The mathematics education reform:
What is it and why should you care?

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1 Introduction

When a mathematician is confronted with recent publications in school mathematics and calculus, the reaction is often one of shock and dismay (cf. e.g., [AN2], [AS3], [MU], and [W1]). The shock comes from the discovery that what passes for mathematics in these publications bears scant resemblance to the subject of our collective professional life. Mathematics has undergone a *re-definition*, and the ongoing process of promoting the transformed version in the mathematics classrooms of K-14 (i.e., from kindergarten to the first two years of college) constitutes the current mathematics education reform movement. As used here, "reform" refers to both the *K-12 mathematics education reform* and the *calculus reform*. This is appropriate because not only is calculus being taught in many schools these days, but the two reforms also share almost identical outlooks and ideology. (For an explanation of this fact, see for example [US].)

There are at least three reasons why mathematicians should know about the reform:

1. The Mathematical Association of America (MAA), the American Mathematical Society (AMS), and the Society for Industrial and Applied Mathematics (SIAM) are all on record as endorsing this new "vision of school mathematics". We owe it to ourselves to find out what this "vision" that we have apparently collectively endorsed is.

2. Over time, the reform will have an enormous impact on the entire undergraduate curriculum. This impact has already materialized in some institutions.

3. There are valid reasons to fear that the reform will throttle the normal process of producing a competent corps of scientists, engineers and mathematicians.

The modest goal of this article is to give an overview of the reform from the standpoint of a working mathematician and, in the process, supply enough details to make sense of the preceding assertions. Some possible courses of action are also suggested.

2 Special features of the reform

The reform is a reaction to the traditional curriculum of the eighties. The latter had defects that were obvious to one and all: it was algorithm-driven and short on explanations, much less proofs, and its over-emphasis on formalism tended to make it sterile and irrelevant. Because of inadequate textbooks and inadequate instruction, even Euclidean geometry became a liability rather than an asset in exposing students to logical thinking. Eventually the ills of this curriculum showed up in the alarming dropout rates in mathematics classes, particularly in the inner cities.

In response, the reform changes the content of the mathematics curriculum as well as the pedagogy and assessment in the classroom. This section describes some of the more disturbing changes; a more detailed discussion of the pros and cons of the reform curriculum vis-a-vis the traditional curriculum will be relegated to [W6]. The comments to follow are based on a representative sample of the better known documents and texts of the reform: [A], [CPM], [DAU], [EI], [HCC], [IM], [IMP1]–[IMP3], [MAF], [N1]–[N5], [NC], [PEL], [SCA], [SE], [UN] and [WAP]. By lumping them together, by no means do I wish to imply that these documents or texts all come from the same mold or are of comparable quality, but they do include some of the best the reform has to offer. For instance, the NCTM Standards [N1] is the leading document of the K-12 reform while the Harvard Calculus [HCC] is commonly recognized as the flagship of the calculus reform effort. Moreover, the Interactive Mathematics Program [IMP1]-[IMP3] has been cited by Luther S. Williams, the head of NSF's education and human resources directorate, as an example of the kind of effort that can lead the U.S. in achieving "the national education goal of global preeminence in math and science" (news release dated Novemeber 20, 1996). It should also be noted that not *all* recent texts both in K-12 mathematics and in calculus share the same mathematical liabilities as those discussed below. Two examples come to mind in this regard: some of the K-12 modules from Education Development Center (Newton, MA) and the *Calculus* of Arnold Ostebee and Paul Zorn (Saunders, 1997). What is true is that these liabilities are common enough among these documents and texts as to be easily recognizable and hence worrisome. *The examples used below have been chosen partly because they are easily understood, are by no means isolated anomalies, and, generally faithful to the tenor of their respective sources.*

Proof-abuse

The first area of concern may be termed *proof-abuse*. One takes for granted that certain theorems are not proved in elementary courses, but one would also take for granted that students are never misled into believing that a plausibility argument is equivalent to a proof. The line between what is true and what appears to be true but is not true must not be crossed in a mathematics education worthy of its name.

The cavalier manner in which the reform texts treat logical argument is nothing short of breath-taking. Heuristic arguments are randomly offered or withheld and, in case of the former, whether these are correct proofs or not is never made clear. Rather than being the underpinning of mathematics, logical deduction is now regarded as at best irrelevant. The following examples from the mathematics of K-12 serve to illustrate this point.

The pre-calculus text [NC], highly praised by some (cf. [TR]), defines the inverse of a square matrix R as a matrix S so that $RS = I$ (p.259). Then it immediately asserts: "It is also true that $SR = I$". Why? It does not say, and does not discuss the uniqueness of the inverse either. But then it says: "The inverse of R is symbolized by R^{-1}, so that $R^{-1} = S$ and $S^{-1} = R$." The gaps hidden in these assertions are never mentioned, much less filled. The 694-page text [SE] on synthetic Euclidean geometry offers not a single proof in its first 562 pages. Finally on p.563, a (poor) presentation of axioms and proofs is begun. (All the teachers that I have consulted said that, in two semesters, they almost never got to p.563.)

The text [IMP1] of the Interactive Mathematics Program adopts a different tack: it offers crude plausibility arguments alongside correct proofs but never states which is which. The same is true of the more recent incarnation [IMP3].

The 9th Grade text *Algebra I* of College Preparatory Mathematics [CPM] makes students verify, by the use of calculators, that for a few choices of integers M and N, $\sqrt{M \cdot N} = \sqrt{M} \cdot \sqrt{N}$ (pp.19-21 of Unit 9). Then, without missing a beat, it asserts that the identity holds in general.

In other words, at Grade 9 level, verifying a few special cases by calculator is equated with understanding. This is clearly an accepted way to teach mathematics nowadays because such examples abound. For instance, in the 8th Grade textbook of the widely used Addison-Wesley series [EI], p.396, students are told that if a number is not a perfect square or a quotient of perfect squares, then its square root is an irrational number (non-repeating and non-terminating decimals). Why? Because one can check this on a calculator. It should come as no surprise therefore that a teacher in Chicago concluded that $\frac{5}{17}$ was irrational because the calculator display of its decimal equivalent showed no pattern of repetition. (The period of repetition of this fraction is 16, but most calculators do not even display 16 digits.)

Let us turn to two examples from calculus. On p.31 of *Approximations* of [DAU], the discussion of finding the power series expansion of $1/(1-x)$ goes like this: One uses the computer to print out the first 50 terms, and of course the print-out reads: $\sum_{n=0}^{50} x^n$. The comment that follows is (p.32):

> Ain't no doubt about it. The expansion of $f[x] = 1/(1-x)$ in powers of x is $1 + x + x^2 + x^3 + \cdots + x^k + \cdots$.

By using computer print outs to replace the elementary derivation of the geometric series, [DAU] and other like-minded texts send out the unmistakable message to K-12 that learning about the geometric series is no longer something of consequence. (This message is echoed on p.181 of the *NCTM Standards* [N1].) A second example is the reasoning given to support (the weak form of) the Fundamental Theorem of Calculus (FTC) in the Harvard Consortium Calculus [HCC], p.171. The following is essentially the complete argument.

> Given F defined on $[a, b]$, partition the latter into n equal subdivisions $x_0 < x_1 < \cdots < x_n$ and let the length of each subdivision be Δt. Then for n large, the change of F in $[t_i, t_{i+1}]$ is approximately $\Delta F \approx$ Rate of change of $F(t)\times$ Time $\approx F'(t_i)\Delta t$. Thus

> the total change in $F = \sum \Delta F \approx \sum_{i=0}^{n-1} F'(t_i)\Delta t$. But the total change in $F(t)$ between a and b can be written as $F(b) - F(a)$. Thus letting n go to infinity: $F(b) - F(a) =$ Total change in $F(t)$ from a to $b = \int_a^b F'(t)dt$.

What is remarkable here is not that no proof is given of such a basic result, but that there is not the slightest hint that the preceding is not a proof but a plausibility argument containing huge gaps.

The above examples are by no means the results of random decisions by individual authors. The Interactive Mathematics Program justifies its expository policy concerning proofs as follows: "... secondary school is [not] the place for students to learn to write rigorous, formal mathematical proofs. That place is in upper division courses in college" ([IMP2]). This sentiment is echoed in the *NCTM Standards* [N1] which hold the opinion that for high school students who do not go to college, "reasoning" rather than proofs should be employed in the teaching of mathematics. So what is the difference between reasoning and proof?

> "... reasoning is the process of thinking about a mathematical question; a justification is a rationale or argument for some mathematical proposition; and a proof is a justification that is logically valid and based on initial assumptions, definitions, and proved results." ([N4], p.61)

In a recent article [PEL] in *Mathematics Teacher*, it is proposed that "the trigonometry teacher can use the graphing calculator in teaching identities". Thus, graphing $\sin 2x$ and $2\sin x\cos x$ and finding that the two graphs coincide on the calculator screen have the supposedly beneficial effect of letting the students avoid "the rote method of pencil and paper" and actually "*see* an identity".[1] (Note that the journal in which [PEL] appeared is an official journal of National Council of Teachers of Mathematics.) Finally, Jerry Uhl offered the following explanation for the absence of proofs in [DAU]: "To coax the students into proof, we call them explanations, but competent mathematicians will recognize most of our explanations as informal (but correct) proofs." It may be relevant to point out that students who read [DAU] are generally not competent mathematicians.

[1] If the authors had said that "*in addition to proving the identity* $\sin 2x = 2\sin x\cos x$*, the graphing of the two functions on a calculator can enhance the students' confidence in the abstract argument*", we could have applauded them for making skillful use of technology in the classroom.

Fuzzification of mathematics

A second area of concern in the current reform is the *fuzzification* of mathematics. Precision is a defining characteristic of our discipline, but the present tendency is to move mathematics *completely* back into the arena of everyday life where ambiguity and allusiveness thrive. One way of fuzzifying mathematics is by intentionally giving incomplete information in the formulation of problems. Thus a so-called *exploration* in [EI] (p.174 of the 7th Grade text) says: "The graph below shows the number of newspapers sold at a newsstand at different times of the day." The graph relates "time of day" to "numbers of papers". Although the domain of the graph goes only from 6 a.m. to 8 p.m. and the graph itself is irregular, one of the questions concerning this graph is: "Predict how many papers would be sold at [sic] 9 p.m.?[2] Explain." One may *guess* that the number of papers sold might decrease further after 8 p.m., but any prediction is out of the question.

There are many other examples of this type. The 1992 California Mathematics Framework ([MAF]) suggests the following problem (p.26):

> The 20% of California families with the lowest annual earnings pay an average of 14.1% in state and local taxes, while the "middle" 20% pay only 8.8%. What does that mean? Do you think it is fair? What additional questions do you have?

These are supposed to be questions in mathematics, and since mathematics does not deal with undefined quantities, "don't know", "don't know" and 'none" are the only possible mathematical answers on the basis of the given data. What is at the heart of such fuzzification is the deliberate attempt to ask questions so vague that students would feel comfortable in tendering partial answers. While this educational strategy can claim obvious short-term advantages — it may boost students' self-esteem, for instance, — it has a pernicious cumulative influence in the long run in shaping both students' and teachers' perception of mathematics. See p.122 of [W2] for a concrete example of its effect on some teachers. ([W2] has a discussion of other problems of this type.)

Slighting of basic mathematical techniques

A third general area of concern is the *slighting of basic mathematical techniques*, especially symbolic computations and formulas. A few examples should suffice to convey the overall picture: the pre-calculus text [NC]

[2] Since there is no such things as "150 copies sold *instantaneously* at 8 a.m.", could the the authors have in mind "the number of newspapers sold *per hour*"?

spends two pages (pp.209-210) discussing the relationship between the measurements of an angle in degrees and radians, but does not give the simple formula relating the two; [NC] does not discuss the geometric series or the binomial theorem; the Interactive Mathematics Program does not discuss the geometric series, and discusses the quadratic formula only in the 12th Grade (cf. also [IMP1]); the calculus text [HCC] does not mention L'Hôpital's rule or the convergence and divergence of infinite series; the 9th Grade text of [CPM] treats the multiplication of two linear polynomials (Units 6 and 9) by the geometric method (cutting a rectangle into four pieces) and by the infamous FOIL algorithm, but *not* by the distributive property, although in an earlier unit (Unit 4) the distributive property is discussed in connection with the expansion of $a(x + b)$ and students are told that the method is "powerful".

Obsession with relevance and real world applications

A fourth area of concern is the current *obsession with relevance and "real world applications"*. The whole IMP curriculum revolves around real world problem, for example ([IMP1] and [IMP2]). The danger of organizing mathematics around such applications is twofold. The *uncertainty in interpreting the data*, which then leads to a multiplicity of possible solutions, is often confused with an intrinsic indeterminacy in mathematics, and an over-emphasis on real world applications robs mathematics of its coherence and internal structure. These dangers, especially the second one, have not been altogether avoided in all the texts cited at the beginning of this section. Thus all the more reason to be alarmed by extreme positions, such as that taken by the Consortium of Mathematics and Its Applications (COMAP), which has been funded by the NSF to develop a complete mathematics curriculum for 9-12: "In ARISE [the projected series of school texts by COMAP], the mathematics truly arises out of applications. The units are not centered around mathematical topics but rather application areas and themes, with the mathematical topics occurring as strands throughout the unit" ([A]).

With the new curriculum comes new pedagogy. At least four points of this pedagogy are worthy of a brief discussion:

1. Over-reliance on the so-called constructivistic educational strategies.
2. Misuse of technology.
3. De-emphasis of drills and the role of memory in some texts.

4. Over-emphasis on the "fun" component of learning without mentioning hard work.

Constructivism is the bedrock on which the whole reform movement, especially the three NCTM volumes [N1]-[N3], rests. Roughly, this is the education philosophy which holds that the acquisition of knowledge takes place only when the external input has been internalized and integrated into one's own mind. Thus learning requires the construction of a mental image in response to the external input. So far so good, except that current proponents of constructivism go further and stipulate that classroom time should be used for the students to re-discover or re-invent the concepts or the methods of solution in order to help along this mental construction, and that the best way to facilitate this process is through group work. In the words of one mathematics educator, "the preeminent characteristics of the present reform effort in school mathematics" include "students frequently working together in small cooperative groups" ([DA]). The teacher is no longer "the sage on the stage" but only "a guide on the side".

While a little bit of group learning and guiding-on-the-side is good in the classroom, too much poses an obstruction to effective dialogue between teacher and students as well as to the efficient transfer of knowledge from teacher to students. In such a climate, gone is the possibility that the teacher can share with students his or her insights or warn them against pitfalls, or that students can learn enthusiastically from their teacher in class and do the mental construction *at home* — with or without a group of friends. Right now *all* the learning must take place instantly in school. But can any substantive mathematics be learned this way? Perhaps the following comments from a high school teacher would shed light on this issue:

> I have seen students put in small groups to measure the radius and circumference of circle after circle, then discuss finding in their small groups, write up their findings, share with the class, and then have the teacher acknowledge the existence of π (as a constant ratio of circumference to diameter). While the experiential approach has some merit, should it really take three class periods for the students to come to such a minimal understanding of the concept? I'd rather find a way to get that concept across to them in 20 minutes (maybe even less) and use the remaining time to discuss π's irrationality, the formula for the area of a circle, the history of man's efforts to determine π precisely, etc. Give me three class periods and my students will have covered circles, cylinders, cones (volumes, surface areas, etc.). As I

> visit the classroom of some of my constructivist colleagues, I see far too much time going to waste. Do students feel good about these classes? Maybe, maybe not. Are they learning very much? I fear not.

The psychological foundation of the theory of group learning should also be mentioned, if only briefly. With such strong advocacy of this particular method of instruction from the reformers, one would believe that its superiority over direct instruction has been firmly established by research data from large-scale studies in the field of cognitive psychology. However, no such data exist and the available evidence even appears to imply that direct instruction is the more effective method of the two. See [GR], especially the discussion of Project Follow Through and the reply of NCTM to a parent's inquiry.

The abuse of technology is evident in some of the examples cited above. Fingers-on-the-buttons has now replaced engagement-of-the-brain with increasing frequency. I should also call attention to the *insistence* in educational documents that students in K-4 be allowed to have access at all times to the calculator ([MAF], pp.57-59; [N1], p.19; [N5], p.vi; [UN]). In theory, giving children an extra tool in the form of a calculator can do nothing but "empower" them mathematically, to use a term that is popular these days, but what if the theory is not born out by hard facts? In anecdote after anecdote, one hears horror stories of the calculator-enriched generation (cf. the Vignettes in [AN2]), but the personal experience of a respected educator would perhaps speak more eloquently on this issue. In the spring semester of 1996, Leon Henkin volunteered to assist in some pre-calculus classes in a local high school in Berkeley which make use of graphing calculators. In his own words ([HE]):

> ... when I first saw what [the calculator] can accomplish I was awed and excited. ... However, after having spent 3 or 4 weeks in the class and seeing how, in practice, the calculator is actually used in the class, I have now concluded that it is about the largest obstacle to their gaining an understanding of the mathematical ideas of the course. The reason is that they have come to rely completely on the calculator to do arithmetic, as well as elementary algebraic calculations. If you ask them to estimate the slope of a function at a certain point when they are looking at the graph, they will punch in four numbers and calculate the difference quotient.. and if you put your hand on the calculator

> to prevent them from doing so, they will assure you with all their might that they cannot multiply and divide without it.
>
> Obviously, this enormously stunts their ability to use graphs in an intuitive way to make conjectures and gain ideas about a problem whose solution they are seeking.

While such experience in high school may not be directly relevant to K-4, it should nevertheless give one pause about the supposed beneficial effects of calculators in general. No one denies that calculators and computers are essential in certain aspects of mathematical instruction, but in the absence of any long-range scientific study of their impact on students, their use in the classroom needs to be accompanied by a great deal of circumspection. Such circumspection seems not to have been exercised thus far.

The de-emphasis of drills and of the need — even in mathematics — to commit certain basic facts and concepts to memory is the natural pedagogical counterpart of the slighting of basic skills and formulas in the curriculum. (Note however that [CPM] has an abundance of drills and exercises,—although one may argue whether the drills and exercises test the desirable skills—and that the number of drills in [HCC] and [NC] would seem to be adequate.) If indeed drills are to be eliminated (cf. [IMP1], [IMP3]), what would replace them to insure the acquisition of technical fluency, so essential in mathematics? A more sensible solution would be to make up better drills in the manner of Chopin, who wrote 24 *Etudes* solely for the purpose of giving pianists better finger exercises. Yes, Chopin created great music in the process, but the important thing is to note that he did not advocate the abolition of finger exercises.

A battle cry of the reform is "Mathematics for all!" In an attempt to make this come true, there is presently a conscientious effort to spread the news that "Math is fun!" While applauding the good intention, we nevertheless must ask whether the constant repetition of this slogan like a mantra helps students learn mathematics. Have students been told that this kind of "fun" includes the fun of working hard to solve difficult problems? Nothing good comes cheaply, and learning mathematics is no exception. We owe it to the students to tell them honestly about the hard work needed to learn mathematics. One mathematician did exactly that. Gelfand wrote in the Foreword to [GE]: "This book, along with the others in this series, is not intended for quick reading. Each section is designed to be studied carefully . . . And if it is difficult for you, come back to it and try to understand what made it hard for you." Alas, there is nothing resembling this in the reform literature.

The justification of some of the practices of the reform, such as the emphasis on real world applications (with its corollary of de-emphasizing abstraction) and the slighting of technical skills, is sometimes laid at the doorstep of cognitive psychology, which purportedly shows that knowledge cannot be decomposed or decontextualized for the purpose of instruction. However, three cognitive psychologists, J.R. Anderson, L.M. Reder and H.A. Simon, have recently challenged such applications of psychology to mathematics education ([AND1], summarized in [AND2]; see also [STO] for a historical perspective). They refute the thesis that "cognition cannot be analyzed into components" and argue instead that "component skills are required in learning". Among their conclusions, the following memorable passage is worth quoting:

> In fact, as in many recent publications in mathematics education, much of what is described ... reflects two movements, "situated learning" and "constructivism", which have been gaining influence on thinking about education and educational research. In our view, some of the central educational recommendations of these movements have questionable psychological foundations. We wish to compare these recommendations with current empirical knowledge about effective and ineffective ways to facilitate learning in mathematics and to reach some conclusions about what are the effective ways. A number of the claims that have been advanced as insights from cognitive psychology are at best highly controversial and at worst directly contradict known research findings. As a consequence, some of the prescriptions for educational reform based on these claims are bound to lead to inferior educational outcomes and to block alternative methods for improvement that are superior.

3 A little background

One can say, with only a slight exaggeration, that the launching of the Soviet *Sputnik* in 1957 also launched the New Math movement of the sixties. (Cf. Chapter 14 of [HI], [RA], or pp.172-175 of [BO] for a brief history.) The New Math was masterminded by mathematicians and, as is well-known, its over-emphasis on sets, abstractions and rigid formalism ("Write the numeral that names the number solving $3x - 7 = 8$.") and its concomitant failure to adequately prepare the teachers for the abrupt shift in content knowledge eventually led to its demise. The pendulum then swung all the way to the mindless drills and algorithms of the Back to Basics movement in the mathematics education of the seventies. By the end of the seventies, deterioration of the schools became too obvious to ignore when catchy phrases such as "Why Johnny can't read" or "Why Johnny can't write" had become part of Americana. The Department of Education then charged the National Commission on Excellence in Education to report on the state of American education. In 1983, the Commision issued *A Nation at Risk: The Imperative for Education Reform* ([NAR]). This slim volume is remarkable for its incisiveness in detailing the ills of the whole education system. As its title suggests, it calls for reform in education.

Responding to the challenge of [NAR], the National Council of Teachers of Mathematics (NCTM) convened its first meeting in 1986 for the purpose of drafting a reform document. The *NCTM Standards* ([N1]) is the outcome.

Such an abbreviated account of the genesis of the current reform is of course over-simplified. *A Nation at Risk* may have spawned the idea of reform, but what ultimately brought it to reality was the business community. When the poor performance of high school graduates in the high-tech workforce began to hamper the development of industries, especially the high-tech industries, business leaders took note. Workers are needed who are capable of more than memorizing a few formulas and cranking out numbers —computers can do that, and do it better. Many such workers are needed, yet barely one in five U.S. workers holds a four-year college degree (see [FS]). One can get an idea of the important linkage of the reform to the work place by reading the papers [FS] and [STF]. As we proceed to look more closely at the *NCTM Standards*, this linkage will be seen to assume a dominant rôle.

4 The manifesto of the reform: the *NCTM Standards*

"*How can we lose when we are so sincere?*"
Charlie Brown

The errant mathematical texts and the mathematical anomalies alluded to in §2 would not by themselves a reform make, had not the *NCTM Standards* [N1] given the reform a voice and an identity. The impact of [N1] has been spectacular. Ever since 1989, when [N1] appeared, one would be hard pressed to find a research paper in mathematics education, an education document, or a school mathematics textbook that does not pay at least lip service to [N1]; and this includes the so-called "traditional" texts such as [RH]. The *NCTM Standards* are the reform.

The goal of [N1] (and the companion volumes [N2] and [N3]) was to redress all the ills of the "traditional" curriculum, some of which were discussed at the beginning of §2. In order to overturn all this in one fell swoop, NCTM came up with a new curriculum, a new pedagogy and new assessment techniques ([N1]–[N3]). Here we shall concentrate mainly on the *NCTM Standards* [N1].

By and large, it is probably not controversial to assert that the curriculum of [N1], if carefully executed, would remove the mindless drills, the dry memorization and much of learning-by-rote. It is more successful than the traditional curriculum in reaching out to the lower 50% of the (mathematics) students. Those who used to be turned off by math now find the *Standards*-inspired materials much more "user-friendly". The *Standards* have also brought renewed enthusiasm to some teachers and made others think hard about education again. Yet, even in the midst of such gains, one senses trouble.

The most common criticisms of *NCTM Standards* [N1] are its inflated prose and its carelessness in the formulation of various recommendations. This carelessness has inspired mischief and lent legitimacy to mischiefs already committed. For example, [N1]'s advocacy of "open-ended" problems "with no right answer"; its unfounded faith in the propriety of technology in mathematics education beginning with *kindergarten* ("There is no evidence to suggest that the availability of calculators makes students dependent on them for simple calculations"); its recommendation to give "decreased attention" to "two-column proofs"; its repeated admonition to downplay "memorization of facts" and "memorization of formulas"; etc. These have all been cited, rightly or wrongly, to justify some or all of the practices described in §2. However, the most substantive defects of the *Standards* are *global* in nature, in the sense that they are not tied to a particular chapter or verse, but concern the sum of all the parts of [N1].

A first major defect lies in its insistence that (real world) problem solving must be the focus of school mathematics ([N1], p.6) *and* the particular way this decision is implemented. A second major defect is that the *floor* of mathematics education has been set far too low (cf. [N1], p.9). The *ceiling*, which is described as "the NCTM Standards for College-Intending Students", is consequently also dragged down. A third major defect is [N1]'s failure to confront the pressing issue of how one single mandated curriculum can be used for *all* students, no matter their mathematical capabilities. All these defects are inter-related.

The eleven-page Introduction of [N1] sets the tone: this is to be a curriculum with an uncompromising emphasis on producing a "technologically competent work force". Subsequent discussions and examples are rooted in this *leitmotiv*. Readers are reminded at every turn that mathematics is a powerful tool to solve real world problems. In contrast, one encounters litte discussion of the need for *mathematical* developments of a mathematical idea. Bluntly put, the *Standards* read like a vocational-training manual that casts occasional side-glances at students' intellectual development. (The discussion in the classic treatise [BE] on this topic is very relevant here.) When the New York Times carried news of the proof of Fermat's Last Theorem (FLT) on the front page, it was acknowledging the fact that there exists an intellectual component to mathematics that even laymen (including high school graduates) must reckon with. Sadly, students coming out of the *Standards* curriculum would have little idea why a quaint statement such as FLT should be of interest to anyone (especially since it obviously holds up when a few integers are plugged into the Fermat equation on a computer).

The preceding paragraph is not an indictment of application-oriented curricula *per se*, only of the particular way [N1] wants such a curriculum implemented: it allows the utilitarian impulse to overwhelm the basic educational mission, with the result that basic ideas and skills not directly related to the so-called real world problems often get left out. By yielding to the temptation of "doing just enough to get the problems solved", the curriculum of [N1] ends up presenting a fragmented and amorphous version of mathematics. What the *Standards* should have done is to bring the idea of *mathematical closure* to the forefront. In other words, if certain tools are developed for the purpose of solving a particular problem, then the solution needs to be rounded off with a discussion of the tools themselves in the context of the overall mathematical fabric. How are they related to other mathematical techniques and concepts? Are they part of a general structure? Is the idea behind them applicable to completely different situations as well? And so on. Unfortunately, the idea of mathematical closure is never broached in [N1].

For the sake of argument, let us take an approach to teaching classical music in school analogous to that used by [N1] to teaching mathematics. Then classical music would be presented only as it serves commercial purposes. The greatness of Beethoven may have to be authenticated by facts such as his 9th Symphony having been used in the Huntley-Brinkley Show and the Beatles' movie *Help!*. Rossini's worth would be shown via *The Lone Ranger*, while Richard Strauss would be immortalized by *2001* and Mozart by *Elvira Madigan*. And so on, *ad nauseum*. No doubt this would make classical music accessible to more students than ever before, and we may even talk ourselves into believing that we have achieved the goal of "Classical music for *all* !" But is this really *all* we want to get out of a classical music education? Why then should we allow the same thing to happen to mathematics?

When an educational document consistently presents mathematics as a toolshed instead of the edifice that it is, its wholesale revision is overdue.

A simple example would perhaps clarify the meaning of mathematical closure. On p.152 of [N1], there is a discussion of the problem of finding the roots of the cubic $5x^3 - 12x^2 - 16x + 8 = 0$ in the context of Grades 9-12. In the view of NCTM, all that the best of the non-college-bound students need to know about this problem is summarized in the following paragraph (my paraphrase of Level 4 on pp.152-3):

> Assign students to a group project of constructing an algorithm for approximating the real roots, such as the bisection algorithm. Pay special attention to the proper expressions used to record this algorithm. Once this is done, test the procedure by computer implementation.

That is all: just a computer procedure to approximate a real root. From the narrow perspective of treating mathematics as a tool to solve real life problems, this is of course sufficient. However, from the point of view of *mathematics*, shouldn't a student be interested in roots of polynomials in general? Fourth degree? Odd degree? Other roots, once one is found? Rational roots? Total number of roots?

Not every detail need be explained, but even the average student will have his life improved by the mere knowledge that there are such questions, often with answers, e.g., that the factor theorem and the quadratic formula predict that the above cubic will be completely solved (approximately) once a single root is found. Students would benefit from the exposure to this bread-and-butter kind of mathematical thinking along with the basic technical skills that are developed in the process. This is the *floor* appropriate for all students in this particular instance. Why is the acquisition of this kind of "higher order thinking skill" not the compelling message of the *Standards*?

Now part of the preceding mathematical discussion is indeed in [N1] (Level 5 on p.153), but (one infers) is reserved in [N1] for what it calls the "college-intending students". It hardly seems reasonable that such simple-minded *mathematical* deductions and questions should not be made available to all students. Furthermore, the ceiling in this case is set far too low. Except for the Fundamental Theorem of Algebra, proofs of most properties of polynomials should be given and even the Cardan formulas and other intellectual triumphs such as the works of Abel and Galois should be discussed to indicate that mankind does not always think of mathematics exclusively as a tool for solving real world problems.

Let me give one more short example of the inappropriateness of either the floor or the ceiling set by [N1]. Consider the comment on p.165 of [N1]:

> "College-intending students also should have opportunities to verify basic trigonometric identities, such as $\sec^2 A = 1 + \tan^2 A$, since this activity improves their understanding of trigonometric properties and provides a setting for deductive proof."

If the proof of such a trivial consequence of $\sin^2 A + \cos^2 A = 1$ is now reserved only for college-intending students, how does NCTM expect the average high school graduate to understand $\sec^2 A = 1 + \tan^2 A$? To graph ($\sec^2 A$) and ($1+\tan^2 A$) separately and observe that the two graphs coincide, as suggested in [PEL]? With this in mind, we find it hard to believe that "the mathematics of [the *Standards*'] core program is sufficiently broad and deep so that students' options for further study would not be limited" ([N1], p.9).

In education circles, a great merit of the *Standards* is seen to be its success in producing a curriculum for *all* students, including all those "who aren't getting it in math". In brief, the coded message behind these words is the elimination of *tracking*, the separation of school students into different classes according to ability. True enough, [N1] sets the floor and describes the ceiling, but it stops short of describing how to implement this abstract idea of addressing both the floor and the ceiling of a curriculum in the same classroom except to offer the disclaimer that "it does *not* imply that students of all performance levels must be taught in the same classroom" (p.130 of [N1]). What it should have done is to make a clearcut recommendation for a *program-with-choices* for the last two or three years of high school. Mathematics courses should bifurcate in those years into the general track and the scientific track, and students should be allowed to choose between the two, the same way college students are in choosing among different kinds of calculus classes. It is a matter of record (according to Zalman Usiskin) that no other developed country practices one-curriculum-for-all in the last years of high school.

What has been happening to reform curricula in the absence of a clear directive for program-with-choices is that the floor takes precedence, the ceiling gets ignored, and the serious and gifted students end up being short-changed. This problem is becoming so serious that it has alarmed not only educators and psychologists (cf. [RE], [GA], and also [PEN]), but also the U.S. Department of Education. Partly in response to the reform, the latter addresses the "quiet crisis" of our neglect of the top students in the refreshingly straightforward document [NA]. I will simply quote a passage from [NA] to serve as a critique of the *Standards*:

> Ultimately, the drive to strengthen the education of students with outstanding talents is a drive toward excellence for all students. Education reform will be slowed if it is restricted to boosting standards for students at the bottom and middle rungs of the academic ladder. At the same time we raise the "floor" (the minimum levels of accomplishment we consider to be acceptable), we also must raise the "ceiling" (the highest academic level for which we strive).

One final comment on [N1]–[N3] may not be out of place here. Any attempt to improve mathematics education must address at least three main issues: to insure that the teachers can do justice to mathematics, to induce students to work hard, and to improve the curriculum and assessment methods. The documents [N1]–[N3] deal with the last of the three, while the first two have fallen by the roadside. However, the critical importance of students' willingness to learn in any meaningful discussion of education has not been overlooked by people outside the reform. The articles [AN1], [AN2] and [BA] (among others) are powerful reminders of the folly of ignoring the student factor in the present reform. One would like to respectfully suggest that the NCTM Standards on assessment and teaching ([N2], [N3]) should waste no time in confronting this topic head-on in their forthcoming revisions. In addition, teachers' inadequate knowledge of their subject was in fact a main concern of *A Nation At Risk* ([NAR]) and is at the root of many serious problems in mathematics education. Yet, in its 195-page volume on teaching ([N2]), NCTM — the National Council of *Teachers* of Mathematics — saw fit to devote only 8 pages (pp.132-139) to this most pressing of all instructional issues. More damaging is the fact that the vignettes in [N2] all seem to point to the failure of pedagogical methods as the root cause of poor performance in mathematics instruction, whereas even a casual inspection of the typical classroom would convince an observer that the lack of a firm grasp of mathematics is most often the culprit. The negligence is the more surprising because the curriculum of the *Standards* in fact makes a greater demand on the teacher's command of the subject matter than the traditional curriculum. Teachers who are already having difficulty with the old curriculum would be even less prepared for the new tasks NCTM sets for them.

Jack Price's Presidential Address at the 1995 Annual NCTM Meeting ([PR]) included a spirited advocacy for mathematicians' support of the *Standards*: "We already have consensus from the major mathematics organizations ..." Price was referring to the statement on p.*vi* of [N1]: "This document is significant because it expresses the consensus of professionals in the mathematical sciences for the direction of school mathematics in the next decade." Further down the page, one finds that AMS, MAA and SIAM are listed as *Endorsers*. This is how we mathematicians enter the picture in education.

5 Why it matters

The most obvious reason why school mathematics education should matter to university professors is that a continuing influx of mathematically incompetent students would decimate the university mathematics curriculum. One can look no further than the United Kingdom to have one's worst fears confirmed. If the report [TM] released by the Council of the London Mathematical Society in October, 1995, is to be trusted, then the UK is some five years ahead of us in a mathematics education reform remarkably similar to our own in its rhetoric. If our reform takes hold, then according to [TM], we can look forward to a generation of students with:

> (i) a serious lack of essential technical facility – the ability to undertake numerical and algebraic calculation with fluency and accuracy;
>
> (ii) a marked decline in analytic powers when faced with simple problems requiring more than one step;
>
> (iii) a changed perception of what mathematics is – in particular of the essential place within it of precision and proof.

To the cynics who say that since such deteriorations in the mathematical preparation of our incoming freshmen have been taking place for so long that it really doesn't matter anymore, one can only say: "Wait till you see what happens next." For example, to the charge that the Harvard Calculus [HCC] was written with a view to pass students through calculus without requiring any algebraic skill, one reply ([PH]) was that given that the students' "symbolic manipulative skills are much weaker than they used to be", the [HCC] curriculum "makes a great virtue out of this necessity [by] eliminating some of the symbolic manipulation from calculus". In the same vein, when the question was raised in a discussion of reform calculus as to whether it was "advocating passing students through calculus with at best a rudimentary knowledge of algebra", the comment from a reformer was that "we were doing this long before calculus reform" ([MCC]).

What we are witnessing here is symptomatic of much of the recent trend of appeasement in education: instead of trying to uphold a certain standard and help mold as-yet-unformed minds, educators simply accept the deteriorations in the classroom as a given. It would only be a small step to apply such a philosophy in earnest to demand a total revamping of undergraduate, and even graduate mathematics programs in order to fit the needs of the new generation. Is such a statement sheer fantasy? Hardly. You think what is taking place in K-12 and calculus does not conform to your conception of mathematics? A remedy has already been proposed: "Change the first two years of collegiate mathematics to match the new K-12 curriculum." ([KA]) Not coincidentally, the opening statement of the Precalculus Project of the Calculus Bridge Consortium Based at Harvard University echoes this sentiment word for word: "Given the success of the reform calculus movement, students and teachers want reformed courses both preceding and following calculus" ([BCH]). More to the point, the mathematics department of a major state university has begun to revised all the upper divisional courses (as of May, 1996) in order to "mesh with the aftermath of the [HCC] reform". Other such examples date as far back as 1993.

But all these allusions to what was said or what has happened are quite unnecessary. The logic of the reform has an inexorability all its own: once the reform is entrenched in K-12, the university courses would have no place to go but to follow suit. Another induction step would result in a demand for reform in the graduate program. Thus in no time at all, the burning question of the day will be: are proofs allowed only in graduate courses?

From a broader perspective, the reason we must object to the reform is that it threatens to bring down the whole education system. Indeed, our students of today will be the teachers of tomorrow, so when the university courses start to deteriorate our children will be taught by teachers who are mathematically worse-equipped than those of today. Then the next wave of students will perform even more poorly, and the poor performance will incite the educators to demand a second mathematics education reform. And the vicious circle will continue. Lest such worries be construed as just paranoia, let me quote a recent (1996) report from the organizer of a workshop for high school mathematics teachers in a Western state:

> "In the afternoon we started talking about the state of students' preparation for calculus and all of them said it is getting worse

> year by year. One of them spoke to me afterwards and said the sentiments expressed at the workshop were echoed in his school by all the other teachers there. There is no question that the backlash has started, and it is gaining strength . . . The picture they painted for me was one in which they (the teachers) are nearly powerless to prevent what they see as a watering down of the curriculum because administrators, untrained in mathematics, are making the decisions based on reports filled with what they describe as NCTM jargon. One teacher said that next year, at his school, every freshman will take the same math course, regardless of background or ability. He predicts that there will be no calculus course in three years because no one will be ready for it. They all say that what used to be called pre-algebra now is called Alg I, and on up the line."

The reform also raises a grave concern in a different context. The economic and social well-being of our nation is critically dependent on the existence of what might be called a robust upper middle class in science and technology. Above and beyond the presence of a high-tech workforce, this nation must insure a continuous supply of competent mathematicians, scientists and engineers in order to stay competitive in the world of the 21st century. A good mathematical training for those school students gifted in science and mathematics provides the foundation on which this scientific-technological class rests. (Again, see the Department of Education document [NA].) Because the reform damages the "ceiling" of the school mathematics curriculum, this foundation is put in jeopardy.

Some of the reformers are of course aware of this problem. For example, Carole Lacampagne worried about "the trend to downplay formal proofs in the schools in favor of communicating ideas and understanding" in the *Summary* of [LA]. It is incumbent on mathematicians to alert the public to the danger of a watered-down curriculum, because, if we don't, who will?[3]

6 What mathematicians can do

In 1962, at the height of the New Math, 75 leading mathematicians (including Lars Ahlfors, Richard Brauer, Marston Morse, George Polya and André

[3] Consider what one Pennsylvania teacher wrote: "The 'other side' is making it very uncomfortable for teachers such as me, and we are dropping like flies. Whereas university professors like you can disagree with impunity, that same privilege is denied to those of us lower on the scale."

Weil) published an open letter in the American Mathematical Monthly to gently chide the New Math for its excesses ([MC]). Thirty-four years later, few if any in the research community know about this salutory statement. The public regards the New Math as the mathematicians' blunder, period.

Faced with the current mathematics education reform, once again some mathematicians are beginning to speak out ([AL], [AN1]–[AN3], [AS1], [AS2], [CU], [ES], [HAI1]–[HAI2], [KL], [KO], [MU], [RO], [ROS], [SCO], [W3] and [W7], among others). If this effort is not to be forgotten thirty-four years from now, as [MC] was forgotten, words must now give way to action. Before embarking on any action, however, a little reflection is in order.

We should first ask ourselves what brought about the present unhappy state of affairs. Our collective indifference to education (cf. [Z]) has allowed the traditional K-12 curriculum and the teaching of calculus to deteriorate, thereby opening the floodgate to a multitude of educational ideas of dubious merit. The reform is the natural product of this indifference.

If we wish to shake off this indifference and enter into a discussion of mathematics education, then we have to enlarge our vision concerning the teaching of mathematics. We have to temporarily abandon the narrow focus of training future mathematicians and embrace the broader and more complicated issue of educating students who have diverse goals in life. We must also learn about the reality in schools where teachers are habitually overworked and have not the luxury of intellectual contemplation. Criticisms of the reform which do not take into account these deviations from our normal "universe of discourse" are not likely to find a receptive audience.

In discussing the reform, we also have to be aware of the existence of the many serious defects in the generic traditional mathematics curriculum in the schools. It would not do to pretend that a return to "business as usual" would be a cure-all. A more extended discussion of the traditional curriculum, together with a direct comparison with the reform curriculum, can be found in [W6].

One last thing we need to be aware of is that, although our professional instincts compel us to insist that everything be rigorously proved, there is no faster way to lose credibility as educators if we build our whole case against the reform on this one theme alone. It is far too easy, for example, to harp on the absence of ϵ-δ proofs of the basic theorems of limit and continuity in the reform calculus texts, but the fact is that a pedantic insistence on rigor is not necessarily the best approach to the teaching of elementary mathematics. Most beginning students cannot learn very well when every step is weighed down with rigor (e.g., ϵ-δ proofs). It would be more realistic to ask that there be careful differentiation between what is actually proved and what is not. Gaps can always be filled later, provided no circular reasoning is involved *and* provided the students are made aware of the gaps.

What then can we do, individually and collectively? Here are a few suggestions.

NCTM is in the process of revising the three *Standards* [N1]-[N3] for a second edition. It has created a Commission on the Future of the Standards and has asked all other member organizations of CBMS, such as MAA, AMS and SIAM to create similar committees to work closely with the Commission over the next three years. These review groups are to provide sustained advice and information. In the coming months or years, we should seek out members of these committees to give them our opinions on the reform in general and on the *Standards* in particular. This is our chance to infuse the *Standards* with more mathematical substance and a more balanced viewpoint.

In the meantime, we would be at our constructive best when we offer our critical comments on the reform. In spite of pleas from many well-intentioned individuals for the mathematical community to "speak with one voice" in support of the reform (e.g., pp. 1117–1118 of [ST1] and [TU]), my personal conviction is that we should keep up the fine critical tradition initiated in [MC]. What is missing in the reform is the commitment to teach mathematics, in all its guises, without violating its integrity. If we mathematicans do not reaffirm this commitment, then who will? Therefore, let our critical comments pour forth.

Something much less easy to achieve but immensely more important is for mathematicicans to help improve the training of prospective school teachers. Many of our students will be the teachers of tomorrow. How we teach them will directly affect how they teach their students. Yet the mathematics education on the college level is, more often than not, aimed exclusively at producing future mathematicians. The usual college mathematics courses drill the students on the technical details of the fundamentals in order to prepare them for graduate work in mathematics. But for those who leave mathematics after their college degree, e.g., school teachers, such courses yield them only brief glimpses of the trees but never the panorama of the forest. In the words of Allyn Jackson, such an experience in mathematics is akin to "finishing a BA in English literature having done a lot of technical analysis of Shakespeare but having no idea about Shakespeare's stature in English literature". Because less than 20% of the math majors go on to do graduate work, what we are doing is in effect *addressing only 20% of our students while pretending to be teaching them all.* A penetrating discussion of this issue of imposing the needs of the few on the many in college education has been given in [AT]. A more detailed discussion within the context of mathematics can be found in [W4]. Such a narrow focus on producing future mathematicians in our present education of the math majors, however admirable from certain standpoints, is a significant factor in the inadequate mathematical preparation of our school teachers.

There is no simple remedy for this educational difficulty. In the larger institutions, it would be relatively easy to schedule different sections of the same course to satisfy the divergent needs of the students (cf. [W4] again). On the other hand, it would seem that this obstacle can be overcome only by extra dedication (and perhaps ingenuity) on the part of the instructor in the smaller colleges. Any improvement in teacher training is of pivotal importance in the education of K-12 in the long run. Inasmuch as *all of us are capable of making a contribution to this important matter just by being more conscientious in carrying out our normal duties*, I strongly suggest that we give it the special attention it deserves.

A third area for possible action is direct participation. This can take many forms. For example:

(A) Be an author of school mathematics texts.

(B) Join a group that engages in curricular activities.

(C) Act as consultant and critic on education.

(D) Speak up as a citizen and do grassroots work.

Perhaps a few brief comments would suffice. Regarding (B), the main difficulty is that there is at present an almost unbridgeable chasm between educators and mathematicians, so any contribution we hope to make here would have to be predicated on the possibility of re-establishing some mutual trust between the two groups. Regarding (C), it should be pointed out that, notwithstanding the exhortation by NSF and AMS for research mathematicians to partake of the education enterprise, there is in fact no support for critical educational writing such as [AN1]-[AN3] or [AS1]. (But NSF funded the writing of textbooks such as *Earth algebra* [SCA]. Life is indeed full of mysteries. Cf. [W5] for a more extended discussion.) It would seem that the most effective method of making one's voice heard in education at present is by way of grassroots efforts. The prime example of this are the groups Mathematically Correct and HOLD in California (cf. the web sites *http://ourworld.compuserve.com:80/homepages/mathman/* and *http://www.rahul.net/dehnbase/hold/*, respectively), consisting mainly of concerned parents. Their work (holding public discussions of mathematics education, working with sympathetic politicians and educators, etc.), together with that of other activists, have been instrumental in bringing about some significant educational changes in California. Both groups have consulted with mathematicians for technical advice on occasions, and that advice was often found useful. If, individually and collectively, we can add our professional voices to the efforts of other such organizations, we can help create a potent force for change within education.

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