20 + 5) \\ &= (43 \times 20) + (43 \times 5) \quad \text{(distributive property)} \\ &= 860 + 215 \end{aligned}$$

This shows why 86 is shifted to the left, because it is really not 86 but 860 that is added to 215. In turn, this is so because the "43×2" in the algorithm is actually 43×20 $(= 860)$ on account of the fact that the 2 in 25 has the value of 20. If in the following addition of

$$\begin{array}{rrrrr} & & 2 & 1 & 5 \\ + & & 8 & 6 & 0 \\ \hline & 1 & 0 & 7 & 5 \end{array}$$

we suppress the 0 in 860, then we would get exactly the same addition as in the multiplication algorithm of 43×25.

In summary, the product of two multi-digit numbers x and y is obtained by adding the products of the multi-digit number x by the (single) digits of y. But we have seen that the product of a multi-digit number by a single digit number is the sum of a sequence of products between single digit numbers, and for the latter, one has to call on the multiplication table. Again, we see why knowing the multiplication table is so fundamental. Be sure to convey this message well.

Division

Division has always been difficult for students, especially the division of fractions and decimals. A central part of the reason has to be that the concept of division is almost never *clearly defined*. With a view to easing this difficulty, at the fourth grade level we introduce the key idea that division is an alternative but equivalent way of writing multiplication. Thus the division statement that *c equals b divided by a* or, sometimes, *a divides b equals c* is to be taught as nothing but *an alternative but equivalent* way of expressing $b = c \times a$ for whole numbers a, b, c (with $a > 0$). In symbols, we write: $b \div a = c$. Thus the two statements

$$b = c \times a \quad \text{and} \quad b \div a = c$$

are the same statement; they go hand-in-hand. This more general way of looking at division turns out to be valid in all of mathematics, in particular, for fractions and decimals.

Students need many concrete examples to be convinced that this definition of division is consistent with their previous understanding of $b \div a = c$ as meaning "b objects can be partitioned into c equal groups, each containing a objects." Indeed, since $b \div a = c$ is the same as $b = c \times a$, which equals $a + a + \cdots + a$ (c times), we see the the preceding interpretation is valid.[4] This should of course be explained using specific values of a, b, and c. For example, how do we teach a third grader how to find $36 \div 4$? We teach them to find the number so that when it multiplies 4 we get 36. In other words, the meaning of $36 \div 4 = 9$ is that $9 \times 4 = 36$. Similarly, the meaning of $78 \div 3 = 26$ is precisely that $3 \times 26 = 78$. And so on. Moreover, since $c \times a = a \times c$ (the commutative property!), $b \div a = c$ is also the same as $b = a \times c$, which equals $c + c + \cdots + c$ (a times). Thus $b \div a = c$ also has the meaning of "if b objects are partitioned into a equal groups, then the number of objects in each group is c."[5]

It is also important to thoroughly discuss the fact that, with this definition, some divisions cannot be carried out (*if we are rstricted to the use of whole numbers only*). For example, we cannot write $7 \div 3 = c$ for any whole number c for the simple reason that there is no corresponding multiplicative statement $7 = c \times 3$ (remember: c must be a whole number). This for division among whole numbers, the division $7 \div 3$ has no meaning. In general, if a and b are whole numbers and a is not a multiple of b, we cannot write $a \div b$ in the context of whole numbers.

[4] This is usually called the ***measurement interpretation of division***.

[5] This is usually called the ***partitive interpretation of division***.

Still with this definition of division, we can now explain why *it is impossible to define division by 0*. Let b be any nonzero whole number. Is it possible to define $b \div 0$ to be equal to some whole number c? So suppose $b \div 0 = c$, then by the definition of division, we have $b = 0 \times c$. But $0 \times c$ is always 0 regardless of what c is, whereas b is nonzero. Therefore it is impossible that $b = 0 \times c$ for any c, and we see that $b \div 0$ cannot be defined. Now of course, this reasoning depends on b being nonzero, and it may be that one can at least define $0 \div 0$. Observe, however, that from $0 = 0 \times 1$ one would conclude $0 \div 0 = 1$ by the definition of division, from $0 = 0 \times 2$ one would similarly conclude $0 \div 0 = 1$, and in general, no matter what c is, from $0 = 0 \times c$ one would always conclude that $0 \div 0 = c$. This means it is impossible to assign a fixed value to $0 \div 0$, so that $0 \div 0$ is also undefinable.

Central to the long division algorithm is the concept of division with remainder which should be understood from the perspective of "getting close" to an answer. For example, the fact that "38 divided by 5 has quotient 7 and remainder 3" is expressed symbolically as $38 = 7 \times 5 + 3$. (The usual symbolic expression of $38 \div 5 = 7\ R\ 3$ is obscure and should be avoided.) Then the *quotient* 7 is by definition the largest possible multiple of 5 which does not exceed 38, and the *remainder* 3 is by definition the difference between 38 and the largest possible multiple of 5 which does not exceed 38. Observe that the quotient is therefore intuitively "the closest that a multiple of 5 can come to 38 without exceeding it", and the remainder will always be a whole number smaller than 5. *These two concepts need to be clearly defined as above.* Significant time should be taken here, using the number line, for students to learn the geometric meaning of division-with-remainder. For example, with 93 fixed, let them plot the multiples of 8 one by one on the number line until they get to the 12th multiple which exceeds 93, i.e., it is the first multiple of 8 that lands to the right of 93 (this multiple is of course equal to 96). Then they have to back up to the 11th multiple, which is 88; this is the last multiple of 8 to the left of 93. Thus the quotient is 11. Since 88 is 5 short of 93, the remainder is 5, and students should verify that 5 is the length of the segment between 88 and 93 on the number line. They should also know that, unless the remainder is 0, division-with-remainder is not a "division" in the sense defined above.

The long division algorithm becomes accessible to students once division with remainder is clearly understood. For large numbers, such as 41548 divided by 29, it is not obvious what the quotient or the remainder ought to be. The long division algorithm provides a step-by-step procedure to *approximate* the quotient of a division-with-remainder one digit at a time. (Once the quotient is known, getting the remainder is of course straightforward.) It is not necessary to spend a lot of time drilling students on long divisions with multi-digit divisors. If they understand the reasoning behind the case of one-digit divisors very well, that should be enough for them to go forward. As illustration, we consider the case of 371 divided by 8.

> We are looking for whole numbers q and r so that $371 = q \times 8 + r$, where r is less than 8. First, can q be a 3-digit number? No, because if it were, then $q \times 8$ would be at least 800, which is larger than 371 whereas the equality $371 = q \times 8 + r$ implies that $q \times 8$ is at most 371. However, q must have two digits because if q is a single-digit number, then q would be at most 9 and therefore $q \times 8 + r$ would be at most $9 \times 8 + r$,

which is at most $72+7$ because the remainder r is at most 7. Therefore $q\times 8+r$ is at most 79, whereas it should be 371. So q is a two-digit number. The first digit of q has place value 10, so that if it is 1, $q\times 8$ would be at least 80, and if it is 2, then $q\times 8$ would be at least $20\times 8=160$, etc. Now $40\times 8=320<371$, while $50\times 8=400$ which is larger than 371. We conclude that the leading digit of q is 4. Remembering that q is a 2-digit number, we may write q as $q=40+q'$, where q' is a 1-digit number.

At this point, from $371=q\times 8+r$, we obtain $371=(40+q')\times 8+r$, and by the distributive property, this becomes $371=\{(40\times 8)+(q'\times 8)\}+r$. By the associative property of addition, we get

$$371=(40\times 8)+\{(q'\times 8)+r\}$$

But $371-(40\times 8)=51$, so we may rewrite the preceding equality as two separate equalities:

$$\begin{array}{rcl} 371 & = & 40\times 8+51 \\ 51 & = & q'\times 8+r \end{array}$$

where r is a whole number less than 8. In terms of the usual representation of the long division algorithm, we have:

$$\begin{array}{r|l} & \;4 \\ \cline{2-2} 8 & 371 \\ & 320 \\ \cline{2-2} & \;\;51 \end{array}$$

Now look at $51=q'\times 8+r$. *This is the division-with-remainder of 51 divided by 8 with q' as quotient and r as remainder.* We repeat the reasoning above and search for q'. This is an easy search and we conclude that q' must be 6 and r is 3. Thus $51=6\times 8+3$. Combined with $371=40\times 8+51$, this gives $371=40\times 8+6\times 8+3=(40+6)\times 8+3$, by the distributive property again. Thus $371=46\times 8+3$, which expresses the fact that 371 divided by 8 has quotient 46 and remainder 3.

The steps of the reasoning given in the preceding paragraphs corresponds exactly to the usual procedures in the long division of $371\div 8$:

$$\begin{array}{r|r} & 46\;\, \\ \cline{2-2} 8 & 371 \\ & 320 \\ \cline{2-2} & 51 \\ & 48 \\ \cline{2-2} & 3 \end{array}$$

Fractions and Decimals

In terms of the California Mathematics Content Standards, the study of fractions starts with the **grade 2** number sense standards

4.0 Students understand that fractions and decimals may refer to parts of a set and parts of a whole:

4.1 Recognize, name, and compare unit fractions from $\frac{1}{12}$ to $\frac{1}{2}$.

4.3 Know that when all fractional parts are included, such as four-fourths, the result is equal to the whole and to one.

5.1 Solve problems using combinations of coins and bills.

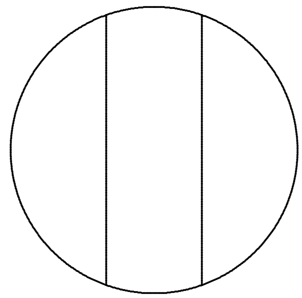
5.2 Know and use the decimal notation and the dollar and cent symbols for money.

Fractions are best introduced to students in this program using dollars and cents, since these are objects of intense interest. This leads naturally to the first model for fractions, what might be called the *set model*, which describes a fraction as a decomposition (or partition) of a collection of objects into equal groups. Thus a dime is decomposed into 10 pennies, a nickel into 5 pennies, a quarter into five nickels, and a dollar into 10 dimes. In these examples, depending on whether the *whole*, i.e., the number 1, is a nickel or a dime, the penny represents a different fraction: it is $\frac{1}{5}$ in case of the nickel, and $\frac{1}{10}$ in case of the dime. Thus the importance of knowing what the whole stands for, i.e., *what the number 1 represents*, comes naturally to the forefront. As another example, if the whole is a collection of eight objects, $\frac{1}{4}$ would represent two objects, and if the whole is a collection of four objects, the same fraction would then represent only one object.

Students should next be introduced to the area model for fractions, again paying careful attention to the relation between the fractional part and the whole.

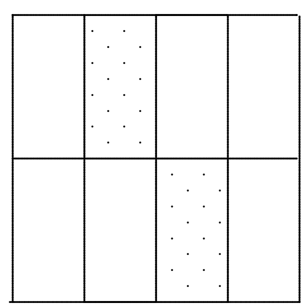
It is important that certain standard kinds of errors in understanding be checked here. In both models it often happens that students can become confused about *the whole*. The meaning of the whole in the area model, for example, has to be carefully explained. It is the *total area* represented by the *unit square* (the square each of whose sides has length 1), so that with this "whole" (i.e., the number 1) understood, the number 2 represents the area of any figure that has twice the total area of the unit square. Likewise, one-third is the area of any region which has the property that three such regions together would have area equal to 1, i.e., equal to the area of the unit square.

Here is an error that some students make that indicates a lack of understanding of the basic assumptions underlying the area model for fractions. Upon being told that the whole is "the circle", they consider the following to be an equal division into thirds:



Such an error is understandable when the whole is presented to them as "the circle" and not "the area of the circle", and when $\frac{1}{3}$ is explained to them as "one part when the circle is divided into three equal parts". Without a clear understanding that it is the *area* of the circle that they have to divide equally into three parts, they may legitimately interpret "three equal parts" as "three parts of equal width". The preceding error is the inevitable outcome of faulty instruction. We have to be explicit about dividing the circle into three parts of *equal area*.

Another thing to note about the area model is that it is difficult for students to equally divide pie-shaped regions into regions of equal area. Rectangular regions are easier to work with for the most common fractions. In the picture below, the dotted region represents $\frac{1}{4}$ if the whole is the area of the square.



Incidentally, if students do not have a clear understanding that the whole in this case is the *area* of the unit square (for example, they may think of the whole as the shape of the square), they will not be able to see that the dotted area represents $\frac{1}{4}$.

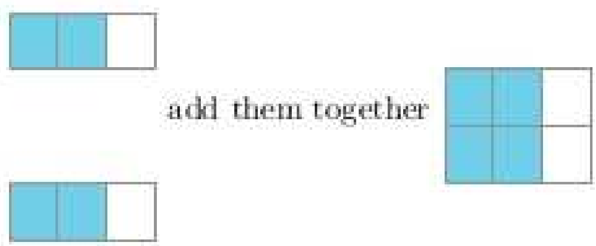
The **grade 3** standards for fractions are a bit more challenging:

3.1 Compare fractions represented by drawings or concrete materials to show equivalency and to add and subtract simple fractions in context (e.g., $\frac{1}{2}$ of a pizza is the same amount as $\frac{2}{4}$ of another pizza that is the same size; show that $\frac{3}{8}$ is larger than $\frac{1}{4}$).

3.2 Add and subtract simple fractions (e.g., determine that $\frac{1}{8}+\frac{3}{8}$ is the same as $\frac{1}{2}$).

Here we come into contact with two substantive concepts: the *equivalence* and the *addition* of fractions. It should emphatically not be assumed that these concepts are either natural or self-evident; rather, they should be clearly explained to students. Here is a classic example of a common misconception of the addition of fractions:

Frank says that $\frac{2}{3}+\frac{2}{3}=\frac{4}{6}$ and uses the picture below to justify his assertion.



Many misconceptions are probably involved here, but one of them would have to be that, since the addition of whole numbers is achieved by counting the union of two groups of objects, one should add fractions in the same way by counting the combination of "part-wholes", by brute force if necessary, even if nobody knows how "part-wholes" should be counted. In the above situation, the thinking must have been that, if we just count everything in sight, then there are 4 colored squares among the six squares, so the sum $\frac{2}{3}+\frac{2}{3}$ "must be" $\frac{4}{6}$.

This example points to the weaknesses in the usual exposition on fractions: the precise meaning of a fraction is never explicated and reasoning based on the fuzzy notions of *part-whole*, *quotient*, etc. inevitably leads to error; the fact that the precise meaning of the addition of fractions must be explicitly given is usually ignored. Moreover, the *addition* of fractions is a delicate issue because not only must it be precisely defined, but it must also retain the basic intuition one gains from the addition of whole numbers as "putting things together."

The Number Line

What is usually done in high achieving countries at this point is to place fractions on the number line and develop the whole theory of fractions. In this setting, a fraction such as $\frac{5}{3}$ is the following point on the number line: divide all the segments from 0 to 1, 1 to 2, 2 to 3, etc., into 3 segments of equal length, so that the number line is now divided into an infinite number of points to the right of 0, any two adjacent points being of length $\frac{1}{3}$ apart. The first such point to the right of 0 is the fraction $\frac{1}{3}$, the second $\frac{2}{3}$, etc., so that the fifth such point is the fraction $\frac{5}{3}$. Equivalently, $\frac{5}{3}$ is the length of the segment obtained by joining 5 segments end-to-end, each segment being of length $\frac{1}{3}$. Now introduce the concept that any two fractions which are placed at the same point on the number line (e.g., $\frac{2}{4}$ and

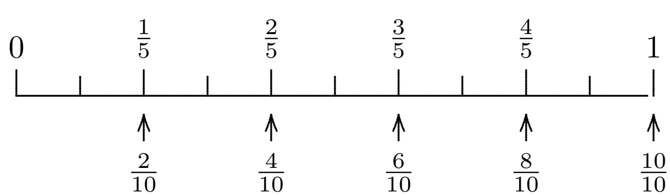
$\frac{1}{2}$, or 3 and $\frac{15}{5}$) are said to be *equivalent* or *equal*. Likewise, if a fraction on the number line is to the right of another fraction, then the first fraction is said to be *larger than* the second. Notice that this definition of "larger than" is consistent with the same concept among whole numbers. Also, *addition of fractions* is now defined by putting the fractions on the number line and then adding the lengths of the segments from 0 to the respective fractions. Again notice that the addition of whole numbers can be phrased in exactly the same way and that this concept of fraction addition literally embodies the intuitive idea of "putting things together." These definitions are quite natural *once they are explained*, but it should not be assumed that students will automatically know them.

A fundamental fact that underlies the development of fractions, usually referred to as *equivalent fractions* is this: two fractions are *equivalent* (i.e., represented by the same point on the number line) if one is obtained from the other by multiplying top and bottom by the same non-zero whole number. This is the fundamental fact that underlies the development of fractions. It is contained in the **grade 4** number sense standard

> **1.5** Explain different interpretations of fractions, for example, parts of a whole, parts of a set, and division of whole numbers by whole numbers; explain equivalence of fractions.

How this works on the number line can be illustrated with, for example, the fractions $\frac{4}{5}$ and $\frac{8}{10} = \frac{2\times 4}{2\times 5}$.

> To get $\frac{4}{5}$, we divide the segment from 0 to 1 into 5 segments of equal length and, counting the division points to the right of 0 from left to right, the fourth is $\frac{4}{5}$. For $\frac{8}{10}$, we do likewise by dividing the segment from 0 to 1 into 10 segments of equal length. Since $10 = 2 \times 5$, the 2nd, 4th, 6th, 8th division points to the right of 0 divide the segment from 0 to 1 into 5 segments of equal length. They are, as a result, the same as the first set of division points. In particular, the 8th division point is $\frac{4}{5}$, i.e., $\frac{4}{5} = \frac{8}{10}$, as the following picture shows.



Another part of standard 1.5 asserts that a fraction, e.g., $\frac{5}{3}$, can be *interpreted as a division*. The meaning of this statement is that, $\frac{5}{3}$ is the size of one part when something of size 5 is partitioned into 3 parts of the same size. More precisely, in terms of the number line, the statement means that the fraction $\frac{5}{3}$, which is by definition the length of 5 segments each of which has length $\frac{1}{3}$, is also the length of a part when a segment of length 5 is divided into 3 parts of equal length. This is an important fact in the study of fractions.

The explanation of this division interpretation is as follows. Think of this segment of length 5 as the joining of 5 sub-segments each of length 1. If each of these sub-segments of length 1 is divided into 3 parts of equal length (so that by the definition of a fraction each part has length $\frac{1}{3}$), then the given segment of length 5 is divided into 15 ($= 5 \times 3$) of these parts of length $\frac{1}{3}$. If we want to divide this segment of length 5 into 3 sub-segments of equal length, it suffices to take the 5th and 10th division points of the original division into 15 sub-segments. But the length of the segment from one end to the 5th division point is the joining of 5 sub-segments each of length $\frac{1}{3}$, and therefore this segment has length $\frac{5}{3}$, which explains the division interpretation of $\frac{5}{3}$.

Notice that the preceding reasoning involves only the numbers 3 and 5, and that these are the numerator and denominator of the fraction $\frac{5}{3}$ *in question.* This is why the same reasoning also suffices to show that a fraction $\frac{a}{b}$ is also the length of a part when a segment of length a is divided into b parts of equal length. Usually one paraphrases this fact as "$\frac{a}{b}$ is one part when a wholes are divided into b equal parts." If $a = mb$ for some whole number m, then we get back the fact that $\frac{a}{b} = \frac{mb}{b} = m$, which then coincides with $(mb) \div b = m$. This explains the phrase "division interpretation", and for the same reason, *the division symbol* $\div$ *is retired at this point and* $a \div b$ *will henceforth be denoted by* $\frac{a}{b}$.

Adding Fractions

Students are now ready to review and further study the addition of fractions with the same denominator as illustrated by the third grade number sense standard 3.2 above. They should add such fractions on the number line, and be able to understand why the following formula

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

for any whole numbers a, b, and c is correct. This is because $\frac{a}{b}$ is *by our definition of a fraction* a copies of $\frac{1}{b}$, and similarly $\frac{c}{b}$ is c copies of $\frac{1}{b}$. Since adding fractions is nothing but getting the combined length of these segments joined together end-to-end, we have to get the length of a copies of $\frac{1}{b}$ together with c segments of $\frac{1}{b}$. Clearly this length is $a + c$ copies of $\frac{1}{b}$, which is to say, $\frac{a+c}{b}$ according to the definition of a fraction. One may paraphrase this reasoning as follows: putting a copies of $\frac{1}{b}$ and c copies of $\frac{1}{b}$ together, we get $a + c$ copies of $\frac{1}{b}$, which is exactly $\frac{a+c}{b}$. Notice how the addition of fractions is now seen to be the *iterated counting* of the number of segments of length $\frac{1}{b}$. In this sense, *adding fractions is no different from adding whole numbers.*

Once students understand fractions on the number line including

- how to place them on the number line
- how to add fractions with the same denominator on the number line
- the fundamental fact of equivalent fractions

- how to interpret a fraction as a division

- the formula for the addition of two fractions with the same denominator

they are ready to be exposed to, and to make use of further formulas. The first key formula that they need to learn is the fundamental fact of equivalent fractions:

$$\frac{a}{b} = \frac{ca}{cb} \quad \text{for } c \text{ any whole number,}$$

where, **by convention**, we write ca for $c \times a$, cb for $c \times b$. The reasoning for this formula is the same as that given above for $\frac{2}{5} = \frac{4}{10}$. Next, they should be given the formula for the addition of two fractions $\frac{a}{b}$ and $\frac{c}{d}$:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

This is true because by equivalent fractions, we can write the given fractions as two fractions *with equal denominator*: $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$. Therefore the addition $\frac{a}{b} + \frac{c}{d}$ now becomes the addition of two fractions with equal denominator, $\frac{ad}{bd} + \frac{bc}{bd}$, which we already know how to do. The preceding formula is the result.

Notice that *no mention in this entire discussion of adding fractions was made of the need to look for the LCM of the two denominators b and d.* The use of LCM is detrimental to the understanding of the *meaning* of the sum of two fractions. What is true is that the addition of fractions is sometimes simplified by the use of the LCM of the denominators, e.g. $\frac{1}{8} + \frac{5}{12} = \frac{3}{24} + \frac{10}{24} = \frac{13}{24}$, so that it makes sense for students to acquire this skill *after* they have thoroughly mastered the correct way to add fractions, as above. However, this specialized skill has nothing to do with the meaning of the sum of two arbitrary fractions.

Considerable practice should be given in using both formulas.

In **grade 5**, embedding fractions on the number line becomes a key part of the number sense standard

1.5 Identify and represent on a number line decimals, fractions, mixed numbers, and positive and negative integers.

In **grade 6**, this extends to the emphasis number sense standard

1.1 Compare and order positive and negative fractions, decimals, and mixed numbers and place them on a number line.

Some comments on ordering fractions and mixed numbers are appropriate at this point.

- First, given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, because $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$, it follows that $\frac{a}{b}$ is to the left of $\frac{c}{d}$ on the number line exactly when $ad < bc$. By the definition of *larger than*, $\frac{c}{d}$ is larger than $\frac{a}{b}$ exactly when $bc > ad$. The same reasoning also shows that $\frac{a}{b} = \frac{c}{d}$ exactly when $bc = ad$. This is sometimes called the *cross-multiplication algorithm* and deserves to be included in the book.[6]

- Next, it is important to demystify the concept of a *mixed number*, normally introduced in grade 4. A mixed number such as $3\frac{2}{5}$ is nothing more than a *shorthand notation* for $3 + \frac{2}{5}$. Since we now know how to add fractions, we have

$$3\frac{2}{5} = 3 + \frac{2}{5} = \frac{3}{1} + \frac{2}{5} = \frac{3 \times 5 + 2}{5} = \frac{17}{5}$$

  Notice that while we end up with the usual "formula for converting a mixed number to a fraction", the difference (and it is an important one) is that here $3\frac{2}{5}$ is clearly *defined* to be $3 + \frac{2}{5}$, and since we know how to add fractions at this point, the formula is a logical consequence of the definition rather than some unsubstantiated facts. *Do not introduce mixed numbers before the addition of two fractions is defined and understood!*

Multiplying and Dividing Fractions

In **grade 5** students are expected to be able to multiply and divide with sufficiently simple fractions:

> **2.5** Compute and perform simple multiplication and division of fractions and apply these procedures to solving problems.

In **grade 6** this is extended to the number sense standard

> **2.1** Solve problems involving addition, subtraction, multiplication, and division of positive fractions and explain why a particular operation was used for a given situation.

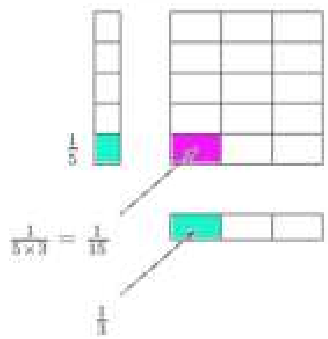
But once more, it is essential that the procedures for multiplication and division be carefully justified and explained. We first deal with multiplication. There are several ways to introduce the multiplication of fractions, but perhaps the simplest is to use the area model (where the unit 1 stands for the area of the unit square) and *define* the product $\frac{a}{b} \times \frac{c}{d}$ to be the area of the rectangle with sides of length $\frac{a}{b}$ and $\frac{c}{d}$. A word should be said about this definition, which undoubtedly seems strange at first sight. We all "know" that the area of such a rectangle is $\frac{ac}{bd}$, so why not just say $\frac{a}{b} \times \frac{c}{d}$ *is* $\frac{ac}{bd}$? One reason for not doing that is because we shall in fact *prove*, strictly on the basis of the above definition of $\frac{a}{b} \times \frac{c}{d}$ and on the basis of the definition of a fraction, that the area of this rectangle is $\frac{ac}{bd}$. Furthermore, this definition of $\frac{a}{b} \times \frac{c}{d}$ in terms of the area of a rectangle is concrete and is also what we have come to expect intuitively of such a product. On the other hand, if

[6] Cross-multiplication is subject to many misinterpretations and has to be handled with care. For example if one has $\frac{a}{b} = \frac{c}{d} + e$, it often happens that students will replace this expression by $ad = bc + e$.

we simply define $\frac{a}{b} \times \frac{c}{d}$ as $\frac{ac}{bd}$, this would always beg the question of why not also define $\frac{a}{b} + \frac{c}{d}$ as $\frac{(a+c)}{(b+d)}$.

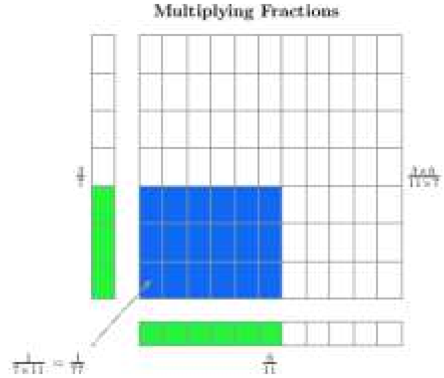
The main objective here is therefore to show that the product $\frac{a}{b} \times \frac{c}{d}$ is equal to as $\frac{ac}{bd}$. For example, let us see why $\frac{1}{5} \times \frac{1}{3} = \frac{1}{5\times 3}$. By our definition, $\frac{1}{5} \times \frac{1}{3}$ is the area of the purple rectangle in the lower left corner of the unit square in the picture below. Since this purple rectangle has area which is $\frac{1}{3\times 5}$ of the area of the unit square (which is 1), the area of the purple rectangle is therefore $\frac{1}{3\times 5}$ by the definition of a fraction. Thus $\frac{1}{5} \times \frac{1}{3} = \frac{1}{5\times 3}$.

Multiplying the unit fractions $\frac{1}{5}$ and $\frac{1}{3}$



Similar reasoning shows why $\frac{1}{b} \times \frac{1}{d} = \frac{1}{bd}$ for any nonzero whole numbers b and d. The reasoning for the general formula $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ is now relatively straightforward. For example, the picture below gives the simple idea of why $\frac{3}{7} \times \frac{6}{11} = \frac{3\times 6}{7\times 11}$.

Multiplying Fractions



Once students understand the *product formula* $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, we can explain why

> if we divide an object of size $\frac{c}{d}$ into b equal parts, then the total size of a of these parts is equal to $\frac{a}{b} \times \frac{c}{d}$.

One comment before we give the reasoning. This interpretation of the product of fractions is important for solving word problems, and is a precise formulation of the usual linguistic usage of the preposition "of." For example, when we say we cut off $\frac{1}{3}$ (by weight) of a piece of ham weighing $1\frac{3}{4}$ pounds, we mean, implicitly, that we will divide the piece of ham into 3 equal parts (by weight) and take 1 of the parts. Moreover, we automatically *assume* that the weight of this cut-off piece would be $\frac{1}{3} \times 1\frac{3}{4}$. What the preceding assertion about the meaning of $\frac{a}{b} \times \frac{c}{d}$ does is to give a firm mathematical foundation for this everyday linguistic usage. Thus in the future, we can confidently reformulate the common expression of "$\frac{3}{5}$ *of* a certain amount" as a precise statement of "$\frac{3}{5}\times$ that amount." Or, more generally, the size of "$\frac{a}{b}$ *of* an object of size $\frac{c}{d}$" is just $\frac{a}{b} \times \frac{c}{d}$.

Now the reasoning. It is a good illustration of the power of the product formula. We place the assertion in the context of the number line, so that we divide the segment from 0 to $\frac{c}{d}$ into b sub-segments of equal length, and we have to show that the total length of a of these sub-segments is $\frac{a}{b} \times \frac{c}{d}$. The idea of the proof becomes most clear if we use concrete numbers, say, $c = 12$, $d = 5$, $a = 3$ and $b = 7$. Thus we must show that if we divide the segment from 0 to $\frac{12}{5}$ into 7 sub-segments of equal length, then the combined length of 3 of these sub-segments is $\frac{3}{7} \times \frac{12}{5}$. First observe that the length of one of these sub-segments is $\frac{1}{7} \times \frac{12}{5}$ because if we combine 7 copies of a segment of length $\frac{1}{7} \times \frac{12}{5}$, we get a segment of length $\frac{12}{5}$. Indeed, the length of 7 such segments is

$$7 \times \left\{ \frac{1}{7} \times \frac{12}{5} \right\} = \frac{7 \times 12}{7 \times 5} = \frac{12}{5}$$

where we have used the product formula twice. This proves our claim. Now, the length of 3 of these sub-segments is, by the product formula once more,

$$3 \times \left\{ \frac{1}{7} \times \frac{12}{5} \right\} = \frac{3}{7} \times \frac{12}{5}$$

We have reached our desired conclusion.

Next, the concept of dividing fractions is, as we have already mentioned in connection with the division of whole numbers, qualitatively the same as the same concept between whole numbers. Recall that for whole numbers a, b, c, with $a \neq 0$,

$$b \div a = c \quad \text{is the same as} \quad b = c \times a$$

For fractions we follow this lead and define: for fractions A, B, C, with $A \neq 0$,

$$B \div A = C \quad \text{is the same as} \quad B = C \times A.$$

If $B = \frac{a}{b}$ and $A = \frac{c}{d}$, then it is immediately verified that the fraction $C = \frac{ad}{bc}$ satisfies $B = C \times A$. Rewriting this multiplication statement as a division statement according to the preceding definition, we get

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

This is the famous *invert-and-multiply* rule. It is important that every student knows why it is true: it is a consequence of the meaning of dividing fractions.

Another thing to note is that in the above discussion of fractions, there is no mention of simplifying fractions. Simplifying fractions is a topic that is terribly confusing to many students, and is not, mathematically speaking, necessary. As long as students understand equivalent fractions and have experience in recognizing equivalent fractions, the very involved procedures of finding greatest common divisors and putting fractions into reduced form can safely be avoided.

Finite Decimals

Among fractions, a special class stands out for historical as well as mathematical reasons. These are the *decimal fractions*, more commonly known as (*finite* or *terminating*) *decimals*. Precisely, they are fractions whose denominators are a power of 10, e.g., $\frac{427}{10^5}$; such a decimal is traditionally abbreviated to 0.00427, where the *decimal point* is placed 5 places (corresponding to the 5 in 10^5) to the left of the last (rightmost) digit of the numerator. In like manner, $\frac{87}{10^4}$ is abbreviated to 0.0087, the decimal point being placed 4 places to the left of the last digit 7 of the numerator. By the same token,

$$\frac{1200}{10^6} = 0.001200.$$

In this case, the last two 0's in 0.001200 are usually omitted so that the decimal is simplified to 0.0012; this is because

$$0.001200 = \frac{1200}{10^6} = \frac{12 \times 10^2}{10^4 \times 10^2} = \frac{12}{10^4} = 0.0012,$$

where we have made use of equivalent fractions.

It is worth repeating that a decimal, usually understood to be *finite* unless specified to the contrary, is nothing but a *shorthand notation* for a fraction whose denominator is a power of 10.

We indicated earlier that money is a very good place to begin the discussion of fractions, but money is especially effective for the instruction on decimals in the early grades. For example, there is the **grade 2** number sense standard

5.2 Know and use the decimal notation and the dollar and cent symbols for money.

Children are taught that there are 100 cents in a dollar, and 32 cents can be written as 0.32 dollars. The explanation for the notation can wait. The study of decimals takes a sophisticated turn when it comes to the **grade 4** number sense standard

2.0 Students extend their use and understanding of whole numbers to the addition and subtraction of simple decimals as well as the more sophisticated **grade 5** number sense standard

> **2.0** Students perform calculations and solve problems involving addition, subtraction, and simple multiplication and division of fractions and decimals

and the **grade 6** number sense standard

> **1.1** Compare and order positive and negative fractions, decimals, and mixed numbers, and place them on the number line.

We will deal with negative fractions presently. For the moment, let us concentrate on decimals. One cannot make sense of these three standards unless one makes use of the definition of a decimal as a fraction. The refusal in most textbooks to present decimals as a special class of fractions is likely the reason for students' well-documented confusion over how to compute or order decimals. Consider the grade 4 standard 2.0 above, which calls for an understanding of the role of whole-number additions in decimal additions. The usual instruction on adding 0.12 to 0.063, for example, is to "line up the decimal point" and then "add the decimals as if they are whole numbers"; no explanation is given for the algorithm. In terms of the definition of decimals as fractions, the reason for the algorithm is clear:

$$0.12 + 0.063 = \frac{12}{100} + \frac{63}{1000} = \frac{120}{1000} + \frac{63}{1000} = \frac{120+63}{1000},$$

which is of course $\frac{183}{1000} = 0.183$.

Similarly, the algorithm of computing $0.00009 \times 2.67 = 0.0002403$ is to first compute the whole-number multiplication $9 \times 267 = 2403$ and then count the total number of decimal places in each factor $(5+2=7)$ to determine where to put the decimal point in 2403. The explanation of this general algorithm can be given only by using the definition of a decimal:

$$0.00009 \times 2.67 = \frac{9}{10^5} \times \frac{257}{10^2} = \frac{2403}{10^{2+5}},$$

which then yields the correct answer 0.0002403.

The ordering of decimals called for in the above grade 6 standard 1.0 becomes the ordering of whole numbers when the definition of a decimal is used. For example, to explain why $0.1 > 0.097$, one recalls that $0.1 = \frac{1}{10}$ and $0.097 = \frac{97}{1000}$. To compare the *fractions* $\frac{1}{10}$ and $\frac{97}{1000}$, we follow the usual procedure of rewriting these fractions as fractions with the same denominator:

$$\frac{1}{10} = \frac{100}{1000} \quad \text{and} \quad \frac{97}{1000}$$

Since obviously $\frac{100}{1000} > \frac{97}{1000}$, we have $\frac{1}{10} > \frac{97}{1000}$. In other words, $0.1 > 0.097$.

This discussion points to the inescapable conclusion that, the teaching of decimals requires that *we give students a thorough grounding in fractions before embarking on any* *extended discussion of the arithmetic operations concerning decimals.* The usual practice of discussing the arithmetic of decimals as if decimals and fractions are unrelated must be avoided at all costs, especially in an intervention program.

Another aspect of decimals that should be brought out is that they provide a natural extension of the place value concept for whole numbers. Again the precise definition of a decimal plays a critical role. For example, since 3.14 is *by definition* the fraction $\frac{314}{10^2}$,

$$3.14 = \frac{314}{10^2} = \frac{300 + 10 + 4}{10^2} = \frac{300}{10^2} + \frac{10}{10^2} + \frac{4}{10^2} = 3 + \frac{1}{10} + \frac{4}{10^2}$$

Thus, for instance, the 2nd decimal digit 4 has place value $\frac{4}{10^2}$ and the 1st decimal digit 1 has place value $\frac{1}{10^1}$. In terms of money, if the unit 1 is a dollar (= 100 cents), then five dollars and 23 cents is $5 + \frac{23}{100} = 5 + \frac{2}{10} + \frac{3}{10^2}$ dollars which, on the one hand, is by definition 5.23 dollars and, on another, displays the amount clearly as 5 dollars, 2 dimes and 3 cents. This is the explanation for the usual notation of money. It also clarifies the **grade 3** number sense standard

3.4 Know and understand that fractions and decimals are two different representations of the same concept (e.g., 50 cents is $\frac{1}{2}$ of a dollar, 75 cents is $\frac{3}{4}$ of a dollar).

The following **grade 4** number sense standards are related:

1.6 Write tenths and hundredths in decimal and fraction notations and know the fraction and decimal equivalents for halves and fourths (e.g., $\frac{1}{2} = 0.5$ or .50; $\frac{7}{4} = 1\frac{3}{4} = 1.75$).

1.7 Write the fraction represented by a drawing of parts of a figure; represent a given fraction by using drawings; and relate a fraction to a simple decimal on a number line.

But note the appearance of *improper fractions* vs. *mixed numbers* in grade 4, number sense 1.6. This is another source of confusion for students. In particular they struggle with things like writing a fraction such as $\frac{7}{4}$ in the form $1\frac{3}{4}$ or 1.75, and the terminology of "improper fraction" makes them believe that a fraction greater than 1 is wrong. It is crucial, at this point that these students be taught that these different notations all mean the same thing, and that the default method of handling these numbers is to *always* write them as ordinary fractions.

Negative Numbers

Once students have a firm grasp of positive fractions, they are ready to tackle negative fractions. Beginning with **grade 4**, negative whole numbers are taught as in number sense standard

1.8 Use concept of negative numbers (e.g., on a number line, in counting, in temperature, in "owing").

In the **grade 5** number sense standard

2.1 Add subtract, multiply, and divide with decimals; add with negative integers; subtract positive integers from negative integers; and verify the reasonableness of the results.

students are asked to add and subtract *integers*, which are by definition the collection of all positive and negative whole numbers together with 0. For the purpose of doing arithmetic, the number -2, for example, should be clearly defined as the number so that $2 + (-2) = 0$. We have already alluded to the need to place the integers on the number line in the grade 5 number sense standard 1.5. Briefly, if we reflect the whole numbers on the number line with respect to 0, we obtain a new collection of numbers to the left of 0. The mirror image of 1 is -1, of 2 is -2, etc. This gives the placement of the negative numbers on the number line. An integer x to the left of another integer y is said to be *smaller than* y. Thus $-5 < -3$.

Arithmetic operations with integers are called for more comprehensively in the **grade 6** number standard

2.3 Solve addition, subtraction, multiplication, and division problems, including those arising in concrete situations that use positive and negative integers and combinations of these operations.

A parallel development takes place with fractions. The fraction $-\frac{2}{5}$, for example, is by definition the number satisfying $\frac{2}{5} + (-\frac{2}{5}) = 0$. On the number line, $-\frac{2}{5}$ is the mirror images of $\frac{2}{5}$ with respect to 0 and the mirror image of $\frac{8}{3}$ is $-\frac{8}{3}$, etc. The collection of positive and negative fractions together with 0 is called the *rational numbers*. The comparison among rational numbers is defined exactly as in the case of integers: $x < y$ if x is to the left of y. The call for placing rational numbers on the number line is in the previously quoted grade 6 number sense standard 1.1. Arithmetic operations with rational numbers are part of the **grade 6** number sense standard

1.2 Add, subtract, multiply and divide rational numbers (integers, fractions, and terminating decimals) and take positive rational numbers to whole-number powers.

We now indicate how to approach the arithmetic of integers. Note that, except for the slight complications in the notation, the discussion is essentially the same for rational numbers. Recall, for every integer x, we have $x + (-x) = 0$, by definition of $-x$. The simplest way to do arithmetic with the integers may be to take for granted that the integers can be added, subtracted, multiplied, and divided (by a nonzero integer), and that the associative, commutative, and distributive properties continue to hold. If students are familiar with these properties for whole numbers and fractions, they would be well-disposed towards such an extension.

On this basis, we can show why $8-5$ is the same as $8+(-5)$. Observe that there is only *one* integer x that can satisfy the equation $x+5=8$, namely the integer 3. But $\{8+(-5)\}+5 = 8+\{(-5)+5\} = 8+0 = 8$, so $8+(-5) = 3$, and consequently $8+(-5) = 8-5$ because $8-5$ is also 3. For the same reason, $x+(-y) = x-y$ if x, y are whole numbers and $x > y$. Reading the equality backwards, this says $x-y = x+(-y)$ if $x > y$. Thus we know how to do the subtraction $x-y$ if x and y are whole numbers and $x > y$. This fact leads us to *adopt as a the definition* that, for any two integers z, w, the subtraction $z-w$ means $z+(-w)$. So once we introduce the integers, *subtraction becomes addition in disguise.*

For multiplication, perhaps the most striking fact that needs confirmation is why is (for example) $(-2)\times(-5)$ equal to 2×5? The reason is very similar to the preceding argument: again observe that if we denote $(-2)\times 5$ by A, then there is only one rational number x that satisfies $x+A=0$, namely $-A$. We now show that both $(-2)\times(-5)$ and 2×5 solve $x+A=0$, and therefore must be equal since both will be equal to $-A$. To this end, we compute twice:

$$\begin{aligned}
\{(-2)\times(-5)\}+A &= \{(-2)\times(-5)\}+\{(-2)\times 5\} \\
&= (-2)\times\{(-5)+5\} \qquad \text{(distributive property)} \\
&= (-2)\times 0 \\
&= 0,
\end{aligned}$$

and also

$$\begin{aligned}
\{2\times 5\}+A &= \{2\times 5\}+\{(-2)\times 5\} \\
&= \{2+(-2)\}\times 5 \qquad \text{(distributive property)} \\
&= 0\times 5 \\
&= 0.
\end{aligned}$$

By a previous remark, this shows $(-2)\times(-5) = 2\times 5$.

The same reasoning justifies the general statement that if x and y are rational numbers, then $(-x)(-y) = xy$. Similarly, $(-x)y = -(xy)$ and $x(-y) = -(xy)$.

The meaning of the division of rational numbers is qualitatively *exactly the same* as the division of fractions, namely, if A, B, C are rational numbers and $A \neq 0$, then

$$B \div A = C \quad \text{has the same meaning as} \quad B = C\times A.$$

If B and A are fractions, i.e., positive rational numbers, then this definition offers nothing new. The interesting thing is to observe what this definition says when one or both of B and A is negative. Let us prove the assertion that "negative divided by negative is positive", as this is almost never explained in school mathematics.

Let us show, for example, why $\frac{-5}{-2} = \frac{5}{2}$. For the sake of clarity, denote $\frac{5}{2}$ by C. By the above definition, to show $\frac{-5}{-2} = C$ is the same as showing $-5 = C \times (-2)$. But by what we know about the multiplication of rational numbers, the latter is true because

$$C \times (-2) = \frac{5}{2} \times (-2) = -(\frac{5}{2} \times 2) = -5$$

Therefore $\frac{-5}{-2} = C$, i.e., $\frac{-5}{-2} = \frac{5}{2}$.

In general, for any rational numbers x, y with $y \neq 0$, the exact same reasoning shows that

$$\frac{-x}{-y} = \frac{x}{y}, \quad \frac{-x}{y} = -\frac{x}{y}, \quad \frac{x}{-y} = -\frac{x}{y}$$

In a sixth or seventh grade classroom, it may not be necessary, having proved some special cases such as $\frac{-5}{-2} = \frac{5}{2}$ or $\frac{15}{-3} = -5$, to give the proofs of these general assertions, but it would not be a good idea to decree $\frac{-5}{-2} = \frac{5}{2}$ or $\frac{15}{-3} = -5$ with no explanation whatsoever.

One consequence of this discussion is that *every rational number is a quotient* (i.e., division) *of integers.* For example, $-\frac{13}{19} = \frac{-13}{19}$. Recall that up to this point, rational numbers are just positive and negative fractions. The fact that rational numbers have an equivalent characterization as a quotient is extremely useful, as we shall see in the following discussion of ratios and rates.

Ratios, Rates, Percents and Proportion

The first serious applications of student's growing skills with numbers, and particularly fractions, appear in the area of ratios, proportions, and percents. These include constant velocity and multiple rate problems, determining the height of a vertical pole given the length of its shadow and the length of the shadow at the same time of a nearby pole of known height, and many other types of problems that are interesting to students and provide crucial foundations for more advanced mathematics.

Unfortunately, when the difficulties that students often have with fractions are combined with the confusion surrounding ratios, proportions, and percents, the majority of U.S. students develop severe difficulties at this point. On the other hand, students in most of the high achieving countries solve very sophisticated problems in these areas from about grade three on. To give some idea of the level of these problems in high achieving countries, here is a sixth grade problem from a Japanese exam:

> *The 132 meter long train travels at 87 kilometers per hour and the 118 meter long train travels at 93 kilometers per hour. Both trains are traveling in the same direction on parallel tracks. How many seconds does it take from the time the front of the locomotive for the faster train reaches the end of the slower train to the time that the end of the faster train reaches the front of the locomotive on the slower one?*[7]

It is important that all of these topics are seen by students as closely related, in fact aspects of just a very few basic concepts. Consequently, we present them is this way here.

Students first need to know what a ratio is: the *ratio* of the quantity A to the quantity B is the quotient (division) $\frac{A}{B}$. Thus ratio is an alternative language for talking about division. If A and B are quantities of different types, then the ratio $\frac{A}{B}$ retains the units of A (in the numerator) and B (in the denominator), and in this case, it is called a *rate*. We will discuss rates specifically later on.

Because the use of the word "ratio" in everyday language is imprecise, the same imprecise usage in a mathematical setting, if left unchecked, can lead to serious errors. We illustrate this with an example. Suppose there are two classrooms A and B in a building. In classroom A, there are 10 boys and 20 girls, and in classroom B, 7 boys and 14 girls. Then the {ratio of boys to girls in clasroom A} is $\frac{10}{20}$, and the {ratio of boys to girls in clasroom B} is $\frac{7}{14}$. Now suppose the students of the two classrooms come together for a joint discussion of a school event, then in the combined class, there are $10 + 7 = 17$ boys and $20 + 14 = 34$ girls, so that the {ratio of boys to girls in the class combining classroom A and classroom B} is $\frac{17}{34}$. However, it is tempting to *assume* that the ratio of boys to girls in the combined class is the sum of the two ratios, i.e., $\frac{10}{20} + \frac{7}{14} = \frac{1}{2} + \frac{1}{2}$. If so, then we are led to a contradiction, to the effect that $\frac{17}{34} = \frac{1}{2} + \frac{1}{2}$, i.e., $\frac{1}{2} = 1$. Let

[7] We discuss this problem at the end of the section on rates.

us spell out the details of this seeming contradiction. In words, the equality $\frac{17}{34} = \frac{1}{2} + \frac{1}{2}$ is the symbolic expression of the statement that

(the ratio of boys to girls *in the class combining classroom A and classroom B*)
= (the ratio of boys to girls *in clasroom A*) +
(the ratio of boys to girls *in clasroom B*).

The misconception that led to this equality is the assumption that these three ratios, which are completely different entities, are related by the operation of addition. Such a misconception is most likely the result of the conditoned reflex, sometimes instilled in the primary grades, which automatically associates the word "combine" and the word "and" with "addition." Students should be taught to unlearn this bad practice, and this example about ratio is a good starting point for this purpose.

Although ratios may first be encountered as quotients of whole numbers, eventually students will have to deal with situations where A and B are arbitrary numbers. In other words, students must in principle learn to divide , for example, the (length of the) diagonal of a square by the (length of the) side, or the circumference of a circle by its diameter. At this point, it is necessary to bring out one aspect of school mathematics that has been suppressed in the literature thus far, and it is that while we expect students to do such divisions, we do not teach them the meaning of these divisions. Instead, *we restrict the teaching to rational numbers, and let them extrapolate all the concepts and computations from rational numbers to arbitrary real numbers.* This is what is called *The Fundamental Assumption of School Mathematics.*[8] There is nothing wrong with this Assumption per se, because numbers which are not rational are too advanced for school mathematics. What should have been done, however, is to make this Assumption explicit, as it is done here, so that students understand that, in the following, although we only deal with rational numbers, they are expect to extrapolate everything to arbitrary numbers.

For the rest of the discussion, we will restrict A and B to be rational numbers, i.e., quotients of integers. Then ratio $\frac{A}{B}$ is a quotient of fractions. Thus, to be successful with ratios, students will need to be comfortable with rational arithmetic.[9] They should understand that if A and B are fractions, then $\frac{A}{B}$ is again a fraction; and specifically, that if $A = \frac{c}{d}$ and $B = \frac{e}{f}$, then

$$\frac{A}{B} = \frac{\frac{c}{d}}{\frac{e}{f}} = \frac{cf}{de}.$$

A quotient of two fractions is called a *complex fraction* in school mathematics. Although a quotient of fraction is just a fraction, as we have just seen , the importance of isolating the concept of a complex fraction lies in the fact that it is convenient to treat complex

[8] For a fuller discussion of this issue, see H. Wu, ***Chapter 2: Fractions*** (Draft), Section 11, ***http://math.berkeley.edu/~wu/***.

[9] Unfortunately, this topic is usually slighted in the upper elementary and middle school curriculum, resulting in glaring gaps in mathematical reasoning in school mathematics. An intervention program must fill in these gaps to promote learning.

fractions as if they were fractions. For example, to add

$$\frac{\frac{3}{5}}{\frac{2}{7}} + \frac{\frac{3}{4}}{\frac{11}{6}}$$

obviously we can first convert each complex fraction to an ordinary fraction and then add them as fractions, thus:

$$\frac{\frac{3}{5}}{\frac{2}{7}} + \frac{\frac{3}{4}}{\frac{11}{6}} = \frac{3 \times 7}{5 \times 2} + \frac{3 \times 6}{4 \times 11}$$

However, it is usually simpler to just treat each of the fractions $\frac{3}{5}$, $\frac{2}{7}$, $\frac{3}{4}$, and $\frac{11}{6}$ as if it were a whole number and then use the usual formula for adding fractions to get

$$\frac{\frac{3}{5}}{\frac{2}{7}} + \frac{\frac{3}{4}}{\frac{11}{6}} = \frac{(\frac{3}{5} \times \frac{11}{6}) + (\frac{2}{7} \times \frac{3}{4})}{\frac{2}{7} \times \frac{11}{6}}$$

The question is: is this legitimate? The answer is yes: in general, if A, B, C, D are rational numbers (i.e., positive or negative fractions, or 0), and $B \neq 0$, $D \neq 0$, etc., then

$$\frac{A}{B} + \frac{C}{D} = \frac{AD + BC}{BD}$$

The *cancellation property* also holds for complex fractions:

$$\frac{AB}{AC} = \frac{B}{C} \qquad \text{when } A \neq 0 \text{ and } C \neq 0$$

In addition, the cross-multiplication algorithm carries over to complex fractions,

$$\frac{A}{B} = \frac{C}{D} \qquad \text{is equivalent to} \qquad AD = BC.$$

and the product formula continues to hold for complex fractions:

$$\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}$$

These assertions about complex fractions are routine (if a bit tedious) to verify. Take the last one, for instance. Suppose $A = \frac{a}{a'}$, $B = \frac{b}{b'}$, $C = \frac{c}{c'}$, and $D = \frac{d}{d'}$. Then

$$\frac{A}{B} \times \frac{C}{D} = \frac{\frac{a}{a'}}{\frac{b}{b'}} \times \frac{\frac{c}{c'}}{\frac{d}{d'}} = \frac{ab'}{a'b} \times \frac{cd'}{c'd} = \frac{ab'cd'}{a'bc'd}$$

while

$$\frac{AC}{BD} = \frac{\frac{a}{a'} \times \frac{c}{c'}}{\frac{b}{b'} \times \frac{d}{d'}} = \frac{\frac{ac}{a'c'}}{\frac{bd}{b'd'}} = \frac{acb'd'}{a'c'bd}$$

Therefore $\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}$. These formulas for complex fractions will be used routinely in subsequent discussions.

Ratios appear as early as third grade in some state standards and in a number of the successful foreign programs. In the California Mathematics Content Standards, ratios first appear in the **grade 3** number sense standard:

2.7 Determine the unit cost when given the total cost and number of units.

As mentioned above, when taking ratios, the order of the two numbers matters, and the ratio of B to A, which is $\frac{B}{A}$, is the reciprocal of the ratio of A to B. Ratios are almost always fractions, but one usually contrives to make the ratio come out to be a whole number for third grade. Thus one may have a problem such as:

If 15 *items cost \$4.50, what is the unit cost?*

It should be made clear to students that the meaning of *unit cost* is the ratio of the total cost to the number of units. So, in this case, if 15 items cost \$4.50, which is 450 cents. Thus the unit cost is $450 \div 15 = 30$ cents. Of course, what is written as $450 \div 15$ for third graders is just the ratio $\frac{450}{15}$. This illustrates the fact that students' first contact with ratios can come before they have studied fractions.

Once students have learned about fractions, ratios should be revisited and the above procedure explained. For example, if 15 items cost 4.5 dollars, then the cost of a single item would be (the value of) one part when 450 cents (i.e., four and a half dollars) is divided into 15 equal parts. By the division interpretation of a fraction, the size of one part when 450 is divided into 15 equal parts is exactly $\frac{450}{15}$. So it is 30 cents. In general, if n items cost x dollars, then the cost of one item is $\frac{x}{n}$ dollars for exactly the same reason.

Ratios appear in grade six (number sense 1.2) and grade seven (algebra and functions 2.3) of the California Mathematics Content Standards; see the discussion below. By grade six, it would be reasonable for students to consider problems such as the following: if 2.5 pounds of beef costs \$22.25, what is the cost of beef per pound? The unit cost is the ratio $\frac{22.25}{2.5}$, and one recognizes this as a quotient of two fractions, $\frac{2225}{100}$ divided by $\frac{25}{10}$.

Dimensions and Unit Conversions

Ratios of units of measurement – feet to inches, meters to kilometers, kilometers to miles, ounces to gallons, hours to seconds – are involved whenever a quantity measured using one unit must be expressed in terms of a different unit. Such ratios are called *unit conversions*, and the units involved are called *dimensions*. Unit conversions usually first appear in the context of money. We will discuss unit conversions in more depth in the section on rates.

Dimensions sometimes appear by grade three in state standards, and sometimes appear even earlier in successful foreign programs. But it is worth noting that, as indicated in the discussion above, they should have already appeared in the study of money. Students should get considerable practice with determining ratios, and with unit conversions. But care should be taken that proportional relationships not be introduced until students are comfortable with the basic concepts of ratio and unit conversion.

> **1.4** Express simple unit conversions in symbolic form (e.g., __ inches = __ feet $\times 12$).

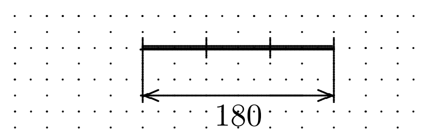
and in the **grade 3** measurement and geometry standard

> **1.4** Carry out simple unit conversions within a system of measurement (e.g., centimeters and meters, hours and minutes).

But it is worth noting that, as indicated in the discussion above, they have already appeared in the study of money. Students should get considerable practice with determining ratios, and with unit conversions. But care should be taken that proportional relationships not be introduced until students are comfortable with the basic concepts of ratio and unit conversion.

Many problems of the following type (taken from a Russian third grade textbook) are appropriate at this grade level:

1. *A train traveled for* 3 *hours and covered a total of* 180 *km. Each hour it traveled the same distance. How many kilometers did the train cover each hour?*



2. *In* 10 *minutes a plane flew* 150 *km. covering the same distance each minute. How many kilometers did it fly each minute?*

Students should see many such problems, and become accustomed to the reasoning used in solving them. For example, in problem 1, since the train travels the same distance each hour, the distance it travels in an hour (no matter what it is) when repeated three times will fill up 180 km. Therefore the distance it travels in an hour is $180 \div 3$, because the meaning of "dividing by 3" is finding that number m so that $3 \times m = 180$. So the answer is 60 km. Problem 2 is similar: the distance the plane flies in a minute, when repeated 10 times, would fill up all of 150 km. So this distance is $150 \div 10 = 15$ km.

Percents

To some, percent is a *function* that takes numbers to new numbers. However, this is difficult for students to grasp before taking algebra. Consequently, it is not productive to pursue this line of thought. We suggest instead the following definition. A *percent* is by definition a complex fraction whose denominator is 100. If the percent is $\frac{x}{100}$ for some fraction x, then it is customary to call it x *percent* and denote it by $x\%$. Thus percents are simply a special kind of complex fraction, in the same way that finite decimals are a special kind of fraction. Percent is used frequently for expressing ratios and proportional relationships, as with the interest on a loan for a given time period.

Percents first appear around **grade 5** in the number sense standard 1.0 and its substandard

1.2 Interpret percents as a part of a hundred; find decimal and percent equivalents for common fractions and explain why they represent the same value; compute a given percent of a whole number. They also appear in the fifth grade data analysis and probability standard

1.3 Use fractions and percentages to compare data sets of different sizes.

At the grade 5 level, one should limit the percents that appear in problems to be ordinary fractions, i.e., only those $x\%$ where x is a *whole number*. Thus we can give problems such as

What percent of 20 *is* 7*?*

We follow the definition to do this and any other problem about percents. So if $a\%$ of 20 is 7 for some fraction a, then by the interpretation of fraction multiplication in terms of the proposition "of", $\frac{a}{100} \times 20 = 7$. We want to know what the number a is. Cancelling 20 from 100 on the left side, we have $\frac{a}{5} = 7$, and if we multiply both sides again by 5, we get $a = 35$. Thus 35 percent of 20 is 7.

What percent is 7 *of* 20*?*

This problem is a linguistic trap, and the linguistic difficulty should be brought to the forefront. Explain clearly that *the meaning of the question* is to *express* $\frac{7}{20}$ *as a percent.* In a mathematics textbook, it is important to separate the linguistic difficulties from genuine mathematical difficulties.

That said, suppose $\frac{7}{20}$ is a percent, for some fraction a. Then $\frac{7}{20} = a\% = \frac{a}{100}$ (so far we have only used the definition of percent). Thus we have $\frac{7}{20} = \frac{a}{100}$, so that (after multiplying both sides by 20) $7 = \frac{a}{100} \times 20$. This shows that *this is the same problem as the preceding one!* So $a = 35$.

The following grade 6 number sense standards cover percent and ratio:

1.2 Interpret and use ratios in different contexts (e.g., batting averages, miles per hour) to show the relative sizes of two quantities, using appropriate notations (a/b, a to b, a:b).

1.4 Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.

By grade 6, students' skills with fractions are more advanced, and more complicated problems can be given. The following problems, which illustrate the preceding standards, are typical of those related to precents that tend to confuse students. However, when students recognize that *percents* are just complex fractions with 100 as denominator, and learn to make use of this precise definition, they should be able to sort out these problems with ease.

What is 211% *of 13?*

By the interpretation of the preposition "of" in terms of fraction multiplication, 211% of 13 equals 211% $\times$ 13. The answer is therefore $\frac{2743}{100}$, which is 27.43.

What percent of 125 *is* 24*? (What percent is* 24 *of* 125*?)*

Let us say, $x\%$ of 125 is 24, where x is a fraction. As in the problems above, we know from fraction multiplication that this means, precisely, that $x\% \times 125 = 24$. In other words, $\frac{x}{100} \times 125 = 24$, or, $\frac{x}{4} \times 5 = 24$. Multiplying both sides by $\frac{4}{5}$, we get $x = \frac{96}{5} = 19.2$. So the answer is 19.2% of 125 is 24. Note that 19.2% is a complex fraction.

32 *is what percent of* 27*?*

Let us say 32 is $x\%$ of 27, where x is a fraction. Then we are given that $32 = x\% \times 27$, i.e., $32 = \frac{x}{100} \times 27$. As usual, multiplying both sides by $\frac{100}{27}$ gives $x = \frac{3200}{27}$, or $x = 118\frac{14}{27}$. Therefore, 12 is $118\frac{14}{27}\%$ of 27.

25 *is* 12 *percent of which number?*

Let us say 25 is 12 percent of some number y. This means, literally, $25 = \frac{12}{100} \times y$. Multiplying both sides by $\frac{100}{12}$ gives $\frac{2500}{12} = y$, and therefore $y = 208\frac{1}{3}$.

The above sequence of problems are among the most feared in elementary and middle school as well as among the most confusing to students. But the one feature common to all the preceding solutions is how matter-of-fact they are *once the precise definition of a percent is followed literally.* One cannot emphasize *the importance of having a precise definition of percent as a complex fraction with 100 as denominator.* Without such a definition, the instruction on percent is commonly reduced to the drawing of pictures. Since picture-drawing is unlikely to provide answers to questions such as "25 is 12 percent of which number?", one can see why the instructional strategy of teaching percents *solely* by drawing pictures is inadequate. Drawing pictures can provide a few *illustrations* of the concept of percent, but they cannot replace teaching the general concept precisely. In the final analysis, the only way to help students in the intervention program is to teach them correct mathematics, clearly and precisely. Always emphasizing definitions in the teaching would be a good start.

Rates

As noted at the outset, rates are ratios with units attached. It is vital to respect the units when doing arithmetic with rates.

UNIT CONVERSION. In **grade 6**, the algebra and function standards

2.1 Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).

2.2 Demonstrate an understanding that rate is a measure of one quantity per unit value of another quantity.

address the issue of units explicitly. For example, we know that 1 yard = 3 feet. In terms of the number line, this means if the unit 1 is one *foot*, then we call the number 3 on this number line *1 yard*. The equality 1 yd. = 3 ft. is usually written also as $1 = 3\frac{\text{ft.}}{\text{yd.}}$. Observe that 1 sq. yd., written also as 1 yd.^{2}, is by definition the area of a square whose side has length 1 yd. Simliarly, 1 ft.^{2} is by definition the area of the square with a side of length equal to 1 ft. Since 1 yard is 3 feet, the square with a side of length 1 yard, to be called the *big square*, is *paved* by $3 \times 3 = 9$ squares each with a side of length equal to 1 ft., in the sense that these 9 identical squares fill up the big square and overlap each other at most on the edges. Therefore the area of the big square is 9 ft.^{2}, i.e., 1 yd.^{2} = 9 ft.^{2}. Sometimes this is written as $1 = 9\frac{\text{ft.}^2}{\text{yd.}^2}$.

How many feet are in 2.3 yd.? Recall that the meaning of 2.3 yd. is the number on the number line whose unit is 1 yd. So

$$2.3 \text{ yd.} = 2 \text{ yd.} + 0.3 \text{ yd.} = 2 \text{ yd.} + \frac{3}{10} \text{ yd.}$$

Now $\frac{3}{10}$ yd. is the length of 3 parts when 1 yard is divided into 10 parts of equal length (by definition of $\frac{3}{10}$), and therefore $\frac{3}{10}$ yd. $= \frac{3}{10} \times 1$ yd. $= \frac{3}{10} \times 3$ ft., using the interpretation of fraction multiplication. Of course 2 yd. = 2×3 ft.. Altogether,

$$2.3 \text{ yd.} = \{(2 \times 3) + (\frac{3}{10} \times 3)\} \text{ ft.} = (2.3 \times 3) \text{ ft.}$$

Exactly the same reasoning shows that

$$y \text{ yd.} = 3y \text{ ft.}$$

which is of course the same as

$$y \text{ ft.} = \frac{y}{3} \text{ yd.}$$

These are the so-called *conversion formulas* for yards and feet, and the seemingly over-pedantic explanation of these formulas above is meant to correct the usual presentation of mnemonic devices for such conversion using so-called *dimension analysis*. For example, since have agreed to write $1 = \frac{3 \text{ ft.}}{1 \text{ yd.}}$, the dimenson analysis would have

$$2.3 \text{ yd.} = (2.3 \times 1) \text{ yd.} = (2.3 \times \frac{3 \text{ ft.}}{1 \text{ yd.}}) \text{ yd.}$$

Upon "cancelling" the *yd.* from top and bottom, we get

$$2.3 \text{ yd.} = (2.3 \times 3) \text{ ft.}$$

The above pedantic explanation shows why this entirely mechanical process also yields the correct answer.

In like manner, 14.2 yd.^2 is

$$14.2 \times 9 \text{ ft.}^2 = 127.8 \text{ ft.}^2$$

Speed As another illustration, the common notion of speed has units miles/hour, or *mph*. Suppose a car travels (in whatever fashion) a total distance of 30,000 ft. in 5 minutes, then by definition, the *average speed* of the car *in this time interval* is the total distance traveled in the time interval divided by the length of the time interval, i.e., the average speed in this particular case is

$$\frac{30,000 \text{ ft.}}{5 \text{ min.}} = 6,000 \text{ ft./min.}$$

Suppose we want to express this average speed as mph. Then, because $1 \text{ m.} = 5280 \text{ ft.}$, the conversion formula in this case gives

$$30,000 \text{ ft.} = \frac{30,000}{5280} \text{ m.} = 5\frac{15}{22} \text{ m.}$$

Also $5 \text{ min.} = \frac{5}{60} \text{ hr.}$, so

$$\frac{30,000 \text{ ft.}}{5 \text{ min.}} = \frac{5\frac{15}{22} \text{ m.}}{\frac{5}{60} \text{ hr.}} = \frac{125}{22} \times \frac{60}{5} \text{ mph} = 68\frac{2}{11} \text{ mph}$$

Or, one could have done it by considering

$$1 \text{ ft./min.} = \frac{\frac{1}{5280}\text{m.}}{\frac{1}{60} \text{ hr.}} = \frac{1}{88} \text{ mph},$$

so that

$$60,000 \text{ ft./min.} = 60,000 \times \frac{1}{88} \text{ mph} = \frac{750}{11} \text{ mph} = 68\frac{2}{11} \text{ mph}$$

Again, once the mathematical reasoning is understood, one can see why the method of dimension analysis applied to this case is valid:

$$\frac{60,000 \text{ ft.}}{1 \text{ min.}} = \frac{60,000 \text{ ft.}}{1 \text{ min.}} \times \frac{1 \text{ m.}}{5280 \text{ ft.}} \times \frac{60 \text{ min.}}{1\text{hr.}}$$

which (after "cancelling" the *ft.* and *min.* from top and bottom) works out to be $68\frac{2}{11}$ mph again.

Motion at constant speed

The **grade 6** algebra and functions standard

2.3 Solve problems involving rates, average speed, distance and time

and the **grade 7** algebra and function standard

4.2 Solve multi-step problems involving rate, average speed, distance, and time or a direct variation.

implicitly bring up the general concept of motion *with constant speed*. This is the key example of rates in school mathematics. The meaning of this kind of motion is that distance traveled in any time interval is equal to the length of the time interval multiplied by a fixed number v. In other words, using the definition of division as an alternate way of writing multiplication, so that $\frac{b}{c} = a$ is the same as $b = ac$, we see that if speed is assumed to be constant, then the average speed over *any* time interval is always equal to v. This number v is called the (*constant*) *speed* of the motion, and the preceding statement is usually abbreviated to "distance is speed multiplied by time" or the self-explanatory formula

$$d = v \cdot t.$$

Emphasize to students that in this kind of motion, the number v remains the same no matter what the time interval (whose length is t) may be, and that the validity of this equality for any time interval is the definition of constant speed.

The concept of constant speed is a subtle one. Understanding why the formula $d = vt$ describes correctly our intuitive idea of "constant speed" requires careful reasoning that is not obvious to most students. The intuitive meaning of "constant speed" is that equal distances are traversed in equal times. We now sketch the multi-step argument that provides the link between this intuitive idea and the above formula. This argument is actually needed in several places in K-12 mathematics, but is usually ignored.[10]

For beginning students, let the time t be a whole number multiple of a fixed unit, e.g., hour, minute, etc. Then this formula is easy seen to be correct, as follows.

[10] We will give another example of this basic argument later in this article when we discuss proportion and scaling.

If the speed v is 55 miles an hour, then the distance traveled after 2 hours is $55+55 = 2\times 55$ miles, after 3 hours is $55+55+55 = 3\times 55$, after 4 hours is $55+55+55+55 = 4\times 55$ miles, etc. After n hours (with n a whole number), the distance traveled d is then $55+55+\cdots+55$ (n times) $= n\times 55 = 55n$. Since $v = 55$ and $t = n$, the formula $d = vt$ is correct in this case. Clearly the speed can be any whole number v instead of 55 and the reasoning remains unchanged. So the formula is correct in general when t is a whole number.

Next, we take up a special case where v is a fractional multiple of the unit time.

Let the distance traveled in unit time be v (miles, feet, etc.). Intuitively, the distance traveled in a time interval of length $\frac{1}{m}$, where m is a positive integer, *should be* $\frac{v}{m}$, and we can see this as follows. If $m = 3$, then whatever the distance traveled in $\frac{1}{3}$ of unit time, we know that 3 times that distance would be the distance traveled in unit time, i.e., v. Therefore the distance traveled in $\frac{1}{3}$ of unit time is $\frac{v}{3}$. If $m = 7$, then whatever the distance traveled in $\frac{1}{7}$ of unit time, we know that 7 times that distance would be the distance traveled in unit time, which is again v. So the distance traveled in $\frac{1}{7}$ of unit time must be $\frac{v}{7}$. Thus for a general positive integer m, the distance traveled in $\frac{1}{m}$ of unit time must be $\frac{v}{m}$. Thus with a motion at a constant speed of v (miles, feet, etc.) per unit time, and $d = \frac{v}{m}$, $t = \frac{1}{m}$, we have verified that the formula $d = vt$ again holds. Next, consider a fractional value of t such as $t = \frac{5}{3}$. Then the time interval consists of 5 sub-intervals of length $\frac{1}{3}$ each. Since we already know the distance traveled is $\frac{v}{3}$ in each of these sub-intervals, the total distance traveled in the interval of length $\frac{5}{3}$ is $\frac{v}{3}+\cdots+\frac{v}{3}$ (5 times), which is $\frac{5v}{3} = \frac{5}{3}\times v$. Thus with $t = \frac{5}{3}$, $d = \frac{5}{3}\times v$, the formula $d = vt$ again has been shown to be correct. The case of t equal to an arbitrary fraction $\frac{n}{m}$ is similar. Knowing the validity of the formula for rational values of t is sufficient for most purposes.

An informal presentation of this argument in the classroom, at least for some concrete values of t, would serve the useful purpose of making the formula more accessible to students. In any case, *the formula should be stated clearly as the correct description of motion at constant speed.*

As a simple illustration of the formula and the grade 6 standard 2.3 quoted above, consider the following problem

A passenger traveled 120 km by bus. The speed of the bus was a constant 45 *km per hour. How long did the passenger travel by bus?*

Thus $d = 120$ km and $v = 45$ km per hour. According to the formula, $120 = 45\times t$, where t is the total time duration of the passenger in the bus. Multiply both sides by $\frac{1}{45}$ and we get $\frac{120}{45} = t$ and so $t = 2\frac{30}{45} = 2\frac{2}{3}$ hours, or 2 hours and 40 minutes.

The other issue of substance is that many ratio problems are accessible without extensive symbolic computations. In particular, the fictitious skill of "setting up proportions."

should be avoided, as we proceed to demonstrate. Consider the problem:

> *A train travels with constant speed and gets from Town A to Town B in $4\frac{2}{3}$ hours. These two towns are 224 miles apart. At the same speed, how long would it take the train to cover 300 miles?*

We first do this the clumsy way. Let v be the (constant) speed of the train. Since the train travles 224 miles in $4\frac{2}{3}$ in $4\frac{2}{3}$ hours, we have (according to $d = vt$) $224 = v \times 4\frac{2}{3}$. Therefore

$$v = \frac{224}{4\frac{2}{3}}$$

Now suppose it takes the train t hours to travel 300 miles. Then also $300 = vt$ for the same number v. This gives

$$v = \frac{300}{t}$$

Comparing the two equations gives

$$\frac{224}{4\frac{2}{3}} = \frac{300}{t}$$

From this, the usual procedure (e.g., cross-multiply) yields $t = 6\frac{1}{4}$ hours.

The last displayed equation is one that students using the method of "setting up the correct proportions" would obtain by the procedure of "writing a proportion for the rate of distance divided by time". What is gained by the above reasoning over this standard procedure is that we derived this equation strictly using the precise definition of constant speed, because the equality of $\frac{224}{4\frac{2}{3}}$ and $\frac{300}{t}$ is due to the fact that the average speeds over the respective time intervals stay the same.

This kind of detailed explanation is needed for beginners as it puts them on a solid mathematical footing. After they have gotten used to this reasoning, then they can be exposed to the normal way of doing such problems, such as the following.

> From the data, the speed is $\frac{224}{4\frac{2}{3}}$ miles per hour, or 48 miles per hour. Therefore to travel 300 miles, it would take $\frac{300}{48} = 6\frac{1}{4}$ hours, or 6 hours and 15 minutes.
>
> Another example of this kind is:
>
> *I spent 36 dollars to purchase 9 cans of Peefle. How much do I have to spend to purchase 16 cans?*

The price per can is $\frac{36}{9} = 4$ dollars, so to buy 16 cans, I would have to pay $16 \times 4 = 64$ dollars.

We emphasize once again that no "setting up a proportion" is necessary.

We are now in a position to consider the constant speed problem in the introduction:

> *The 132 meter long train travels at 87 kilometers per hour and the 118 meter long train travels at 93 kilometers per hour. Both trains are traveling in the same direction on parallel tracks. How many seconds does it take from the time the front of the locomotive for the faster train reaches the end of the slower train to the time that the end of the faster train reaches the front of the locomotive on the slower one?*

In this case what matters is that the faster train must travel the distance equal to the sum of the lengths of the two trains *further* than the other train in the given time interval. This extra distance is $132 + 118 = 250$ meters or $\frac{1}{4}$ km. We now apply the defining property of constant speed motion, namely, $d = vt$. Suppose it take t hours for the faster train to travel $\frac{1}{4}$ km. Then in these t hours, the faster train travels $93t$ km and the slow train $87t$ km, so that the faster train has traveled $(93t - 87t)$ km more than the slower train at the end of the t $93t - 87t = \frac{1}{4}$ or $6t = \frac{1}{4}$ and $t = \frac{1}{24}$ hours which is $2\frac{1}{2}$ minutes or 150 seconds. hours. But this difference is equal to $\frac{1}{4}$ km. Therefore,

Finally, we point out that the **grade 6** statistics, data analysis and probability standard

> **3.3** Represent probabilities as ratios, proportions, decimals between 0 and 1, and percentages between 0 and 100 and verify that the probabilities computed are reasonable; know that if P is the probability of an event, 1-P is the probability of an event not occurring.

pulls together almost everything we have discussed thus far.

Percentage Increase and Decrease Problems

These are represented by the **grade 7** number sense standard

> **1.6** Calculate the percentage of increases and decreases of a quantity.

Percentage increase, percentage decrease problems and related problems can appear to be quite tricky and somewhat non-intuitive at the beginning. The fact that an $x\%$ increase, followed by the same percentage decrease, is *not* going to get back to where you started, is extremely mystifying to students. It is especially important at this point to be clear about the terminology, emphasize clear definitions, and give illustrative examples. Part of the problem is that when this topic appears, students generally lack the algebraic skills to go through the argument. A second difficulty is that students often lack skill and practice in dividing fractions.

Students should convince themselves, via direct calculation, that a 20% increase, followed by a 20% decrease *does not* get one back to where one started. For example, suppose a coat costs \$200. A 20% *increase in price* means precisely that the new price is what one gets by adding to the original price (\$200) an amount equal to 20% of this price (\$200). The new price is therefore $200 + (20\% \times 200) = 200 + 40 = 240$ dollars. Now at this price of \$240, a 20% *decrease in price* means precisely that the new price is what one gets after subtracting from this price an amount equal to 20% of the price (\$240). Thus the new price is now $240 - (20\% \times 240) = 240 - 48 = 192$ dollars. Notice that 192 is less than 200, the original price of the coat, and this is because 20% of a higher price (\$240) is bigger than 20% of the lower original price (\$200) By looking at further examples, say 10% and 15%, students should come to understand that an y percent increase followed by the same percent decrease, for any positive rational number y, will always give *less* than what they started with, and they should be able to give a heuristic argument to justify this.

At the seventh grade level we can explain this phenomenon *precisely* by drawing on ideas from the section on Symbolic Manipulations in the chapter on *The Core Processes of Mathematics*, as follows.

> A $y\%$ increase in price over a given price of P dollars means that the new price is $P + (y\% \times P) = (1 + y\%)P$, by the distributive property. A $y\%$ decrease in price of a given price Q means, as defined above, that the new price is $Q - (y\% \times Q)$, which equals $(1 - y\%)Q$, by the distributive property again. So if we follow a $y\%$ increase by a $y\%$ decrease of P dollars, the new price is a $y\%$ decrease of $(1 + y\%)P$ dollars, and is therefore
>
> $$(1 - y\%)\{(1 + y\%)P\} = \{(1 - y\%)(1 + y\%)\}P = (1 - (y\%)^2)P$$
>
> dollars, where we have made use of the standard identity $(a + b)(a - b) = a^2 - b^2$. Normally, $0 < y < 100$, so $0 < (y\%) < 1$, and hence also $0 < (y\%)^2 < 1$. It follows that
>
> $$0 < (1 - (y\%)^2)P < P$$
>
> This is the reason that, if $0 < y < 100$, a $y\%$ increase in price followed by the same $y\%$ decrease always result in a price less than the original one.

Of course the conclusion is the same if we begin with a $y\%$ decrease in price (instead of increase), to be followed by a $y\%$ increase in price. Again concrete examples should be given.

Related to the preceding considerations is the **grade 7** standard

1.7 Solve problems that involve discounts, markups, commissions, and profit and compute simple and compound interest.

Compound interest is likely to be too involved at this stage for our needs and serious thought should be given as to whether it should be discussed or not.

Proportions

A *proportion* is an equality of ratios: four *ordered* numbers A, B, C and D define a proportion if

$$\frac{A}{B} = \frac{C}{D}.$$

We have already come across an example of a proportion in the problem of a train going from Town A to Town B in the preceding section. We now approach this topic more systematically, making implicit use of the idea of a linear function in order to add clarity to the discussion. Though not strictly necessary, we nevertheless recommend a revisit of this section after students have learn about linear functions (in the volume on *Core Processes of Mathematics*).

Proportions are a major source of difficulty in the K-8 curriculum. Part of the trouble may come from the fact that they involve division of fractions, which is one of the parts of arithmetic least understood by students. Beyond that, however, lies another difficulty. Just as ratios are a way of talking about division without using the term, proportions are a way of talking about linear functions without really mentioning them. That is, very often when one is talking about proportions, one has in mind two quantities which can take different values, but the ratio of the two quantities is a fixed number (or rate). If the quantities are A and B, then we are given that $\frac{A}{B} = k$, k is a fixed number, called the *constant of proportionality*. Multiply both sides of this equality by B and we get $A = kB$. We have come across this kind of equation before, e.g., A is the distance traveled in a time interval of length B, so that k would be the (constant) speed of this motion. Other examples of this kind will show up below. When two quantities A and B are related by $A = kB$ for some fixed number k, we say A and B are *in direct proportion*.

There is a large class of examples in daily life of two quantities in direct proportion. If A is the cost of n identical dresses and $B = n$, then k in this case would be the cost of one dress; the constant of proportionality is then called the *unit cost*. If A is the total number of legs in n identical living creatures (e.g., a crab) and again $B = n$; then k is the number of legs in one such creature. If A is the total number of wheels in n tricycles and $B = n$; then k is 3. And so on. Such examples share a common characteristic: B can only be a whole number, so that it is possible to examine the two quantities A and B one by one. These examples are relatively easy to understand. The more difficult case is where B can take on arbitrary (positive) values, such as in motion or water flow problems. We will concentrate our discussion on the more difficult case.

Proportions arise from the equation $A = kB$ by taking two values A_1 and A_2 of A and two corresponding values B_1 and B_2 for B. The equations $A_1 = kB_1$ and $A_2 = kB_2$ imply that

$$\frac{A_1}{B_1} = k = \frac{A_2}{B_2}.$$

This just says that A_1, B_1, A_2 and B_2 form a proportion. The fact that proportions are treated in the curriculum before it is commonly thought appropriate to discuss linear functions and linear equations means that the heart of what is going on cannot be directly confronted. The difficulty is compounded by discussions in textbooks of solving problems about proportions by "setting up a proportion", such as $\frac{A_1}{B_1} = \frac{A_2}{B_2}$ above, without being able to explain what it takes to "set it up" or why the procedure is correct. However, if students can learn about proportions in a mathematically correct way, as we are suggesting here, this knowledge can be a tremendous aid when they come to an algebra course.

Basic proportional relationships appear as early as **grade 3**, in the standard

2.2 Extend and recognize a linear pattern by its rules (e.g., the number of legs on a given number of horses may be calculated by counting by 4s or by multiplying the number of horses by 4).

However, standards of this kind should just be regarded as preparation for proportions. The first major standard that addresses proportions is the **grade 6** number sense standard

1.3 Use proportions to solve problems (e.g., determine the value of N if $\frac{4}{7} = \frac{N}{21}$, find the length of a side of a polygon similar to a known polygon). Use cross multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

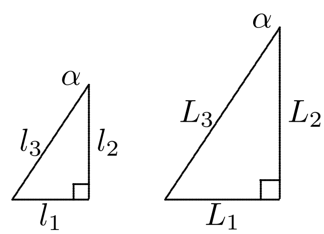
Consider the following problem:

If a building at 5:00 PM has a shadow that is 75 feet long, while, at the same time, a vertical pole that is 6 feet long makes a shadow that is 11 feet long, then how high is the building?

What is implicitly assumed here, and what *must* be made explicit when teaching this material, is the fact that

$$\frac{\text{height of object}}{\text{length of its shadow}} = \text{constant}$$

This comes from considerations of similar triangles which will be taken up in a high school course. However, since such problems already occur in much lower grades, there should be some discussion of the concept of similarity already in grade 6. Briefly, what students need to understand is first that the situation above gives two triangles with pairs of equal corresponding angles but different lengths for the corresponding sides, as shown below. The left represents an object and its shadow and the right is a second object with its shadows, and in each case, the hypotenuse represents the sunlight. On the left, l_1 is the length of the shadow of the object of height l_2, and on the right, L_1 is the length of the shadow of the object of height L_2.



From the theory of similar triangles, we know that these triangles are similar and, therefore, the ratio $\frac{l_i}{L_i}$ is a constant k independent of i, so $L_i = kl_i$, for $i = 1, 2, 3$. Thus,

$$\frac{L_2}{L_1} = \frac{l_2}{l_1}$$

because both sides are equal to $\frac{kl_2}{kl_1}$. This equality is exactly the above statement about the ratio of the height of each object to the length of its shadow.

Now we can do the problem. Let A be the height of the object and let B be the length of its shadow. Then we have $\frac{A}{B} = K$, where K is a fixed number. If the object is the building, then A is the height of the building (i.e., what we are trying to find out) and B is 75 ft. by the given data, and we know $\frac{A}{75}$ is equal to this constant K. On the other hand, if the object is the pole, then the ratio $\frac{6}{11}$ is also equal to K. Therefore the two numbers $\frac{A}{75}$ and $\frac{6}{11}$, being both equal to K, are equal, i.e.,

$$\frac{A}{75} = \frac{6}{11}$$

The solution of the problem is now straightforward: multiply both sides by 75 to get $A = 75 \times \frac{6}{11} = 40\frac{10}{11}$ feet. (Or cross-multiply and get $11A = 6 \times 75$, so $A = \frac{450}{11} = 40\frac{10}{11}$ feet.)

It would not do to teach the solution of this and similar problems by asking students to learn the "skill of setting up the correct proportion" of $\frac{A}{75} = \frac{6}{11}$ but without making explicit the fact that $\frac{A}{B}$ is a fixed number k. We repeat: "Setting up a proportion", in the sense it is currently understood in textbooks, is emphatically *not* a mathematical concept.

As we have seen, problems of motion at constant speed provide a good illustration of proportions. From our discussion of such problems we know that the distance d traveled in a given time t at constant speed v is determined by the equation $\frac{d}{t} = v$; that is, the ratio of distance to the length of the time interval of travel is equal to the constant v no matter what time interval is chosen. An entirely analogous discussion can be given for work or water flow, two of the staple topics concerning rates in school mathematics. For the case of water flowing out of a faucet, we say the water *flows at a constant rate* if the amount of water, say w gallons, coming out of the faucet during a time interval of t minutes always satisfies $\frac{w}{t}$ = a fixed number, say r, no matter what time interval is chosen. Note that r is called the *rate of the water flow*, and it has units of $\frac{\text{gallon}}{\text{minute}}$. The intuitive reason why the equation $\frac{w}{t} = r$ correctly translates the concept of "constant rate of water flow" is identical to the reason given above for "constant speed."

Here is an illustrative problem:

> *Water is coming out of a faucet at a constant rate. If it takes 3 minutes to fill up a container with a capacity of 15.5 gallons, how long will it take for it to fill a tub of 25 gallons?*

Suppose it takes x minutes to fill the tub, then both $\frac{25}{x}$ and $\frac{15.5}{3}$ are equal to r, and therefore

$$\frac{25}{x} = \frac{15.5}{3}.$$

Notice that both are quotients of fractions because 15.5 is really $\frac{155}{10}$ and x is expected to be a fraction and not a whole number. Nevertheless, knowing that the cross-multiplication algorithm holds also for complex fractions, we cross-multiply to get $15.5x = 75$ so $x = \frac{75}{15.5} = 4\frac{26}{31}$ minutes.

As before, the displayed equation above was not "set up as a proportion." Rather, it is a statement about the constancy of the rate of water flow. At the risk of excessive repetition, we want students to understand that *all claims about two things being proportional amount to saying that certain quotients are constants.*

Direct and Inverse Variation

The final topics that we mention are

(1) Direct variation: Two quantities that change in such a way that their ratio remains constant are said to be be *directly proportional* or to *vary directly*. Thus this is simply further vocabulary for proportion

(2) Inverse variation: Two quantities that change in such a way that their product is constant are said to vary *inversely*.

As we have already mentioned in the section on proportions, the term *direct variation* is the school math vocabulary for saying that one quantity is a linear function (without constant term!) of another quantity. It applies to constant speed motion: $\frac{d}{t} = v$, or $d = vt$; to constant flow problems: $\frac{w}{t} = r$, or $w = rt$; to cost-quantity relationships: $C = pN$ (where N is the number purchased of some item, C is the total cost of the order, and p is the unit price); and to many other interesting and important situations.

Inverse variation occurs in problems like the following taken from a sixth grade Russian text:

Three workers can perform a certain task in six hours. How much time will two workers need to perform this same task if all workers work at the same speed?

Let us first explain in what way this is a problem about inverse variations. Since all workers work at the same speed, the amount of work done by one worker in each hour, to be denoted by T, is a constant independent of the worker. The amount of work done by one worker in h hours is then hT, and the total amount of work done by n workers in each hour is therefore nhT. Now if it takes n workers to perform a given task in h hours, then the amount of work done by n workers in h hours is a fixed constant K, which is the amount of work required to finish the task. So $nhT =$ a constant K. Since T is also a constant, so is $\frac{K}{T}$. Thus

$$nh = \text{ a constant } \frac{K}{T}$$

We see that n *varies inversely with* h. The given data is that if $n = 3$, then $h = 6$. If $n = 2$, let the numbers of hours it takes two workers to perform this task be x. Then $3 \times 6 = 2x$, both being equal to $\frac{K}{T}$. It follows that $x = 9$, i.e., it will take two workers 9 hours to perform this task.

Both direct and inverse variation are captured by the same equation, namely $c = ab$. The difference between them is which term stays constant. If a remains fixed, then c varies directly with b. On the other hand, c remains fixed, then a and b vary inversely. Thus, in the Russian problem above, we had a fixed job, and a variable source of labor, with consequent variation in the time needed. Time and amount of labor for a given job are inversely related. On the other hand, the same equation might be used to calculate, with fixed labor input, the amount of work that can be done in a given amount of time. Then one would be in a direct variation situation: amount of work done would vary directly with time allotted.

At present, there is substantial confusion about these concepts in schools. To illustrate the problem we offer this example of an actual fifth grade state standard referring to direct and inverse variation in which almost every aspect of the mathematics is incorrect:

Identify and describe relationships between two quantities that vary directly (e.g., length of a square and its area), and inversely (e.g., number of children to the size of a piece of pizza).

(Of course, while the difficulties with the first example are severe, the difficulties with the second are easily fixed.)

It should be realized that problems like the "three-worker" question above involve important techniques and problem solving skills. But the tension between vocabulary and the core concepts that the vocabulary is supposed to summarize must be resolved before instruction can improve. In the case of direct and inverse variation, it would seem better to entirely suppress the vocabulary and simply discuss the equation

$$c = ab,$$

particularly the special cases where a is a constant or c is constant.

The Core Processes of Mathematics

Introduction

Mathematics tends to be taught as sequences of rigid rules in K - 8 instruction in this country. A typical example is the break-up of solving linear equations into the following four types:

1. one-step, $x + a = b$, $ax = b$, where only one basic operation is needed to replace the equation by one of the form $x = c$ (called isolating the variable),

2. two-step, $ax + b = c$, $ax + bx = c$, where two basic operations are needed, and

3. three-step, $ax = cx + d$,

4. four-step, $ax + b = cx + d$.

Traditionally, each type of equation is taught separately, and typical texts never indicate the core insight that in all cases there is a common objective, *isolating the variable* – the variable being the (as-yet-unknown) number – by using the standard properties of number operations (distributive property, associative property, etc.). Consequently, general skills with symbolic manipulation are not developed. Almost uniformly, American students are taught to multiply binomials of the form $(a+b)(x+y)$ by using the memonic FOIL (first, outer, inner, last). That is the *only* way they are taught, and the procedure itself is called "foiling." Students become adept at foiling, but don't understand that this is just three applications of the distributive law. As a result they are unable to generalize foiling and expand an expression like $(a + b)(x + y + z)$.

This is unfortunate, since it is precisely in this area of symbolic manipulation that the power of mathematics in general, and algebra in particular, comes to the forefront. If students cannot handle these processes, they will not be able to use mathematics in effective ways. Yet, extremely few students manage to become proficient in symbolic manipulation on their own, and the need for remediation in this area is widespread.

Symbols

It often happens that we want to determine a quantity that satisfies a number of conditions. A method for doing this which has been extremely fruitful is to let a symbol stand for the unknown quantity, and to express the conditions via equations involving the symbol. In many situations these equations can then be manipulated using a small set of principles to find the value (or values) of the quantity.

Symbolic manipulation begins with the use of symbols. The standards in the California Mathematics Content Standards that directly address this issue are the following algebra and functions standards:

Grade 3

1.0 Students select appropriate symbols, operations, and properties to represent, describe, simplify, and solve simple number relationships.

Grade 4

1.1 Use letters, boxes, or other symbols to stand for any number in simple expressions or equations (e.g., demonstrate an understanding and the use of the concept of a variable).

Grade 5

1.2 Use a letter to represent an unknown number; write and evaluate simple algebraic expressions in one variable by substitution.

Grade 6

1.2 Write and evaluate an algebraic expression for a given situation, using up to three variables.

Grade 7

1.1 Use variables and appropriate operations to write an expression, an equation, an inequality, or a system of equations or inequalities that represents a verbal description (e.g., three less than a number, half as large as area A).

It will be noticed that all these standards are vaguely similar, and differ from each other only in the level of sophistication with which the symbols are put to use. For the moment, we shall ignore the references to "evaluate" expression and concentrate on the *use* of symbols instead.

At the outset, a symbol or *variable* (such as x or a) is just a number, in exactly the same way that the pronoun "it" in the question "Does it have five letters?" is just a word (in a guessing game of "What is my word?") This x or a may be unknown for the time being, but there is no doubt about the fact that x or a is a number and therefore can be added, subtracted, multiplied, and divided. For example, it makes perfect sense to write $3 + x$ or $5x$ as soon as we specify that x is a number.[11] Third graders should be taught to use letters to represent numbers instead of using blanks all the time, i.e., use $3 + x = 5$ sometimes instead of $3 + \_ = 5$. Students should be taught the good habit of *always* specifying what a symbol means instead of just writing something like $27 - x = 14$ without saying what x is. It could be, for instance, "Find the number x so that $27 - x = 14$." Or, "What number x would satisfy $27 - x = 14$?" But a common mistake one finds in many textbooks is to just thrust something like $27 - x = 14$ on

[11] $5x$ means "5 times x", but this symbolism should only be used starting at the fourth grade level.

students with no explanation. An error of another kind is to legitimize the writing of things like $27 - x = 14$ without any explanation of what x is by introducing the concept of an *open sentence*, thereby making students learn an unnecessary concept. We can avoid both types of issues by helping students learn the good habit of specifying each symbol they use from the beginning.

Third grade is a good starting point for students to learn the use of symbols, but because the main thrust of these standards is for students to learn to use symbols fluently, the exact grade level of each standard is not a primary concern. What is important is that the technical sophistication of the exercises they are asked to do increases gradually. And it must be said that *exercises are the heart of these standards*; students must achieve fluency in the use of symbols through practice. Here are some sample suggestions.

Third grade level problems

1 Write a number sentence for a number y so that 21 minus y is equal to 7.

2 Write a number sentence to express: 21 cars are parked and y cars drive off; only 7 cars remain.

3 Express in symbolic form: a number x when added to 21 is larger than 45.

4 There were 112 birch trees and x aspens in a forest. Explain what the following expressions denote: $112 + x$; $112 - x$; $x - 112$.

Fourth grade level problems

1 Paulo reads a number of pages of a 145-page book, then he read 43 pages more so that only 38 pages remain. If p is the number of pages Paulo read the first time, write an equation using p to express the above information.

2 I have a number x and when I first subtract 18 and then 9 from it, I get 7. Write an equation to express this information. What is x?

3 18 meters of wire was cut from a reel, and then another 9 meters of wire was cut. 7 meters of wire then remained on the reel. If there are w meters of wire on the reel originally, write an equation that expresses the preceding information. What is w?

4 Make up an equation for each problem and solve it. (a) Some number is 20 greater than 15. Find the number. (b) 27 is 13 less than some number. What is the number?

Fifth grade level problems

1 Starting with a number x, Eva multiplies it by 5 and then subtract 9 from it to get a new number. If x is 3, what is the new number? If x is 10? If x is 14? If $x = 19$?

2 Suppose that y is a number so that when 5 is subtracted from 3 times y, we get 31. Write down an equation for y. What is y?

3 Let x be the number of oranges in a basket. Write a story about the equation $x - 5 - 11 = \frac{1}{2}x$.

Sixth grade level problems

1 Johnny has three siblings, two brothers and a sister. His sister is half the age of his older brother, and three fourths the age of his younger brother. Johnny's older brother is four years older than Johnny, and his younger brother is two years younger than Johnny. Let J be Johnny's age, A the age of Johnny's older brother, and B the age of his younger brother. Express the above information in terms of only J, A, and B.

2 Helena bought two books. The total cost is 49 dollars, and the difference of the squares of the prices is 735. If the prices are x and y dollars, express the above information in terms of x and y.

Seventh grade level problems

1 We look for two whole numbers so that the larger exceeds the the smaller by at least 10, but that the cube of the smaller exceeds the square of the larger number by at least 500. If the larger number is x and the smaller number is y, write expressions relating x to y.

2 Erin has 10 dollars and she wants to buy as many of her two favorite pastries as possible. She finds that she can buy either 10 of one and 9 of the other, or 13 of one and 6 of the other, and in both cases she will not have enough money left over to buy more of either pastry. If the prices of the pastries are x dollars and y dollars, respectively, write down the inequalities satisfied by x and y.

Mathematical Preliminaries to Symbolic Manipulation

Observe that the main emphasis of the above standards (and the included examples) is on the *use* of symbols. If a solution is asked for in any of the examples, it can be obtained by simple arithmetic or even mental math. The next stage of the development of students' command of symbolic language is the acquisition of symbolic manipulative skills to solve the equations or inequalities. Before taking up the latter, we would like to point out a very important aspect of the use of symbols: the symbolic representations of the associative, commutative, and distributive properties, as given in the following algebra and functions standards

Grade 2

1.1 Use the commutative and associative rules [of addition] to simplify mental calculations and to check results.

Grade 3

1.5 Recognize and use the commutative and associative properties of multiplication (e.g., if $5 \times 7 = 35$, then what is 7×5? and if $5 \times 7 \times 3 = 105$, then what is $7 \times 3 \times 5$?)

Grade 5

1.3 Know and use the distributive property in equations and expressions with variables.

Students should be taught how to express these rules in symbolic form no later than fourth grade. For example, the associative property of addition can be rephrased symbolically as follows: *for any numbers x, y, z,* it is always true that

$$x + (y + z) = (x + y) + z.$$

This symbolic representation can be motivated by pointing out that concrete statements about the associativity of addition in terms of explicit numbers are inadequate, e.g., we can go on listing equalities like these: $(2 + 3) + 13 = 2 + (3 + 13)$ (both sides equal 18), $(17 + 5) + 43 = 17 + (5 + 43)$ (both sides equal 65), $(8 + 613) + 11 = 8 + (613 + 11)$ (both sides equal 632), etc. But no matter how many such equalities are written down, they still do not completely convey the fact that associativity works for *any* three numbers. For this reason, the preceding symbolic representation becomes a necessity if we want to express the associative property *precisely* and correctly. We reiterate that this property says for *any three numbers*, adding the first to the sum of the second and third is the same as adding the sum of the first two to the third.

As was mentioned earlier, every symbolic expression must be accompanied by a statement of what the symbols mean. In the case of associativity for addition, the statement *for any numbers x, y, z* should be explained with some care to students because this will be their first encounter with the concept of **generality**. They have only written out symbolic statements for specific numbers, e.g., $(2 + 3) + 13$, or the number x so that $x - 17 = 8$ to this point. By contrast, the associativity of addition does not make a statement about one or several triples of numbers, but about *all* triples x, y, z. When we begin to assert that something is true for all numbers, we are introducing students to the heart of algebra and the core of mathematical reasoning. This is why we should be careful here.

The symbolic representations of the commutativity of addition and multiplication and the associativity of multiplication should be similarly presented to students no later than the fourth grade level, and distributivity no later than the fifth grade level.

Students can, and should be shown the power of generality even at this stage. For example, the calculation $(19,805+80,195)+2,867,904 = 100,000+2,867,904 = 2,967,904$, is simple enough to be done by mental math. What we want to demonstrate is that, if one knows the commutativity and associativity of addition, then the rather formidable addition problem of

$$(80,195 + 2,867,904) + 19,805$$

can also be done by mental math. This is because one recognizes that the numbers $19,805$ and $80,195$ "fit together in terms of addition", so that one mentally maneuvers to bring them together using commutativity and associativity:

$$\begin{aligned}
(80,195 + 2,867,904) + 19,805 &= 19,805 + (80,195 + 2,867,904) \\
&= (19,805 + 80,195) + 2,867,904 \\
&= 100,000 + 2,867,904 \\
&= 2,967,904
\end{aligned}$$

It is necessary to point out to students that, impressive as this example may seem, it is a rather trivial justification of why they should learn about these general properties. The real justification comes from applying them to *unknown* numbers x, y, and z when we try to solve equations. This is what we do below.

Another noteworthy feature of the symbolic representation of these properties is a reinforcement of the comment made earlier about the need to form the good habit of *always* giving meaning to the symbols. Consider the following two statements:

For any numbers x, y, z, it is true that $x + (y + z) = (x + y) + z$.

and

There are some numbers x, y, z for which it is true that $x+(y+z) = (x+y)+z$.

Notice that although both statements contain the same equation, $x+(y+z) = (x+y)+z$, they mean completely different things simply because the meaning we give to these symbols are different in the two statements. We repeat: *always make sure students explain the meaning of the symbols they use.*

Finally, students should be aware that the associative, commutative, and distributive properties remain valid no matter how many numbers are involved. For example, the validity of the associative property for the addition of four numbers a, b, c, d, states that all possible ways of adding these four numbers are equal:

$$\begin{aligned}(a+b)+(c+d) &= ((a+b)+c)+d \\ &= a+((b+c)+d) \\ &= (a+(b+c))+d \\ &= a+(b+(c+d))\end{aligned}$$

The equality

$$(a+b)+(c+d) = ((a+b)+c)+d,$$

can be seen to be the application of the original associative property to the three numbers: $(a+b)$, c, and d. The equality of the others is similar. While reasoning of this kind is (admittedly) boring, it must be recognized that the more general form of the associative property is what makes it possible to write $a+b+c+d$ *without* the use of parentheses (because the parentheses don't matter). To push this line of reasoning one step further, students should at least see why the distributive property for any three numbers, i.e., $a(b+c) = ab+ac$, would imply

$$a(b+c+d+e) = ab+ac+ad+ae.$$

This is because

$$a(b+c+d+e) = a\{(b+c)+(d+e)\} = a(b+c)+a(d+e) = (ab+ae)+(ad+ae),$$

and the last is equal to $ab+ac+ad+ae$ because of the associative property for four numbers.

One should not over-emphasize this kind of generality to students in grades 6 or 7, but to the extent that they will be seeing expressions such as $12+87-2+66+54$ or $44 \times 17 \times (-23) \times 91$ often (polynomials of high degree, for example), these facts should be explained to them.

Evaluating Expressions

The **grade 7** algebra and functions standard 1.2 introduces a new dimension to students' growing proficiency with symbolic expressions

> **1.2** Use the correct order of operations to evaluate algebraic expressions such as $3(2x+5)^2$.

Two comments are in order. First, we have mentioned previously that *order of operations* should be de-emphasized. It is now possible to be more precise at this point: with symbolic expressions of the type $5x^2 + 7(2x-1)^2 - 2x^3$ for a number x, the notation itself almost suggests the correct order of doing the operations: first do the exponents (i.e., x^2, $(2x-1)^2$, and x^3), then do the multiplications (i.e., $5x^2$, $7(2x-1)^2$, and $2x^3$), and finally the additions. Because subtraction is just a different way of writing addition (e.g., $-2x^3$ is just $+(-[2x^3])$), and because division is expressed in terms of fraction multiplication (e.g., $2x \div 5$ is really $\frac{1}{5}(2x)$ in disguise), [12] this rule is sufficiently comprehensive. Anything more complicated should use parentheses for the sake of clarity. For example, *monstrosities such as* $2 \times \frac{5}{7} \div 13 \div 3 + 15 \div 16$ *should be avoided at all costs.* Second, we strongly suggest the use of numbers which are not integers for the exercises in connection with the evaluation of algebraic expressions, i.e., the seventh grade standards 1.2 and 1.3 above, and the algebra and functions standards 1.2 in grade 5 and 1.2 in grade 6 (which were quoted at the beginning of this section).

Here are some sample problems for the evaluation of expressions. In each case, an expression involving a number x and sometimes other numbers y and z are given and students are asked to evaluate the expression for the value of x (and y and z) specified in each case. Note that some of the exercises are set up in such a way that if the distributive property or some general property of rational numbers is properly applied, then they become extremely simple.

Grade 5

1. $8 \times (x \div 7) - (x - 2)$. $\quad x = 84$.
2. $(3x + 5) - 4 \times (7 - x)$. $\quad x = 6$.
3. $(4 + (2x - (9 - x)))$. $\quad x = 5$.
4. $\frac{3}{4} \times (x - \frac{1}{2})$. $\quad x = \frac{2}{3}$.
5. $5 \times (x^2 + \frac{2}{5}) - x$. $\quad x = 1\frac{1}{2}$.

Grade 6

1. $x(3y - 2z) + x(2z - 3y)$. $\quad x = 213,\ y = 71,\ z = 102$.
2. $8xy - 5xz + x^2$. $\quad x = 35,\ y = 1,\ z = 7$.
3. $63x - 49x + 5x - 8x$. $\quad x = 21$.
4. $24x^2 - 3x - 21x^3 + 6x^2$. $\quad x = \frac{1}{2}$
5. $\frac{3}{7}x^2 + 2\frac{1}{3}x - \frac{5}{3}x - \frac{2}{7}x^2$. $\quad x = 21$.

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[12] By the seventh grade, students should know that parentheses stand for multiplication, i.e., $\frac{1}{5}(2x)$ means the product of $\frac{1}{5}$ and $2x$.

Grade 7

1. $\frac{1}{4}x^2 - 3\frac{1}{3}x - \frac{3}{2}x^2 + \frac{1}{3}x. \quad x = 6.$

2. $5x^2 + 18 - \frac{2}{3}x - (\frac{1}{2} - 5x). \quad x = \frac{1}{4}$

3. $2x(1 - \frac{1}{2}x) + \frac{1}{2}x(2x + 4). \quad x = \frac{85}{4}.$

4. $x(2 - \frac{1}{x}) - 47\frac{1}{x}. \quad x = \frac{47}{2}.$

Symbolic Manipulation

Students learn about rational numbers in the seventh grade. As suggested in the chapter on *Fractions and Decimals*, rational numbers are a system of numbers which is assumed to satisfy the associative, commutative, and distributive properties. Students need to be reminded of this fact, and in particular, the fact that the distributive property now includes not just addition but also subtraction, namely, for all rational numbers x, y, z, the following holds:

$$x(y - z) = xy - xz$$

This is a consequence of that fact that $y - z$ is by definition nothing but $y + (-z)$ and that $-xz = x(-z)$, so that if we appeal to the distributive law for addition, we have:

$$\begin{aligned} x(y - z) &= x(y + (-z)) \\ &= xy + x(-z) \\ &= xy - xz \end{aligned}$$

This and more are part of the following **grade 7** algebra and functions standard

1.3 Simplify numerical expressions by applying properties of rational numbers (e.g., identity, inverse, distributive, associative, commutative) and justify the process used.

Here something new and immensely significant has been added to the subject - *simplifying expressions*. This is one of the two basic components of symbolic manipulation on the introductory level, to which we may regard the exercises above on evaluation as a prelude. The topic of simplifying expressions will be taken up at greater length in a course on algebra, but a few pertinent comments at this point would help to pave the way for students' future work. What matters is not just simplifying expressions but *manipulating them*, changing a mathematical expression possibly involving variables into an equivalent expression, or correctly deriving a more useful expression from a given one. Put this way, this is one of the most important steps students must take in developing mathematical proficiency. Let us illustrate the most elementary aspect of simplifying expressions, which is nothing but the application of the distributive property

Suppose we are given an expression $(251 \times 69) + (76 \times 12^2) + (32 \times 251) - (12^2 \times 16)$. Rather than doing the multiplications 251×69 and 32×251 involving the same number 251 separately, it would save labor to combine the two operations if possible. The distributive and commutative properties imply that

$$(251 \times 69) + (32 \times 251) = (251 \times 69) + (251 \times 32) = 251\,(69 + 32) = 251 \times 101 = 25351$$

Similarly,

$$(76 \times 12^2) - (12^2 \times 16) = 12^2\,(76 - 16) = 12^2 \times 60 = 8640$$

Thus the original expression equals $25351 + 8640 = 33991$. This is one of the instances where the process is more important than the end result. The key idea of this process is that the general properties of the number operations should be applied whenever possible to achieve implifications:

$$(251 \times 69) + (76 \times 12^2) + (32 \times 251) - (12^2 \times 16) = 251\,(69 + 32) + 12^2\,(76 - 16)$$

If we replace the numbers 251 and 12 by any other numbers, the simplification would be similar. This means that for any two numbers x and y, we always have:

$$69x + 76y^2 + 32x - 16y^2 = (69 + 32)x + (76 - 16)y^2,$$

Notice that, **as a matter of convention**, we have written $69x$ for $x69$ and $16y^2$ for $y^2 16$. More generally, given (rational) numbers a, b, c, d, the equality

$$ax + by^2 + cx + dy^2 = (a + c)x + (b + d)y^2$$

holds for all numbers x and y. The extension to any sum of this type can be similarly formulated. This technique is known as "collecting like terms" in algebra and is one of the rigid rules that is normally emphasized in the teaching of algebra, but students should understand that so-called *"collecting like terms" is no more than a sensible application of the distributive property to achieve an efficient organization of one's work.* The basic idea of applying the distributive property whenever possible will also be a key step in the solution of linear equations below.

We now turn to the other basic component of symbolic manipulation which has to do, not with expressions, but with equations. It is based on the two basic algebra and functions standards in **grade 4**.

2.1 Know and understand that equals added to equals are equal.

2.2 Know and understand that equals multiplied by equals are equal.

Students should know how to express these in symbolic language. Standard 2.1 means that if a, b, x, y are any four numbers, and $a = b$, $x = y$, then $a + x = b + y$. Similarly, 2.2 means that under the same assumption on a, b, x, y, $ax = by$. Note that these two statements remain valid even as the meaning of "numbers" (i.e., a, b, x, y) becomes more inclusive as the grade level progresses: in grade 4, numbers are basically whole numbers, in grade 5 they include fractions, in grade 6 they include integers, until finally in grade 7 a "number" means any rational number.

What is important about these two statements in the context of symbolic manipulation is that, at the level of grade seven, they give rise to the following basic skills in manipulating equations. First, suppose a, b, c are rational numbers so that $a = b + c$, then $a - c = b$, *and conversely*. The passage from $a = b + c$ to $a - c = b$, or that from $a - c = b$ to $a = b + c$, is commonly referred to as *transposing* c.[13]

> To show that $a = b + c$ implies $a - c = b$, observe that from $-c = -c$ and $a = b + c$, we obtain $a + (-c) = (b + c) + (-c)$. The associative property implies $(b + c) + (-c) = b + (c + (-c)) = b + 0 = b$ and $a + (-c)$ is by definition of $a - c$. Therefore we obtain $a - c = b$.
>
> Conversely, $a - c = b$ implies $a = b + c$ because, from $a - c = b$ and $c = c$, we get $(a - c) + c = b + c$, which is $(a + (-c)) + c = b + c$. The associative property for addition shows $(a + (-c)) + c = a + ((-c) + c) = a$ so that we get $a = b + c$.

Thus the two equations $a = b + c$ and $a - c = b$ are interchangeable in the sense that knowing either one means knowing the other. It is common to abbreviate this fact by saying that $a = b + c$ *is the same as* or *is equivalent to* $a - c = b$.[14] In more suggestive language, one can say that the number c in the equation $a = b + c$ may be *transposed to the other side* and the equation would remain the same.

For exactly the same reason, if $b \neq 0$ and $ab = c$, then by looking at $ab = c$ and $\frac{1}{b} = \frac{1}{b}$, we see that $ab = c$ is equivalent to $a = \frac{c}{b}$.

These two facts,

$$a = b + c \quad \text{is equivalent to} \quad a - c = b$$

and

$$\text{if} \quad b \neq 0, \quad \text{then} \quad ab = c \quad \text{is equivalent to} \quad a = \frac{c}{b}$$

are fundamental to the symbolic manipulative aspect of solving equations. Because they are so important, many concrete examples and exercises on these manipulations should be given. For example: if a number a satisfies $(3125 - 467) = a + 100$, what is $a + 567$? Or, if b is a number so that $3b + 2345 = 287$, what is b?

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[13] Of course this fact can be taught in grade 4 if a, b, c are restricted to be whole numbers so that $a \geq c$. A similar statement can be made about grade 5 and grade 6, but we want to state the most general statement possible up to this point for convenience.

[14] It would be good practice to suppress the terminology "the same as" and use "equivalent to" exclusively since, in common usage, the word "same" has a different meaning than its usage here.

Isolating Variables in Simple Linear Equations

We are finally in a position to address the following **grade 7** algebra and functions standard

4.0 Students solve simple linear equations and inequalities over the rational numbers

In the intervention program we concentrate on equations. Once the procedures for solving equations are understood, students will not have much difficulty with inequalities.

To *solve a linear equation* such as $12x - 5 = 6x$ means to find all numbers c that make $12c - 5$ equal to $6c$. Such a number c is called a *solution*.[15] Thus 2 is not a solution of $12x - 5 = 6x$ because $12 \times 2 - 5 \neq 6 \times 2$ as $19 \neq 12$. However $\frac{5}{6}$ is a solution because $12 \times \frac{5}{6}$ and $6 \times \frac{5}{6}$ both equal to 5. Other examples of linear equations are $2x = 7x + 14$, $11 - 6x = 54x + 8$, and $85 = 3x + 24 - 25x$. For example, a solution c of $11 - 6x = 54x + 8$ is a number so that $11 - 6c$ is equal to $54c + 8$. Notice the relevance of evaluating expressions to our attempt to determine whether or not a number is a solution of a given equation.

The critical observation is that if the equation is presented as simply $3x = 25$, then we can directly make use of the fact that

$$\text{if} \quad b \neq 0, \quad \text{then} \quad ab = c \quad \text{is equivalent to} \quad a = \frac{b}{c}$$

to conclude that

$$3x = 25 \quad \text{is equivalent to} \quad x = \frac{25}{3}.$$

(Needless to say, one can also directly verify that $3(\frac{25}{3}) = 25$.) Therefore, $\frac{25}{3}$ *is the* (*one and only*) *solution of the equation* $3x = 25$.

In like manner, the equation $12x = 3$ has solution $x = \frac{3}{12} = \frac{1}{4}$, the equation $\frac{2}{5}x = 6$ has solution $x = 6/\frac{2}{5} = \frac{6 \times 5}{2} = 15$, and, in each case, the solution is the only one. In general, if c, d are rational numbers, then for $c \neq 0$,

$$cx = d \quad \text{has the one and only one solution} \quad \frac{d}{c}.$$

To proceed further, define in general two equations to be *equivalent* if after a finite number of transpositions and arithmetic operations (i.e., $+$, $-$, $\times$, and $\div$), one can transform either equation into the other. For example, $12x - 5 = 6x$ and $6x = 5$ are equivalent. Equivalent equations clearly have the same solutions.

[15] Generally, equations have more than one solution, though linear equations usually have at most one solution. Since linear equations are so common in school mathematics, one often finds reference to ***the*** solution for general equations. In the case of equations with more than one number in the solution set, it is also common to call the ***entire set of solutions*** "the solution." These differing usages cause difficulties for students, and more care must be taken at this point.

Suppose now we are given an equation $11x + 6 = 3x - 6$ where x is some unknown number. The preceding discussion shows that if we can *isolate the variable* x in the sense of getting an equivalent equation with all the x's on one side and the other numbers on the other side (e.g., $11x - 3x = -6 - 6$), then after an application of the distributive property, we can get an equation of the form $cx = d$ for some numbers c and d (e.g., since $11x - 3x = (11 - 3)x = 8x$, we get $8x = -6 - 6$ for the preceding example), and the determination of the solution set for the equation would follow.

To isolate the variable, it is therefore a matter of transposing all the x's to one side and all other numbers to the other side. To this end, we make repeated use of the fact proven earlier:

$$a = b + c \quad \text{is equivalent to} \quad a - c = b$$

Here are some examples, in increasing order of difficulty.

1 Solve $7x = 3x - 16$. We begin by transposing $3x$ to the left side: $7x - 3x = -16$. By the distributive property, $7x - 3x = (7 - 3)x = 4x$, so that $4x = -16$. The solution is then $\frac{-16}{4} = -4$.

2 Solve $5x + 1 = 2x - 11$. We first transpose $2x$ to the left side: $(5x + 1) - 2x = -11$, so that $3x + 1 = -11$. Now we have to transpose $+1$ to the right side: $3x = -11 - 1$, and $3x = -12$. Thus the solution is $\frac{(-12)}{3}$, which is -4.

3 Solve $3x + 14 = 2 - 8x - 17$. As usual, it is convenient to first transpose the x's to one side: $(3x + 14) - (-8x) = -15$ (we have made use of $2 - 8x - 17 = 2 - 17 - 8x = -15 - 8x$), so that $3x + 14 + 8x = -15$, and $11x + 14 = -15$. We next dispose of 14: $11x = -15 - 14$, and therefore $11x = -29$. The solution is then $\frac{-29}{11} = -\frac{29}{11}$.

However, one must not give students the idea that the x's must be on the left, *so we will manipulate the symbols differently to get the same solution.* We begin by transposing $3x$ to the right side: $14 = (2 - 8x - 17) - 3x$, which is $14 = 2 - 17 - 8x - 3x$, i.e., $14 = -15 - 11x$. Now transpose -15 to the left: $14 - (-15) = -11x$, so that $14 + 15 = -11x$, or $-11x = 29$. The solution is then $\frac{29}{(-11)} = -\frac{29}{11}$, which is of course the same as before.

4 Solve $-\frac{2}{3}x + 4 = -\frac{1}{5}x + 5\frac{1}{3}$. As usual, we collect all the x's to the left: $(-\frac{2}{3}x + 4) - (-\frac{1}{5}x) = 5\frac{1}{3}$. On the left we have $-\frac{2}{3}x + 4 + \frac{1}{5}x = -\frac{2}{3}x + \frac{1}{5}x + 4 = \frac{-7}{15}x + 4$. Thus the equation becomes $\frac{-7}{15}x + 4 = 5\frac{1}{3}$. We next transpose 4 to the right side: $\frac{-7}{15}x = 5\frac{1}{3} - 4$, and so $\frac{-7}{15}x = \frac{4}{3}$. Thus the solution is $\frac{4}{3}/\frac{-7}{15} = -\frac{20}{7}$.

It is important to also learn a second way of dealing with equations some of whose coefficients are fractions: *clear the denominators*, in the following sense. The denominators of the fractions in $-\frac{2}{3}x + 4 = -\frac{1}{5}x + 5\frac{1}{3}$ are 3 and 5, so if we multiply both sides by $3 \times 5 = 15$, we get (using the distributive law): $-10x + 60 = -3x + 80$. (Notice that $15 \times 5\frac{1}{3} = 15 \times \frac{16}{3} = 80$, or else $15 \times 5\frac{1}{3} =$

$(15 \times 5) + (15 \times \frac{1}{3}) = 75 + 5 = 80$.) Thus $-10x + 3x = 80 - 60$, and we get $-7x = 20$, or $x = -\frac{20}{7}$ as before.

To summarize, to solve a linear equation, *first transpose all the x's to one side and all the other numbers to the other side.* This would isolate the variable x, obtaining an equation of the form $cx = d$, where c, d are rational numbers. *If $c \neq 0$, then the solution is $\frac{d}{c}$.*

It remains to note that the above method of solving linear equations makes use of the associative, commutative, and distributive properties in full generality. To see this, let us examine in detail how we went in Example 3 from $(3x + 14) - (-8x)$ on the left side to $11x + 14$. It goes as follows:

$$
\begin{aligned}
(3x + 14) - (-8x) &= (3x + 14) + 8x \\
&= 8x + (3x + 14) && \text{(commutativity of +)} \\
&= (8x + 3x) + 14 && \text{(asociativity of +)} \\
&= ([8 + 3]x) + 14 && \text{(distributivity)} \\
&= 11x + 14
\end{aligned}
$$

The point to be emphasized here is that, in each of the above applications of these properties, the number x *is an unknown number*, but the commutative, associative and distributive properties are applicable to x anyway because they are valid for *all* numbers. This harks back to the earlier comment about the importance of the *generality* that is inherent in these rules. However, while students should be exposed to this kind of reasoning, perhaps more than once before they get to grade 8 so that they become aware of the concept of generality, it would be inappropriate to hold them responsible for it.

Functions and Equations

We discuss the concept of a function, its graph, and its relation with the study of equations.

Functions and graphs are much misunderstood by students. Students need to understand functions, not as formulas, but as rules that associate to each object of one kind an object of another kind. Such an understanding cannot be achieved overnight, but should be fostered through all the grades. In kindergarten, they encounter the function which associates to each ball its color. In grade 2, they encounter the function which associates to a given number of horses the total number of legs. In grade 4, they encounter for the first time a linear function expressed in symbols in terms of x and y, and with a similar nonlinear one which associates with each square of side s its area s^2. And so on. If a concerted effort is made to bring these functions to students' attention, they will begin to see the central role of functions in mathematics. Gradually, the functions being studied become increasingly complex and students need aids to understand them. Graphs are initially introduced for this reason as visual representations of functions. Later on, for functions which associate a number to another number, their graphs are curves (or lines) in the plane, and through them, geometric considerations become an integral part of the study of function.

Functions

A function is a rule that associates to each element in a set one and only one element in a second set. In the context of the middle grades, most of the time functions associate a number from one set of numbers to a number in another set. As an example, the equation $2x + y = 1$ in terms of two numbers x and y determines one and only one y for each value of x, namely $1 - 2x$, and thus determines a function associating the number $1 - 2x$ to each number x. Students have seen functions repeatedly from the earliest grades. In the California Mathematics Content Standards, the study of functions begins in **kindergarten** with the emphasized algebra standard

1.1 Identify, sort, and classify objects by attribute and identify objects that do not belong to a particular group (e.g., all these balls are green, those are red).

It is again taken up the Statistics, Data Analysis and Probability standards in **grade 1**:

1.1 Sort objects and data by common attributes and describe the categories.

as well as the emphasized standard

2.1 Describe, extend, and explain ways to get to a next element in simple repeating patterns (e.g., rhythmic, numeric, color, and shape).

In **grade 2**, this topic is again taken up, but with more specificity in the Statistics, Data Analysis, and Probability emphasis standard:

2.1 Recognize, describe, and extend patterns and determine a next term in linear patterns (e.g., 4, 8, 12 . . . ; the number of ears on one horse, two horses, three horses, four horses).

In **grade 3**, functions move back into the algebra strand with the emphasized standard:

2.1 Solve simple problems involving a functional relationship between two quantities (e.g., find the total cost of multiple items given the cost per unit).

and the almost as important

2.2 Extend and recognize a linear pattern by its rules (e.g., the number of legs on a given number of horses may be calculated by counting by 4s or by multiplying the number of horses by 4).

Then, in the **grade 4** algebra standards, a working definition of a function is given as the emphasis standard:

1.5 Understand that an equation such as y = 3x + 5 is a prescription for determining a second number when a first number is given.

Students should become familiar with certain basic functions, particularly the functions x, and x^2. They should take note of the fact that some functions, such as $\frac{1}{x}$ are not defined for every value of x, but there is no need for them to learn the terminology such as domain and range at this point. Such terminology will be introduced in an algebra course.

Some linear functions occur in daily life, as in the **grade 6** algebra and functions standards

2.1 Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).

Conversion of the type mentioned in standard 2.1 above leads to linear functions. For example, the conversion of miles to feet is described by the linear function $m \mapsto 5280m$, and the conversion of Celsius to Fahrenheit in temperature is described by the linear function $C \mapsto \frac{9}{5}C + 32$.

Functions and Their Graphs

The *graph of a function* from numbers to numbers is defined as the set of *all* pairs of numbers (a, b) so that b is the number the function associates with a. Graphing, in the generalized form of representing a function via a picture or diagram first appears in **grade 1** in the Statistics, Data Analysis, and Probability standard:

1.2 Represent and compare data (e.g., largest, smallest, most often, least often) by using pictures, bar graphs, tally charts, and picture graphs.

Graphing in this generalized sense also appears in the **grade 2** Statistics, Data Analysis, and Probability standards we find the emphasis standards

> **1.1** Record numerical data in systematic ways, keeping track of what has been counted.
>
> **1.2** Represent the same data set in more than one way (e.g., bar graphs and charts with tallies).

as well as in the **grade 3** Statistics, Data Analysis, and Probability standard

> **1.3** Summarize and display the results of probability experiments in a clear and organized way (e.g., use a bar graph or a line plot).

In **grade 4**, graphing becomes an emphasis topic in the Measurement and Geometry standards:

2.0 Students use two-dimensional coordinate grids to represent points and graph lines and simple figures:

> **2.1** Draw the points corresponding to linear relationships on graph paper (e.g., draw 10 points on the graph of the equation y = 3x and connect them by using a straight line).

At this point, students must understand the concept of the *graph of an equation* as the collection of *all* the ordered pairs of points (x, y) satisfying the equation. The failure to come to grips with the quantifier "all" may account for students' common error of not recognizing why the graph of $y = 5$ is a (complete) horizontal line or that the graph of $x = -3$ is a (complete) vertical line. For example, let us show that the horizontal line 5 units above the x-axis, to be denoted by H, is (identical to) the graph of $y = 5$. First, how to show that H is part of the graph of $y = 5$? Now a point (a, b) is on H precisely when $b = 5$ no matter what a is. So a point being on H means it has coordinates $(a, 5)$, where a is some number. Since the graph of $y = 5$ consists of *all* the points with y-coordinate equal to 5, every point of the form $(a, 5)$ has to be a point of the graph. Thus H is part of the graph of $y = 5$. There remains the possibility that there are points on the graph of $y = 5$ which are not part of H. Thus we have to further show that an arbitrary point on the graph of $y = 5$ belongs to H. Now such a point is of the form $(x, 5)$ and, since any point with y coordinate equal to 5 is on H, an arbitrary point of the graph of $y = 5$ is also a point on H. This then explains why *the graph of* $y = 5$ *is the (complete) horizontal line 5 units above the* x*-axis*, i.e., each point of the former is a point of the latter, and *vice versa.*

In the **grade 5** Statistics, Data Analysis, and Probability standards the use of coordinates in the plane is the content of the emphasized topics:

1.4 Identify ordered pairs of data from a graph and interpret the meaning of the data in terms of the situation depicted by the graph.

1.5 Know how to write ordered pairs correctly; for example, (x, y).

The first time students are asked to plot the graph of a function in the coordinate plane is in the **grade 5** emphasized algebra standard:

1.5 Solve problems involving linear functions with integer values; write the equation; and graph the resulting ordered pairs of integers on a grid.

In **grade 7**, graphs of functions are taken more seriously and the relevant standards are in the algebra-and-functions strand:

3.0 Students graph and interpret linear and some nonlinear functions

3.1 Graph functions of the form $y = nx^2$ and $y = nx^3$ and use in solving problems.

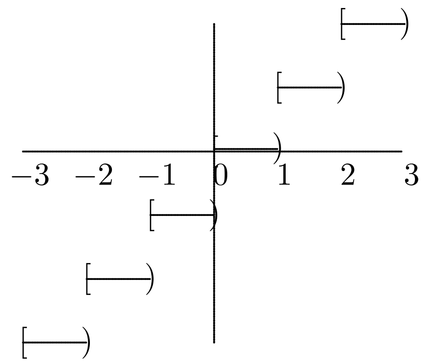
3.3 Graph linear functions, noting that the vertical change (change in y-value) per unit of horizontal change (change in x-value) is always the same and know that the ratio ("rise over run") is called the slope of the graph.

3.4 Plot the values of quantities whose ratios are always the same (e.g., cost to the number of an item, feet to inches, circumference to diameter of a circle). Fit a line to the plot and understand that the slope of the line equals the quantities.

The fact that the graph of a linear function is a straight line cannot easily be demonstrated for students without using properties of similar triangles, so students will have to take it on faith (for now) that the graph is a straight line. But this fact should be carefully explored with exercises and concrete examples. For instance, ask students to graph $y = \frac{1}{2}x$ by plotting the points on its graph with x-coordinates equal to 1, 2, 4, 8, 16, and ask them to observe how the y-coordinates also correspondingly grow by a factor of 2. Then the fact that the graph of $y = \frac{1}{2}x$ is a straight line becomes at least believable. Then do the same to $y = \frac{1}{2}x + 3$ and other similar examples.

Students should graph functions of the form $ax + by = 0$, $x^2 + a$, and $\frac{1}{x}$. They should also explore the graph of a function such as $x \mapsto$ its "integer part", i.e., the largest integer

$\leq x$ (e.g., $1.05 \mapsto 1$, $5 \mapsto 5$, $-2.1 \mapsto -3$, etc.):



Students should get lots of practice plotting functions such as $y = \frac{2}{3}x - 5$ or $y = x^2 - 25x - 18$. A very important point that should be impressed on students is that, in order to develop any feeling for the graphs of functions, they have to learn to visualize a graph by plotting judiciously chosen points *by hand*. This mean that they have to get used to computing the number that the given function associates to a chosen number. There is no substitute for learning about graphs through plotting points.

Another point worth mentioning is that students should be alerted to the possibility of viewing a function simply as the collection of all ordered pairs of number $\{(a,b)\}$ on its graph, where b is the number the function associates to a. There is no need to over-emphasize this fact, but they should be aware that in higher mathematics, this collection of ordered pairs is used as the formal definition of a function.

Solving and Graphing Linear Equations

In the discussion of symbolic manipulation, solving linear equations was the key example. When given a linear equation of the form $ax + b = cx + d$ with $a \neq c$, students know how to solve it by isolating the variable and obtain $x = \frac{d-c}{a-c}$.

The linear equation with two variables

$$ax + by = c$$

should now be introduced. The study of its graph will be one of the main topics in algebra, but what needs clarification for students at this point is the relationship between the graph of this *equation* and the graph of a linear *function* that is the subject of the seventh grade algebra and function standards 3.3 and 3.4 above.

Recall that the graph of a linear equation $ax + by = c$ is the collection of all the points (x', y') which *satisfy the equation*, in the sense that $ax' + by' = c$. When $b \neq 0$, then for each number x' we can in fact find a number y' so that (x', y') satisfies $ax + by = c$. The explicit process of determining the number y' that makes the equation $ax + by = c$ true for a given x' when $b \neq 0$ will now be developed. Thus given x', we have to solve for a y' so that $ax' + by' = c$. Remembering that x' is just a number, we can solve for y in $ax' + by = c$ the same way we solve for x in linear equations of a single variable, as follows. Transpose the number ax' in $ax' + by = c$ to the right side to isolate y gives $by = c - ax'$. Now $c - ax'$ is again just a number, so $y = \frac{c-ax'}{b}$, which is the same as $y = (\frac{1}{b})(c - ax')$, so that by the distributive law, $y = -\frac{a}{b}x' + \frac{c}{b}$. Thus with x' given, if we set $y' = -\frac{a}{b}x' + \frac{c}{b}$, the point (x', y') satisfies $ax + by = c$. Notice in particular that that $c = 0$ in $ax + by = c$ is equivalent to the graph of $ax + by = c$ containing the origin $(0, 0)$. The special case $y = c$ should be discussed as well, and also the special case of $x = c$ when $b = 0$.

Now observe that when $b \neq 0$, the graph of the equation $ax + by = c$ is exactly the same as the graph of the linear function $x \mapsto -\frac{a}{b}x + \frac{c}{b}$. This means one should check that every (x', y') in the graph of the equation $ax + by = c$ is also a point in the graph of the linear function $x \mapsto -\frac{a}{b}x + \frac{c}{b}$, and vice versa. The reasoning is as follows:

> (x', y') being on the graph of $ax+by = c$ means $ax'+by' = c$, and this means (x', y') can be rewritten as $(x', -\frac{a}{b}x' + \frac{c}{b})$. The latter is a point on the graph of the function which associates to each x the number $-\frac{a}{b}x + \frac{c}{b}$. Conversely, if (x', y') is a point on the graph of the linear function that associates to each x the number $-\frac{a}{b}x + \frac{c}{b}$, then by definition $y' = -\frac{a}{b}x' + \frac{c}{b}$. Since $ax' + b(-\frac{a}{b}x' + \frac{c}{b}) = c$, we see that (x', y') is a point in the graph of $ax + by = c$.

Proportions Revisited

In the chapter on *Ratios, Rates, Percents and Proportion*, we started the discussion of proportions. We can now clarify exactly what a proportion means using the concept of a linear function. In general terms, one can say that any problem that can be solved by *setting up a proportion* is a problem about a linear function *without constant term*. The meaning of the latter is this: if a linear function $x \mapsto ax + b$ is given, where a, b are fixed numbers, then b is called the *constant term* of the function. To say that this function *has no constant term* means that $b = 0$, i.e., the function is of the form $x \mapsto ax$ for some fixed number a. The most important example of a linear function with no constant term (in school mathematics) is the function that describes motion with constant speed: the function $t \mapsto vt$, where v is a fixed number, gives the distance traveled after t units of time, and v is the (constant) speed.

> *Suppose a train travels at constant speed can cover a distance of 150 miles in* $2\frac{1}{4}$ *hours. How long will it take the train to go 55 miles?*

A problem similar to this one has been solved without explicitly using the concept of a linear function in the chapter on *Ratios, Rates, Percents and Proportion.* We now solve it again, this time using the linear function $t \mapsto vt$. In the long run the new method will be seen to be more widely applicable even though it initially seems more complicated. So we have a linear function $t \mapsto vt$ which gives the distance traveled after t hours. Recall that v is the (constant) speed. Suppose this function associates to a time t_0 the value 55; the question is what is t_0. For this we need to know v. Since the given data says $2\frac{1}{4} \mapsto 150$, we know $v \cdot 2\frac{1}{4} = 150$ and therefore $v = \frac{150}{2\frac{1}{4}} = 66\frac{2}{3}$. So if $t_0 \mapsto 55$, we have $vt_0 = 55$ and $t_0 = \frac{55}{v} = \frac{55}{66\frac{2}{3}} = \frac{33}{44}$ hours, or 49 and a half minutes.

The usual solution in terms of "setting up the proportion" is to equate the ratio of distances to the ratio of corresponding times,

$$\frac{150}{55} = \frac{2\frac{1}{4}}{t} \ ,$$

in order to solve for t. Such a procedure suppresses entirely the fact that it is the linear function $t \mapsto vt$ that underlies everything. Indeed, the preceding proportion is equivalent to

$$\frac{150}{2\frac{1}{4}} = \frac{55}{t}$$

(cross-multiply both proportions, for instance), which displays more clearly the fact that the division

$$\frac{\text{distance traveled in } t \text{ hours}}{t \text{ hours}}$$

is always equal to a fixed constant v, the (constant) speed. In giving problems that traditionally require "setting up proportions" to students, especially those in the intervention program, an effort therefore must be made to be explicit about the underlying linear-function-without-constant-term because *this is part of the data students must be given in order to solve the problem in a mathematical manner.*

We conclude with another example of how the presence of a linear function without constant term is camouflaged in a problem. Consider the following:

> *In a certain movie, the dinosaur was a scale model and so was the sport utility vehicle overturned by the dinosaur. The vehicle was made to the scale of 1 inch to 8 inches. The actual vehicle was 14 feet long. What was the length of the model sport utility vehicle?*

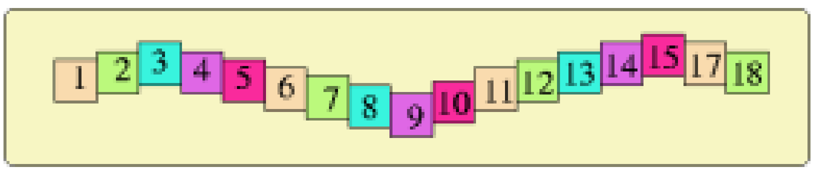
The key phrase here is that the "vehicle was made to the scale of 1 inch to 8 inches." Unless this phrase is precisely explained, students will not understand the meaning of the phrase "made to a certain scale", and will therefore not be able to process the information correctly. What needs to be made explicit as part of the data of the problem (*at least when this phrase comes up for the first time*) is that, if the vehicle is x inches long, then the model is $\frac{1}{8}x$ inches. Thus it should be explained at the outset (in a way that is grade level appropriate) that the problem assumes the existence of a linear function without constant term which associates to each x inches of a sport vehicle the length of $\frac{1}{8}x$ inches of the model, i.e., the function $x \mapsto \frac{1}{8}x$. With this understood, the problem wants to know what number would this function associate to $x = 168$. (14 feet is 168 inches.) The answer is of course $\frac{1}{8} \times 168 = 21$ inches.

At present, the suggested solution in most books is to “form the proportion” $\frac{1}{8} = \frac{\ell}{168}$ (where ℓ is the length of the model). Such a solution is unacceptable because it begs the question *why*?

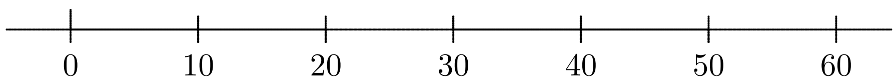
Measurement

One of the critical areas where mathematics connects with applications is measurement. There are two kinds of measurement that students need to understand. The first is the measurement of real objects, including length, areas, angles, and volumes. Such measurements require an appreciation of magnitude and reasonable estimates, an understanding that all such measurements have *errors*, and an understanding of how errors build up and can affect things. The second comprises the *exact* measurements of objects in mathematics. If we have a right triangle with leg lengths l_1 and l_2 then we know that the length of the hypotenuse is exactly $\sqrt{l_1^2 + l_2^2}$. Likewise, if we have an equilateral triangle, then we know that each interior angle is exactly 60 degrees, and if we have a square with side length $\frac{1}{4}$ meter, then we know that the area is exactly $\frac{1}{16}$ square meter. The similarities and distinctions between these two kinds of measurement must be solidly understood by students if they are to effectively use mathematics.

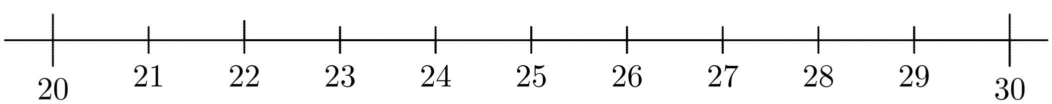
Before students can work with measurement and the core topics that develop from it, they must have some idea of the number line and means of making measurements. The number line is prepared for by sequentially ordering numbers as the following illustration shows:



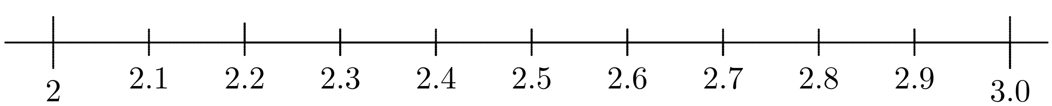
Next the method of placing numbers on the number line can be described, first working perhaps with multiples of 10, as equi-spaced points, as shown:



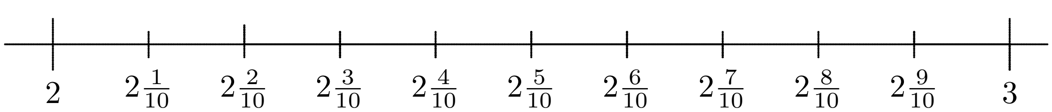
then expand out and magnify a single region or two, by further sub-dividing into equi-spaced points:



Finally, discuss, briefly the same process but starting with intervals of 100. It is also worth noting that the same process can be continued for unit intervals:



and just for emphasis



Preparation for the number line often begins with using rulers, but it would be advisable to use only the metric portion, i.e., the portion marked in centimeters and millimeters, as the portion marked in inches might confuse children with its markings in sixteenths. Moreover, it appears that many students do not fully understand the conventions used in measuring with rulers, so care has to be taken here, and it might be good practice to bring in the number line first and then rulers, or perhaps both at about the same time. One thing to be particularly careful about is

> Some students will not understand how rulers work, confusing tick marks and intervals, for example. Confusion between intervals and tick marks – points and distances – is a fundamental difficulty.

Once the usage of rulers for measuring and the number line are in place, the discussion of measurement can begin.

Measurement involves length, weight, capacity, and time, and standards involving measurements are present from Kindergarten on. The elementary aspects of measurement involve knowing:

(1) the different systems of units used for measurement,

(2) when it is more appropriate to use one or the other of these systems,

(3) how to translate between these systems.

In the California Mathematics Content Standards, the conversion of measurement units is the subject of the **grade 3** algebra and function standard

> **1.3** Express simple unite conversions in symbolic form.

and the **grade 3** measurement and geometry standard

> **1.4** Carry out simple unit conversions within a system of measurement (e.g., centimeters and meters, hours and minutes).

Then there is the **grade 6** algebra and functions standard

> **2.1** Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).

The last standard was already discussed in *Functions and Equations* in connection with linear functions. Two additional relevant standards are the **grade 7** measurement and geometry standards

> **1.1** Compare weights, capacities, geometric measures, times, and temperatures within and between measurement systems (e.g., miles per hour and feet per second, cubic inches to cubic centimeters).

1.3 Use measures expressed as rates (e.g., speed, density) and measures expressed as products (e.g., person-days) to solve problems; check the units of the solutions; and use dimensional analysis to check the reasonableness of the answer.

These standards, while they deserve to be covered with care, should not offer great challenges for most students.

The next two aspects of measurement are on a higher level: students should know

(4) how to use and make measurements in abstract mathematical situations

(5) how to make measurements in real life situations.

Perhaps the most basic issue here is the distinction between (4) and (5). In abstract, mathematical situations, measurements are almost always assumed to be exact. However, actual measurements in the real world always are inaccurate, and the resulting error can seldom be ignored. (When building a house or even a shed, if we are not aware that our measurements have errors, the errors will gradually accumulate, and core parts will simply not fit together.) Real world measurements involve making estimates and keeping track of the resulting errors. If your measurements are accurate only up to the nearest millimeter, then adding two such measurements would lead to a possible inaccuracy of one millimeter, and multiplying two such measurements a mm and b mm would lead to a possible error of $a + b + 1$ mm. Estimations are routinely discussed in school without reference to the attendant errors of estimations. This oversight should be corrected.

Thus, students have to make the distinction between the measurements of types (4) and (5), but all too often neither (4) nor (5) is done correctly, and the distinction is not made. The discussion below recommends ways to improve this situation.

Area

The first time measurement is emphasized is in the **grade 2** measurement and geometry standard

1.3 Measure the length of an object to the nearest inch and/or centimeter.

Area and perimeter are brought in with the **grade 3** emphasis topic

1.2 Estimate or determine the area and volume of solid figures by covering them with squares or by counting the number of cubes that would fill them.

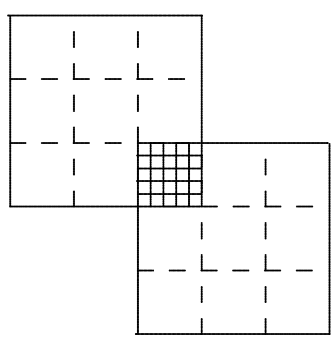
1.3 Find the perimeter of a polygon with integer sides.

These are, mostly, real-world measurement standards. They do not involve the precise, formal measurements of geometry. But in developing these standards it is very important that students understand what area and perimeter are. Perimeter can be readily understood via direct measurement of the perimeters of some basic figures, but area is a different matter.

> One of the most common errors that students make is to believe that area is defined by formulas, so that if they see a figure for which they do not know a formulaic method of determining the area, they will have no idea how to proceed.

Students should know some basic properties of area. They should begin by working with figures made out of non-overlapping squares, all of the same size; for convenience, we will say that those squares *pave* the figures. The *area* of such a figure should be *defined* for students as the the sum of the areas of the individual squares. To go further into the discussion of area, we have to consider the issue of numerical measurement of area. To this end, it should be emphasized that, as in the discussion of fractions (see the chapter *Fractions and Decimals*), *one must fix a unit of area* throughout the discussion. The most common area-unit is a *unit square*, where "unit" refers to the length of a side and it could be 1 cm, 1 mm, or 1 of anything. The area of the unit square is traditionally assigned the value of 1 *area-unit*. If the side of the unit square is 1 cm, then the area-unit is called 1 *sq. cm*, or 1 cm^2; if the side of the unit square is 1 ft., then the area-unit is called 1 *sq. ft.*, or 1 ft.^2, etc. It follows that the area of a rectangle with sides 5 units and 7 units has value 5×7 area-units because it can be paved by $7 + 7 + 7 + 7 + +7 = 5 \times 7$ unit squares. In general, if a, b are whole numbers, the same reasoning gives that the area of a rectangle with sides of lengths a, b is ab area-units.

If two figures overlap, then the area of the resulting figure is *less* than the sum of the individual areas. Consider the following example where each of the two big squares is a unit square and is each paved by nine smaller squares, as shown:[16]



The area of the resulting figure is, by the definition above, the sum of the areas of the resulting smaller squares. There are 17 of the latter, so the area of the figure will be determined as soon as the area of each small square is. Since the area of either unit square (which is 1) is the sum of the areas of 9 congruent smaller squares, the area of each smaller

[16] We recall the Fundamental Assumption of School Mathematics that, in K-12, we only deal explicitly with rational numbers, so that for the purpose of illustration here, care should be taken with the overlapping squares to ensure that the overlap is a fraction of (the area of) the two squares.

square is $\frac{1}{9}$ of the area-unit. Consequently, the total area of the above figure is $17 \times \frac{1}{9} = \frac{17}{9}$ area-units. When the overlap of two figures is more complicated than a square, then the numerical determination of the total area would not be easy.

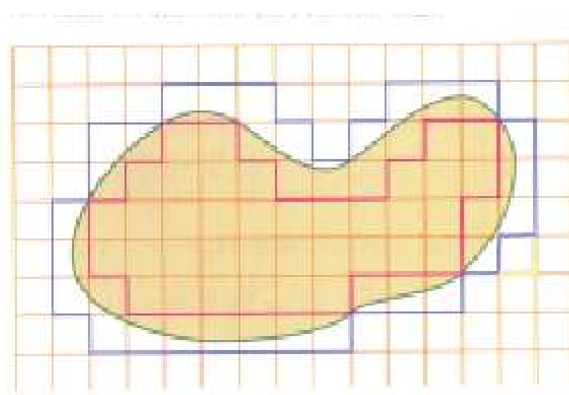
The preceding reasoning leads to the numerical determination of areas of rectangles whose sides have fractional lengths. Suppose a rectangle has sides of lengths ℓ and w, where ℓ and w are fractions. Then according to the definition of the product of fractions in the Chapter *Fractions and Decimals*,

the area of a rectangle with sides of length ℓ and w is ℓw

What is important for explicit computations is the fact that if $\ell = \frac{a}{b}$ and $w = \frac{c}{d}$, then $\ell w = \frac{ac}{bd}$. The reasoning for this so-called product formula, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ can be found in the chapter on *Fractions*. In any case, with the understanding as always that we deal principally with rational numbers, we have now *explained* why "area of a rectangle is length times width."

Practice with measuring the approximate area of more complex regions can now be given:

> The edge of each square in the figure below is 5cm. Count the number of squares inside the liver-shaped figure; there are 37. Then count the number of squares which intersect the liver-shaped figure; there are 72.



It is intuitively clear that the area of the liver-shaped figure is bigger than the area of the figure consisting of the squares inside the liver-shaped figure, but smaller than the area of the figure consisting of all the squares which intersect the liver-shaped figure. Therefore the area of the liver-shaped figure is between 37×5^2 cm^2 and 72×5^2 cm^2. If we use smaller squares, that is if the edge of each square is 1 cm or even 0.01 cm instead of 5 cm, it is also intuitively clear that the area of the liver-shaped figure will be approximated by two numbers closer together than 37×5^2 and 72×5^2. By using smaller and smaller squares, it is easily believable that the bigger and smaller numbers will get closer and closer together until at the end they coincide. This common number is formally what is called the *area* of the liver-shaped figure. In other words, the area of the liver-shaped figure is, *by definition*, this common number. The same can be said for any figure that appears in daily life. One consequence of this definition of area is that one gets the area of a given figure by *approximating it with smaller and smaller squares that lie completely inside the figure* without worrying about those squares that merely intersect the figure. In the classroom, especially in the lower grades of 3 – 5, this may be the more practical approach and may be the one that should be presented to students.

Exact Measurement in Geometry

The exact measurements of geometry are prepared for in the **grade 4** measurement and geometry standards:

1.1 Measure the area of rectangular shapes by using appropriate units, such as square centimeter, square meter, square kilometer, square inch, sqrae yard, or square mile.

2.2 Understand that the length of a horizontal line segment equals the difference of the x-coordinates.

2.3 Understand that the length of a vertical line segment equals the difference of the y-coordinates.

Then comes the basic **grade 5** measurement and geometry standards

1.1 Derive and use the formula for the area of a triangle and of a parallelogram by comparing it with the formula for the area of a rectangle (i.e., two of the same triangles make a parallelogram with twice the area; a parallelogram is compared with a rectangle of the same area by pasting and cutting a right triangle on the parallelogram).

1.2 Construct a cube and rectangular box from two-dimensional patterns and use these patterns to compute the surface area for these objects.

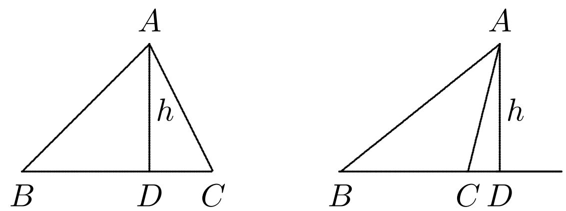
1.3 Understand the concept of volume and use the appropriate units in common measuring systems (i.e., cubic centimeter [cm^3], cubic meter [m^3], cubic inch [in.^3], cubic yard [yd.^3]) to compute the volume of rectangular solids.

We now demonstrate the area formula for a triangle (i.e., standard 1.2 above), which requires a significant advance on the previous *exact* determination of area in the case of figures constructed out of squares with restricted overlaps.

We will take as a given the intuitive fact that if two figures intersect only at their boundaries, then the area of the combined figure is the sum of the respective areas. For right triangles, we can use this fact to determine their area by observing that if a right triangle is doubled in the usual way to yield a rectangle, then the area of the right triangle is half the area of the rectangle. Since the "height" and "base" of a right triangle are the lengths of the legs, and hence the lengths of the sides of the rectangle so produced, the area of the right triangle is half the product of height and base (because the latter is the area of the rectangle). This justifies the special case of

the area of a triangle is $\frac{1}{2}$ (base $\times$ height)

when the triangle is a right triangle. For the general case, let triangle ABC be arbitrary and let AD be the perpendicular from the vertex A to the line containing BC. Then there are two cases to consider: D is inside the segment BC, and D is outside the segment BC. See the figures:



In either case, AD is called the *height* with respect to the *base* BC. By the usual abuse of language, *height* and *base* are also used to signify the *lengths* of AD and BC, respectively. With this understood, we shall prove the area formula of a triangle in general. For the case on the left, the area of triangle ABC is clearly the sum of the areas of triangle ABD and ADC. Since the latter two triangles are right triangles (whose areas we already know how to compute), the general formula is easy. Next we observe that the area of triangle ABC is now the area of triangle ABD *minus* the area of triangle ACD for the case on the right hand side above. A similar calculation again yields the general formula.

It is important not to leave out the case on the right, i.e, the case where the perpendicular from the top vertex meets the line containing the base at a point outside the base.

Once we get the area formula of a triangle, one can, in principle, compute the areas of all polygons in the plane. The fundamental reason behind this assertion is that every polygon is paved by triangles obtained by joining appropriate vertices of the polygon. Although the proof of this general fact is too intricate for school mathematics, its validity in simple cases is quite obvious. For instance, if the polygon is a quadrilateral, then adding a suitable diagonal would create two triangles that pave the quadrilateral. The area of the quadrilateral is thus the sum of the areas of these two triangles and the areas of the latter can be computed using the area formula we have just derived. If the quadrilateral is a trapezoid or a parallelogram, such a computation actually leads to a simple formula for the area of the quadrilateral itself, as is well-known. The derivation of these standard formulas for trapezoids and parallelograms should also be given in detail.

More Advanced Topics

In **grade 5**, angles are also brought in:

2.1 Measure, identify, and draw angles, perpendicular and parallel lines, rectangles, and triangles by using appropriate tools (e.g., straightedge, ruler, compass, protractor, drawing software).

2.2 Know that the sum of the angles of any triangle is 180 degrees and the sum of the angles of any quadrilateral is 360 degrees and use this information to solve problems.

At this stage, students need only do *experimental geometry* in the sense of verifying geometric assertions by direct measurements (and it is important that they do). So measuring angles of many triangles should convince them of the likely truth of the fact that the sum of the angles of any triangle is 180 degrees.

In **grade 6**, the following two non-emphasized standards can be easily misunderstood:

1.2 Know common estimates of π (3.14, $\frac{22}{7}$) and use these values to estimate and calculate the circumference and the area of circles; compare with actual measurements.

1.3 Know and use the formulas for the volume of triangular prisms and cylinders (area of base times height); compare these formulas and explain the similarity between them and the formula for the volume of a rectangular solid.

There are two aspects to 1.2. On the one hand, there is a number π so that the area and circumference of a circle of radius r are πr^2 and $2\pi r$, respectively. On the other hand, the value of this number π cannot be given by a finite decimal and so we are usually forced to use an approximate value. Consequently, 1.2 asks for *experimental geometric* activities such as estimating the area of a circle using grid paper, for instance, and comparing with πr^2 using an approximate value of π.

The volume formulas of 1.3 deserve some comments. First, the concept of volume is entirely analogous to that of area and should be clearly presented. As in the case of area, we need to fix a *unit for volume*, which is usually taken to be the *unit cube*, i.e., the cube with sides of length 1 unit (cm, in., mm, ft., etc.) The volume assigned to the unit cube is 1 volume-unit. If the length unit that is used is cm, then the corresponding volume unit will be called 1 *cubic cm*, or 1 cm^3; if the length unit is ft., then the corresponding volume unit is 1 *cubic ft.*, or 1 ft.^{3}. And so on.

A figure is said to be *paved* with a collection of (not necessarily unit) cubes if it is the union of these cubes and if any two cubes in the collection intersect at most at their boundaries. The *volume* of such a figure is by definition the sum of the volumes of these cubes. It remains to determine what the volume of a cube, or more generally, the volume of a rectangular prism should be. If we have a rectangular prism whose sides have lengths equal to whole numbers, e.g., 2, 3, and 5, then it is clearly paved by exactly $2 \times 3 \times 5$ unit cubes and therefore has volume equal to $1 + 1 + \cdots + 1$ ($2 \times 3 \times 5$ times), which is of course equal to $2 \times 3 \times 5$. Thus the volume of a rectangular prism with sides (of length) 2, 3, and 5 is $2 \times 3 \times 5$ volume-units. The same reasoning shows that if the sides of a rectangular prism have lengths which are whole numbers a, b, and c, then its volume is abc volume-units.

If the lengths of the sides of a rectangular prism are *fractions* a, b, and c, then we want to show that its volume is again abc volume-units. The idea of the proof can be seen in the the following simple special case, which is also one that can be safely presented to students in grade 6.

> Suppose a rectangular prism has sides 2, 4, and $\frac{1}{3}$. Think of this prism as having a rectangular base of 2 by 4, and having a height of $\frac{1}{3}$. Then the base of the prism can be paved by 2×4 unit squares, and the prism itself therefore can be paved by 2×4 small rectangular prisms whose base is a unit square and whose height is $\frac{1}{3}$. Thus each of these small rectangular prisms has dimensions 1, 1, and $\frac{1}{3}$. Denote such a small rectangular prism by $\mathcal{P}$. Now three of these $\mathcal{P}$'s stacked on top of each other form a unit cube. Since the volume of the unit cube is 1, by the definition of a fraction, the volume of each $\mathcal{P}$ is $\frac{1}{3}$ volume-unit. Since the original rectangular prism is paved by 2×4 of these $\mathcal{P}$'s, its volume is $\frac{1}{3} + \cdots + \frac{1}{3}$ $(2 \times 4$ times$)$, which is of course $(2 \times 4) \times \frac{1}{3}$, exactly as claimed.

In general, we have the following theorem about rectangular prisms:

The volume of a rectangular prism whose sides have length a, b, and c is abc.

We have explained the most important special case of this theorem for school mathematics, namely, the case where a, b, c are fractions.

There is another way to look at this formula. We can think of such a rectangular prism as a solid whose (rectangular) base has lengths a and b, and whose height is c. Then $abc = (ab)c =$ (area of base) $\times$ height. We may therefore rewrite the volume formula as

volume of rectangular prism = (area of base) $\times$ height

Why this re-writing is important can be explained as follows. Suppose we modify the foregoing solid by fixing the height but modifying the shape of the base so that, instead of a rectangle, we now have a geometric figure where $\mathcal{S}$ is arbitrary. Then the resulting solid is called a *cylinder over the base* $\mathcal{S}$, and the volume of the cylinder will be given by this same formula, i.e.,

volume of cylinder over base $\mathcal{S}$ = (area of $\mathcal{S}$) $\times$ height

The reason for the validity of this formula can only be given in more advanced courses. In particular, if $\mathcal{S}$ is a circle of radius r, the cylinder is commonly called a *circular cylinder of radius* r. Therefore,

$$\text{volume of circular cylinder of radius } r \text{ and height } h \;=\; \pi r^2 h$$

If the base $\mathcal{S}$ is a triangle, the cylinder is called a *triangular prism*. If this triangle has a base of length b and corresponding height h, its volume is then

$$\begin{aligned}&\text{volume of triangular prism whose triangular base has base } b \text{ and height } h\\ &= \tfrac{1}{2} bh \times \text{ (height of prism).}\end{aligned}$$

Exact properties of the measurement of angles are used in **grade 6** measurement and geometry standard

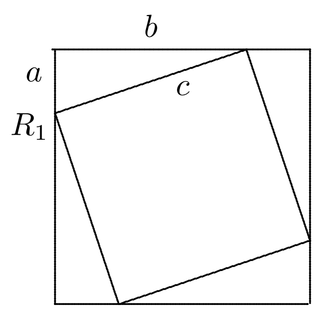
2.2 Use the properties of complementary and supplementary angles and the sum of the angles of a triangle to solve problems involving an unknown angle.

In **grade 7**, further exact properties of measurement are brought forward in the measurement and geometry standards

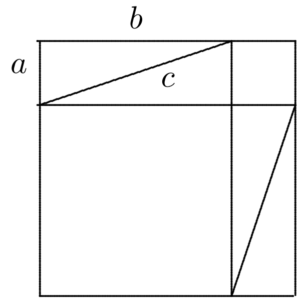
3.3 Know and understand the Pythagorean theorem and its converse and use it to find the length of the missing side of a right triangle and the lengths of other line segments and, in some situations, empirically verify the Pythagorean theorem by direct measurement.

3.4 Demonstrate an understanding of conditions that indicate two geometrical figures are congruent and what congruence means about the relationships between the sides and angles of the two figures.

With the concept of area developed carefully for students, and assuming he fact that the sum of angles of a triangle is 180 degrees, one can give one of the simplest proofs of the Pythagorean theorem without having to appeal to any algebraic identities. Given a right triangle with sides of the legs being a and b and that of the hypotenuse being c, consider the following picture obtained by placing four identical copies of this triangle at the corners of a square whose sides have length $a + b$, as shown:



Since the straight angle at R_1 is 180 degrees, and the sum of the two non-right angles in each copy of the right triangle is (180 - 90) = 90 degrees, it follows that the interior angle of the quadrilateral inside the square is 90 degrees at R_1. A similar argument holds for the other vertices - each is 90 degrees - and, since each side of the interior quadrilateral has the same length c, it follows that the interior quadrilateral is actually a square. The area of the big square minus the areas of the four congruent right triangles is then the area of the inner square, which is c^2. Now rearrange the four triangles so that we have the following decomposition of the original square.



Observe that the area of the big square minus the areas of the *same* four congruent right triangles is now the sum of two smaller squares, one in the upper right and the other in the lower left, and is therfore $a^2 + b^2$. So $c^2 = a^2 + b^2$.

An example of a problem involving idealized measurements in geometry is the following:

> *We are given an isoceles triangle with height* 7 *and base length* 12. *What are the lengths of the remaining two edges?*

Here, one uses the Pythagorean theorem and a symmetry argument with respect to the angle bisector of the top angle to observe that the answer is $\sqrt{7^2 + 6^2} = \sqrt{85}$. This is an exact answer, and such an answer only occurs in the idealized world of mathematics. It would be typical to write the decimal approximation of the answer $\sqrt{85}$ as 9.21954, or more simply 9.22 after rounding, but the latter figures are *not exact*. If one were to draw an isoceles triangle with the given height and base, and then measure the length of one of the remaining edges, the measurement would give an answer of about 9.2 units if one is truly precise, so people tend to be quite content with the approximation of 9.21954 or 9.22 to the exact value of $\sqrt{85}$.

In summary, we repeat that approximations are characteristic of all real world measurements. They inevitably have errors. In most current texts, there is little if any attention paid to the distinction between the precise measurements in geometry and the approximate measurements that result when one actually measures real world objects. This results in serious confusion on the part of students.