Some Lessons from California

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> *The right to search for truth also implies a duty; one must not conceal any part of what one has recognized to be true.*
>
> Albert Einstein

Starting with year 2000, California has been funding extensive professional development in mathematics in the form of summer institutes. H. Wu has taught five of these summer institutes for teachers in grades 4 to 6 with Beverly Braxton, Mary Burmester, Jaine Kopp, Bruce Simon, and Ada Wada: three Number Institutes in 2000–2002, respectively, and two Geometry Institutes in 2001–2002. For convenience, the discussion in this article will be restricted to the NUMBER INSTITUTE of 2000 and the GEOMETRY INSTITUTE of 2001, which were both taught by Wu and Braxton, Burmester and Wada. Wu is a professor of mathematics at the University of California at Berkeley. Braxton is at present an Elementary Mathematics Specialist in the Teacher Education and Professional Development unit of the University of California Office of the President, but was a teacher in K–8 before that. Burmester is a teacher at Rosa Parks Elementary School at Berkeley, and Wada is a teacher at Martin Luther King Jr. Middle School at Berkeley. Both

[0] November 29, 2001; **revised May 25, 2004.**

of these institutes were originally funded until 2004, but California's spectacular budget deficit caused the funding to be cut starting in year 2003. These summer institutes are therefore a relic of the past as of 2003.[1] Nevertheless, the experience of the first two years[2] has already yielded some insight into professional development that may be of general interest. For reasons to be given in §1, this insight is particularly relevant to the university preparation of mathematics teachers. In this article, Burmester and Wu will share what they have learned from two different perspectives: from Burmester's perspective as both an elementary teacher in the audience of Wu's lectures and a teacher leader who taught a small group in the institutes, and Wu's as a university mathematician and as the lecturer of the institutes. The following two sections were written by Wu (§1) and Burmester (§2) separately.

Both authors gratefully acknowledge the long hours of discussions with our friends Beverly Braxton, Jaine Kopp, Bruce Simon, and Ada Wada which form the basis of this article. Wu also takes this opportunity to express his profound gratitude to his five co-workers. Whatever success these institutes may have achieved is in large part due to their dedication, effective teaching and, above all else, unstinting support.

1 Wu's Perspective

The purpose of this section is two-fold: to give an overview of the structure of the two institutes under discussion, and to make some observations concerning professional development in mathematics from the perspective of a university mathematician.

Wu's knowledge of California's professional development in mathematics is based on his involvement in the California Mathematics Project (CMP) as a member of the Advisory committee from 1996 to 2000 (he was co-Principal Investigator in 1999–2000). Until year 2000, CMP was the only state sponsored agency that provided professional development for mathematics teachers. What he saw there convinced him that the most urgent need

[1] While the funding lasted, the summer institutes were spread over all of California. In the summer of 2001, for example, about 150 such institutes were given over 32 geographic locations in California.

[2] **Note added March 15, 2003**: This article was originally written in November of 2001.

of mathematics teachers (in California) was content knowledge rather than new pedagogical techniques or clever classroom projects. Teachers needed *systematic* exposition on the basic topics of whole number algorithms, fractions, and area and volume formulas, but they could not get it from the short-term workshops of the time which, even at their best, were mostly devoted to discussions of one or two small topics. Although the CMP institutes often lasted three weeks, they did not provide teachers with the kind of mathematics instruction that teachers needed either. One of several reasons for this failure is that, due to the lack of funding, participating teachers were never adequately remunerated. This lack of compensation made it almost impossible to ask teachers in the institute to work hard and, without hard work, learning mathematics was out of the question. So it came to pass that CMP could not, or did not do what had to be done, which was to strengthen teachers' content knowledge. In addition, the lack of funding sometimes induced the CMP institutes to teach teachers of all grades, K–12, in one group. Such indiscriminate grouping was clearly counterproductive to the kind of mathematics instruction teachers needed. The idea that there should be intensive summer institutes focused on mathematical content, in which teachers get reasonable payment for attendance, then became painfully obvious. Because mathematics is not learned overnight under pressure, there should also be follow-up sessions throughout the succeeding academic year to help teachers internalize the new-found knowledge. Furthermore, and this is more important, because the goal of professional development is not to help teachers learn mathematics but to make them better teachers, the follow-up sessions serve the dual purpose of providing a forum for the continuing discussion of how to put the content knowledge to use in the classroom. These ideas were presented in a 1999 article ([Wu1999]).

In year 2000, Governor Gray Davis of California introduced a series of initiatives which, for the first time in the state's history, fund professional development on a large scale. By coincidence,[3] the basic requirements for funding in mathematics are entirely similar to the views expressed in the preceding paragraph. Specifically, fundable institutes:

1. Should be grade-level specific: grades 4–6, grades 7–8, or 9–12.

[3] **Note added May 25, 2004**: I should have mentioned from the beginning that I was a member of the committee that formulated the basic policy for funding the professional development institutes described in the five points below.

2. Should be three-week institutes in the summer except for those in grades 9–12,[4] eight hours a day and five days a week.

3. Must be devoted to the instruction of mathematics.

4. Must have five follow-up Saturday sessions in the succeeding school year, each Saturday again being a full day of instruction.

5. Must pay each teacher $100 per day of attendance.

By special arrangement with a local university, the Dominican University, the teachers in the institutes of Wu et al. can get 6 university (education) credits for completing a three-week institute at $45 a credit. If they also complete the five follow-up Saturday sessions and do some supplementary work, they can get an additional 3 credits.

Further details about these institutes can be obtained at the web site http://www.ucop.edu/math by clicking on MATHEMATICS RFP . On the same web-page, one can also find de fault statements of the contents of some of the institutes by clicking on the items under CONTENT STATEMENTS . In particular, the Number Institute and the Geometry Institute reported in this article follow closely the content statements of

Revised Elementary Number and Operation May 2002
Revised Elementary Geometry August 2002

which are found near the bottom of the web-page.

The fact that each day of such an institute should be devoted to instruction in mathematics clearly is not meant to be interpreted literally as a call for full-day lectures. Even research mathematicians find it diffcult to maintain concentration on mathematics for eight hours a day, five days a week. In point of fact, the structure of a typical day of the institutes of Wu et al. is roughly the following: The day begins at 8:30 and ends at 4. With time taken away by lunch, breaks, and miscellaneous items, there are about six hours of mathematics instruction. Of these six,

the first three to four hours are devoted to lectures by Wu, and

[4] In special situations some variations are permissible. Incidentally, there is a reason for the decision on *three* weeks: most teachers would balk at giving up a whole month of their summer vacation, and two weeks are not enough to do serious mathematics.

> the remaining two to three hours in the afternoon are devoted to small group sessions with the three teacher leaders (Braxton, Burmester, and Wada) to review the lectures, review homework problems of the day before, discuss new problems, and discuss how to apply the new knowledge to classrooms.

The original idea was to limit Wu's lectures to three hours, but often teachers' unforeseen difficulties in the form of questions raised during the lectures would throw the schedule off balance. Three hours then stretched to four. It must also be admitted that sometimes teachers' grasp of the new material was too tenuous for them to contemplate the possibility of applying it to their own teaching. In that case, they concentrated their effort on absorbing the new knowledge, and would turn their thoughts to classroom applications only during the subsequent Saturday follow-up sessions during the regular school year.

What is not obvious in this description of the daily activities is the importance of the afternoon sessions led by the three teacher leaders Braxton, Burmester and Wada. By speaking the teachers' language, they put the teachers at ease and got them to open up. They were also very effective with the use of appropriate manipulatives to supplement the lectures. It was in the more intimate setting of a small group that the learning difficulties were exposed. Wu has the suspicion that, in fact, most of the learning in the institutes took place not during the lectures but in the afternoon sessions.

To see why teachers had such a hard time coming to grips with the new material, let us look at what was taught. For the Number Institute, the schedule of the fifteen days was as follows:

days 1–4: whole numbers

days 5–11: fractions

days 12–13: decimals

days 14–15: rational numbers

Here fractions refer to "positive rational numbers", so that the last two days of the institute was essentially about negative rational numbers. The content of the first eleven days is now available in the form of the first two chapters of a forthcoming monograph ([Wu2001a] and [Wu2001b]). The teachers of the year 2000 institute, it must be pointed out, received only embryonic forms of these chapters and their primitive exposition might have contributed to teachers' difficulties. Due to the lack of time on Wu's part, no notes were passed out for the last two topics of decimals and rational numbers. A weakness in this sequencing of topics was eventually realized by the teaching staff. Both the existence and uniqueness of the simplest form of a fraction as well as the theorem about which fractions have finite decimal expansions were taught in the institute, but the basic number theory needed for the proofs (the Euclidean algorithm and the fundamental theorem of arithmetic) had to be assumed. In a later incarnations of this institute (years 2001–2002), it was decided to spend the last four days on the needed number theory and decimals. The treatment of rational numbers was then relegated to the succeeding Saturday sessions instead. This new sequencing seems slightly more natural.

Most teachers could understand the material on whole numbers. Given all the controversy surrounding the long divison algorithm in elementary mathematics education, the method adopted in [Wu2001a] was shockingly popular. The teachers all seemed to understand this algorithm and love it. The love was further deepened when they saw how the properly formulated algorithm in §3.5 of [Wu2001a] leads *automatically* to the decimal expansion of a fraction, in the sense that the decimal point appears in exactly the right place without having to invoke any artificial rules. The most difficult part for the teachers was, not surprisingly, fractions. In the approach adopted in [Wu2001b], a fraction is a point on the number line and every concept or assertion about fraction is explained on this basis. At the beginning, not many teachers liked the idea of being tied down to the number line. In addition, the possibility of explaining everything about fractions on the basis of a single definition is an idea entirely foreign to most teachers, so that the advantage of doing this — the ability to ground everything on reason rather than on arbitrary decrees — was consequently not apparent to them. In the same vein, the fact that finite decimals are nothing but special kinds of fractions was initially also difficult for many teachers to accept. As a result, the summer institutes were engaged also in the remolding of teachers' preception of mathematics in addition to the teaching of content. How well we succeeded in doing that can be gleaned in part from the next section.

The Geometry Institute can be said in hindsight to be the less successful of the two. Before going into the explanation, let us look at the schedule:

days 4–5: Explorations in 2 and 3 dimensions

days 6–7: Points, lines and planes; distance and convexity, including a proof of Euler's polyhedron theorem

days 8–10: Transformations in dimension 2; use of transformations to prove elementary theorems on parallelism, perpendicularity, and circles

days 11–12: Dilation and similarity

days 13–15: Length, area and volume

The omission of "days 1–3" is not an oversight: because of a miscalculation on Wu's part, the first three days of the institute were devoted to the number theory omitted from the Number Institute of the year before. There is no doubt that these geometric topics — which would be taxing to teachers even in a fifteen-day institute — should never have been compressed into a twelve-day span. When the Geometry Institute was done again in 2002, the same material was taught in fifteen days, with two more days given to proofs on parallelism, perpendicularity, and circles, and the remaining day split between similarity and the area-volume discussion. The reception (not reported below) was appreciably better as a result,

Although notes were also passed out during the 2001 Institute (except for dilation and similarity), they were exceedingly terse.[5] Because these notes have not been made publicly available, some additional comments about the schedule would be appropriate. The explorations of days 4-5 were entirely hands-on activities designed to put the teachers at ease with geometric objects. We were concerned about the weak geometric background of the teachers and wanted to provide as gentle an introduction to geometry as possible. For exactly this reason, the discussion of lines, planes, distance, etc., in days 6-7 was also essentially devoted to explorations and introducing the geometric vocabulary. The one exception was the proof of Euler's polyhedral formula of $V - E + F = 2$; it was a proof that gave the main geometric ideas without insisting on the formal details ("graphs" were used

[5] The notes in the 2002 institute, while still sketchy, were far more complete than the earlier version. It was difficult to gauge whether the added information made a difference in the teachers' learning, however.

without a formal definition).[6] Up to this point, the main emphasis was on pointing out some attractive features of geometry that are easily accessible to the teachers without any worry about proofs. One example of the kind of thing we had in mind was Pappus' theorem on hexagons whose vertices lie on two lines, and especially the Steiner point arising from the Pappus lines (cf. [Salmon], p. 380). The whole configuration can be drawn using nothing more than a (large) piece of paper, a ruler, and a pen. Geometry proper begins with the topics of days 8-10. The standard rigid motions of the plane (translations, rotations and reflections) were introduced intuitively, and they were then used as the starting point for proofs of standard theorems about parallelograms, rectangles, circles, etc. We believe that teachers should know some basic geometric facts such as why opposite sides of of a parallelgram (defined as a quadrilateral with parallel pairs of opposite sides) have equal length, and why the tangent to a circle at a point is perpendicular to the radius through that point. Instead of usual axioms of Euclidean geometry, the more intuitive rigid motions seem to be easier to grasp by beginners. The usual congruence criteria (ASA, SSS, SAS) were then proved as illustrations of the power of rigid motions, but extensive use of these criteria for further proofs was not emphasized. After all, it was not our intention to give a short course on Euclidean geometry. The discussion of similarity was based on the notion of dilation with respect to a fixed center. The AA criterion for similar triangles was then discussed and proved in a special case. The discussions of length, area, and volume were based on an intuitive idea of limit, and then the standard formulas for the areas of triangles, disks, and the volume of a ball were derived (the latter assumes Cavalieri's principle). The institute concluded with a discussion of the effect of dilation on area and volume.

[**Note added March 15, 2003**: It may be of some interest to point out the salient differences between the Geometry Institute of 2002 and its predecessor. In the geometric explorations of the first three days, we asked teachers in 2002 to sketch geometric figures *without* the use of ruler and compass; for example, draw a triangle and try to draw also the circumcircle of the triangle. This the teachers found to be helpful in fostering their geometric intuition. The proof of Euler's polyhedral formula, no matter how informal, was found to be too ambitious, so the time was spent instead on learning

[6] **Note added May 25, 2004**: Wu has finally conceded that giving a proof of $V - E + F = 2$, in any form at this level, is probably a mistake given the time constraint.

how to *use* this formula. Teachers seemed excited, for example, to discover that $V - E + F$ is 0 for a torus and -2 for a double torus. More time was spent on explaining what rigid motions are as well as on explaining proofs of simple geometric theorems using rigid motions (e.g., opposite sides of a parallelogram are equal in length). The more relaxed pace in proofs was, needless to say, welcomed by one and all. More time was also given to explorations with dilations, e.g., asking teachers to actually trace out the dilation of a curved figure with dilation ratio equal to 2, so that they witnessed for themselves how the magnified figure did look "similar" to the original figure. The price we paid was to give up discussing many proofs about similarity, and the discussion of volume also became more abbreviated.]

The main difficulty with the Geometry Institute, and the relative lack of success thereof, was the teachers' unfamiliarity with *anything* geometric. With but mild exaggeration, some teachers literally trembled at the sight of ruler and compass or when they were handed a geometric solid. As mentioned in the preceding paragraph, we were prepared for teachers' being ill-at-ease with geometric reasoning and lack of geometric intuition, but not for the degree to which both were true. School education in geometry is in deep trouble. It goes without saying that, given such a low starting point, every step of the instruction in the institute was met with considerable resistance. It was especially true of the teachers' encounter with proofs about parallelograms, rectangles, and circles, although this difficulty is less surprising when one considers that our high school teachers also have the same difficulty.

With hindsight, we now see that maybe both institutes tried to do too much. What to take out is of course a much more difficult issue. In the Geometry Institute, it may be possible to go more lightly on similarity and the proof of the Euler formula.[7] However, there is no doubt that the topics in the Number Institute represent the absolute minimum that every elementary teacher must know in order to be effective in the classroom. At the moment, we do not know how to resolve this difficulty.

The preceding description of the the mathematical content of what was taught in these institutes should suffice to explain why our experience with the institutes is relevant to the universities' preparation of mathematics

[7] See the Note added on March 15, 2003, two paragraphs above.

teachers. Whereas the more standard kind of inservice professional development is concerned with enrichment or refinement of teachers' knowledge in a small area of school mathematics, our institutes were devoted to a systematic revamping of teachers' knowledge in two basic domains — numbers and geometry — *from the ground up*. Aside from the time restrictions, i.e., three weeks of our institutes versus a semester or a year course in a university, there is virtually no difference between the two. We can go further by asserting that our institutes may be more difficult to bring off than a standard university course because the pressure of covering so much new ground each day in a three week period can be — and was — draining to the teachers. What we learned from these institutes therefore should have a bearing on professional development — pre-service and inservice — as a whole.

What then did we learn? First we learned something about the most glaring weaknesses in elementary teachers' mathematical knowledge:

(A). It is difficult for them to abide by precise definitions.

(B). They do not have habit of asking *why*, much less finding out why.

(C). They are afraid to use theorems, e.g., the cross-multiply algorithm, the cancellation law of fractions, etc.

(D). "Proof by contradiction" and the concept of "uniqueness" are very difficult for them.

(E). They have little or no hands-on experience with geometric objects, much less the ability to reason with them.

The last item, (E), is the easiest to discuss. School mathematics education must take the teaching of geometry seriously starting with upper elementary school in order to remove students' fear of geometry sufficiently early. In the context of pre-service professional development, an after-the-fact remedy may be to devote time, two to three weeks perhaps, to do geometric activities such as building Platonic solids and drawing geometric figures with ruler and compass. Then slowly add simple geometric deductions to bring prospective teachers "back to the fold". As to the difficulty with "proof by contradiction" and "uniqueness", this phenomenon is actually well-known with all beginners of mathematics. The use of legal arguments as examples (suggested to us by Richard Askey) could help to overcome the difficulty with "proof by contradiction". The problem with "uniqueness" would seem to be one caused by insufficient exposure to the concept. Given the fundamental importance of this concept, there is some urgency in getting this concept carefully discussed in high school whenever the opportunity arises (e.g., the uniqueness of the remainder and quotient in the integer division algorithm and the polynomial division algorithm, the uniqueness of the parallel line through a point, the uniquenss of the prime factorization of a whole number, etc.). The phenomenon exhibited in (C) is puzzling, and Wu suspects that it has something to do with the injunction of the recent mathematics education reform against "learning by rote". Surprising as it may seem to mathematicians, the use of any mathematical tool (e.g., a computational algorithm) is often equated with a "lack of conceptual understanding" or "learning by rote" in the prevailing reform climate. One example is the lack of emphasis on the cross-multiply algorithm in the more recent reform texts. We can understand the reluctance to promote mindless computation, but if the new instructional strategy has been shown to throw out the baby with the bath water, then some re-evaluation is in order.

As a result of these institutes, Wu has serious doubts that for elementary school teachers, 9 semester-hours of mathematics courses provide enough mathematical preparation. Each of our institutes, including the five Saturday sessions, had about 55 hours of lectures and 65 hours of small section meetings on problem solving and pedagogical discussions.[8] Altogether, we got to work with our teachers for about 120 hours. A regular semester course has 45 hours, so even if extra hours are scheduled, it takes two such semester courses to approximate the 120 hours made available to us. It is of course difficult to equate the intense and short-duration atmosphere of a summer institute with a more relaxing semester offering, with both pros and cons, but it would be a stretch to claim that what we did in 120 hours could be done in a semester. In addition, what we did with Number and Geometry in 240 hours still leaves some critical topics untouched: probability and some basic algebra, for instance. Our impression is therefore that a requirement of 12 semester-hours of mathematics may be closer to the mark for elementary teachers.

[8] Each day of the institute has about 3 hours of lectures, so 15 days add up to 45; about 2 hours of each of the 5 Saturdays are devoted to lectures, so there are 10 more hours. There are about 3 hours of each day in the institute are devoted to problem solving and pedagogy, so again there are about 45 hours altogether; about 4 hours of each of the Saturdays do the same, so this accounts for another 20 hours.

Another lesson we learned is that three hours of lecture each day, even when broken up by many exchanges between the lecturer and his audience, do not lead to good learning (see comments in §2). In the summers of 2001 and 2002, Wu gradually changed the lecture format to 10–15 minutes of problem solving after each hour of lecture. Teachers reacted quite positively to the change. It must be noted that, as a consequence, each day of the institute now consists of about four hours of lecture-and-problem-solving and two hours of small group meetings.

Perhaps our most rewarding experience with these institutes is to have observed the evolution of some of the teachers' attitude towards the importance and efficacy of emphasizing definitions and logical explanations in the teaching of mathematics. Their initial rejection of the use of precise definitions and their initial reluctance to ask for reasons in mathematics were almost completely turned around in a year's time. This is most vividly illustrated by the anonymous, final evaluations (written in the last Saturday follow-up session in May of 2001) of the teachers who participated in the Number Institute of year 2000. These evaluations are transcribed in full in the Appendix.

To put these evaluations in the proper perspective, we should note that this institute started with 27 teachers in the summer of 2000, and by the time of the last Saturday session in May of 2001, only 12 remained. One should understand that, on the one hand, the difficulty for a teacher to give up five Saturdays in a school year cannot be overstated. So the attrition from 27 to 12 is largely a reflection of human nature and the reality of teaching in schools. On the other hand, because the 12 teachers who chose to stay to the bitter end were clearly among the more motivated and mathematically stronger of the whole group, their self-selection should explain their somewhat enthusiastic endorsement of the institute. What seems to us worthy of pointing out, however, is the fact that our institutes asked for daily anonymous evaluations by the teachers and, all through the three weeks as well as the five Saturdays, the teachers were never known to hesitate to speak their minds (cf. §2). In this light, these final evaluations are valuable to us because they seem to give an indication of a fundamental change in these teachers' perception of mathematics. Clearly the possibility of such a change has implications in how we approach pre-service professional development. It gives us hope that we can teach prospective teachers honest mathematics without having to bend over backwards to coddle them.

We do not want to overdo this, but a passing comment about assessment in professional development may not be out of place. In the absence of any good measurement, as yet, of what teachers learn in a professional development program or how their mathematical knowledge impacts their teaching, we believe that such anonymous evaluations should be taken into account, for now, as a key factor in the overall assessment of a professional development program. Needless to say, the value of such evaluations very much depends on an atmosphere of openness in the program.

[**Note added May 25, 2004**: The need of *independent, in-depth* evaluation of funded projects does not seem to be realized by the federal funding agencies. The kind of activities that pass for evaluations up to this point are far from independent and most often not in-depth. If we are serious about improving mathematics education, then we should decouple evaluations from the funded projects themselves and fund serious, independent evaluations.]

Finally, Wu would like to add two personal observations of his own. One is that an overwhelming majority of the teachers seemed extremely eager to learn. Admittedly he did not have a large sample to work with: the Number Institutes had 27 and 24 teachers, and the Geometry Institute 24. With this limitation understood, the teachers' eagerness to learn stands out in his mind as one of the most inspiring experiences in his many years of teaching. The other observation is that, if his experience in teaching these institutes is at all typical, then the input of a school teacher in both pre-service or inservice professional development — such as what he received from his teacher leaders — is indispensable. It would be unthinkable for him to go through the institutes without the constant advice and support from his staff members.

2 Burmester's Perspective

Let me first say that it has been an honor to assist Wu in the summer institutes for teachers. I came to the project with much the same perspective as Wu: that elementary teachers need staff development in the mathematics itself, much more than in pedagogy or "activities." If there is one thing that elementary teachers do well, it is to teach. And if there is one thing that they may not know, it is probably mathematics.

The purpose of this section will be to show many of the actual comments made by the teachers in the final evaluation of the summer institutes, with interpretation as to general trends that seemed to be evident throughout the comments. With but one exception, the comments that I quote here are from the end of the three-week Geometry Institute, before the year's follow-up Saturdays have taken place. I will also make reference to the comments found in the Appendix, which are from the Number Institute of the previous summer, after a year of follow-up Saturdays and application to the classroom. As you will see, there are some significant differences between the two groups.

As I did during the talk that Wu and I gave at the National Summit in Washington DC (November 2001), I will first show a theme that emerged from the comments and follow with quotes of the actual comments and a discussion before going on to the next theme. I am indebted to the entire team for these conclusions, Braxton, Wada, and Wu, as many hours of reflection were put in by all of us, although my comments here are strictly my own. And of course we are all indebted to the participants (teachers) for their caring and honest evaluations!

Theme #1. Mathematics behind the formulas/algorithms — the WHY is exciting.

In the institutes Wu put a daily emphasis on knowing why the mathematics taught in elementary school works the way that it does, so that student questions can be answered from a basis of mathematical knowledge. This theme came out clearly in the evaluations:

> **The institute's effect on my teaching and thinking has been immeasurable. I used to teach my students by rote — "this is the way you do it." I think I never even really considered that it was important to understand why. Now I first demand logical explanations of everything from myself which I can use with my students. I am no longer so frustrated and frightened by their questions. I am better able to break concepts down into small pieces and relate concepts**

> **to other concepts. I really feel like I have developed some mastery over my subject. Other teachers in my school have come to rely on me for explanations of the concepts they are teaching.**

In our experience, teachers are grateful to know more mathematics. They are very aware when they don't know enough and often are embarrassed in the classroom or when talking to parents. They told us this over and over.

> **I believe that my teaching of mathematics — especially geometry — will be improved as I have a greater understanding of the whys.**

Another participant writes:

> **The most significant thing for me about the institute was being able to form a beginning understanding for the WHY in geometry. All of the things I learned in high school were taken apart and proved. I don't just have to accept these theorems on faith — I know why they work. The continued importance of definitions and consistency with these was continually reinforced.**

Definitions as a basis for building up the mathematics understanding was a new way of working for most teachers and it proved most difficult to get the teachers to *rely* on definitions for explanations. It was a foreign language.

At this point, I want to take a side trip: two things related to the learning of mathematics proved quite difficult in the institutes because they were so unknown, and certainly not encountered in other mathematics staff developments, and both are basic in the field of mathematics. First, reading mathematics sentences or formulas in fully articulated English (or any other language for that matter), so that *meaning* is constructed, and second, the even more mysterious process of building meaning from certain specific assumptions and definitions and then using these assumptions and definitions to prove other conclusions (theorems). The definitions themselves seemed to have come from Mars! To remedy the first problem of reading mathematics for meaning, we partnered up and simply read aloud to each other from a randomly picked passage in Wu's notes (cf. [Wu2001a], [Wu2001b]). The participants had many questions about notation, and had to be nudged to really see every mark on the page. Mathematics professors must be very familiar with this problem. The second problem of internalizing definitions and reasoning with them improved somewhat with daily practice but never became routine.

So now, let us go back to participants learning the mathematics:

> Most significant — **I think the most outstanding was learning! Seeing the rationale for using certain formulas, realizing that the mathematics is actually behind the formulas, not (just in) the formulas themselves.**
>
> **I think my teaching of geometry will come from not just a book, but from some concrete understanding now that I have been through the institute.**
>
> **Having spent these three weeks immersed in math, I now feel more confident to teach it. In fact, I will look forward to teaching it! I think the kids will be much more enthusiastic because of my enthusiasm!**
>
> **I hope I have learned enough for this to really impact my teaching. . . at least having a grasp (however tentative) of the concepts, I hope I can better explain what the students are learning so it makes some sensible whole.**

Again and again the appreciation for new understanding of concepts comes through.

Theme #2. More time for processing the math — "embedded think time".

The following complaints were common all through the institutes:

> **Have a longer "process" time for the theorems, etc., and a chance to redo the proofs in class or even to go through a guided "proofing" during p.m. sessions.**
>
> **More embedded think time, break up the lecture with group work, i.e. less time in p.m.= more time for group work in the midst of the lecture. We can do the same problems we would do in breakout groups, but we wouldn't have to wait until the end of the day. The end could be a review activity.**

Wu had a very ambitious agenda for each of the math institutes (number and geometry), which required homework and reading in the evening and pushing on ruthlessly every day! The overall structure of the mathematics was very tight and every individual proof was necessary at the end for a final important theorem. The participants wanted very much to understand the material, and interrupted the lectures with important questions, which would put us behind, or in a rushed situation. The three weeks, all-day format was very intense, but was probably still the best arrangement for working teachers.

Theme #3. How the institute helped the participants.

> **The realization that many teachers lack the background to teach math accurately and how this is affecting our students (helped me). I have a strong math background but now know I need to learn more. My teaching will be affected in that I will now question more, have my students**

> **question more and lastly I will pursue further math development via institutes like this.**
>
> **Emphasizing from the very start that this is for** TEACHER KNOWLEDGE**, not for teaching strategies will again clarify expectations of the class' subject matter and purpose.**

Underlying this last comment is the fact that at least one participant left in the first week because there were not enough presentations of teaching activities, and another probably left because the material was too difficult. So some were not helped in the way that they expected to be by the institute. Those that stayed, however, repeatedly told us that the material was very valuable and their efforts to understand were very evident to us.

Theme #4. Integrating mathematics learned with classroom practice.

> **I think my sensitivity to the struggling students has been heightened. I have seen the importance of telling students where we're going and showing how one thing relates to another. In elementary school it's important for their foundation to be strong and seeing the connections strengthens that foundation.**

This group had to struggle with the mathematics, and apparently regained the appreciation for the student experience!

> **I can see now that geometry is much more than area and perimeter and names of shapes and solids. I will incorporate the simpler concepts and proofs in my teaching. Especially, since all of the curriculum at my school is differentiated for those kids at the more advanced levels.**

Since the Geometry Institute of this past summer has not yet finished with a year of follow-up Saturdays, the integration of the material into classroom teaching is still in the future for these teachers. More comments on this subject can be found in the end-of-the-year evaluations of the first year Number group in the Appendix.

Theme # 5. Proof by contradiction/ Visualization.

> **At the beginning of each day there should be exercises on contradictory thinking and visualization to help prepare us for the content. For some it's been a long time since college and I just don't think either way.**

Although only one person commented on this, we struggled a lot with "seeing" things. As mentioned in §1, the first three days of the Geometry Institute were hands-on making of three-dimensional shapes including all of the Platonic solids, and naming and counting faces, side, vertices, etc. Then as people kept asking for help, we began to present simple visualization puzzles at the beginning of the afternoon sessions. Many people reported a distinct improvement in visualizing as the days went on. However, when Wu presented his first proof by contradiction, we hit the wall again! We tried, with limited success, to think of everyday examples of showing something is true by contradiction, and Wu made his now famous quote: "In mathematics, there is no gray area; if something is not true, then it must be false." But the concept was elusive.

Theme #6. More proofs and theorems vs. Make it more basic.

> **The length of the day is about 60 minutes too long. The level of instruction needs to be more basic and not so many theorems.**

> **During the Saturday sessions I would like to explore more proofs and theorems but also work more on applications in the classroom. I don't want to lessen the depth — just share ideas of using it with my students.**

Our geometry group was very heterogeneous in their knowledge of mathematics. Wu had a system that he employed almost daily to ask anonymously whether he was going too slow or too fast, or needed to explain more, or move on. He would give every teacher a collection of cards numbered 1, 2, and 3, and he would write, for example, "1 = too fast, 2 = too slow, 3 = just right" on the board, and then cards would be collected from the teachers and counted. This was very helpful but there were very real differences in ability and understanding that remained. And as I mentioned above, the mathematics was very tightly constructed and could not really be skipped or changed substantially.

Theme #7. Different Worlds: the culture of higher mathematics vs. the culture of elementary school.

I have chosen this theme to illustrate an interesting dynamic that presented itself in the Geometry Institute. A participant would ask a question, Wu would ask a question back and expect a definition-based answer, and an atmosphere of challenge would develop. This challenge method of improving students' approach and grasp of the material is no doubt familiar to college mathematics instructors, but was sometimes interpreted as insulting by elementary teachers. Reasons for this abound I am sure, but one seemed rather obvious: not many elementary teachers use challenge as a method of helping students. We probably specialize in hand-holding, if anything! The challenge of going back to the definitions and making the connections step by step up to the current theorem or result was very difficult and many teachers expected to be convinced by informal explanation. When Wu insisted on precise statements, difficulties multiplied and tension developed. In the afternoon sessions, we asked teachers to go to the board repeatedly and tell us first, what they were going to show, then second, to show us, and then third, to recap and restate what they had just showed. This was a slightly easier way to improve precision in language, but we were not really able to resolve the whole issue! It was very interesting and unsettling at the same time.

In conclusion, I would like to contrast the evaluations from the Geometry and the Number Institutes. The end-of-the-year evaluations from the Number Institute are located in the Appendix to this article. My general impression is that the participants in the Number Institute were more selfassured and secure with their knowledge, less confused by the mathematics and knew what to do with the information in the classroom. After all, it had to do with fractions, decimals and the number system, and these are things they were familiar with. Moreover, there was a certain degree of self-selection in this group because not all teachers were able to attend the follow-up Saturday sessions, and the group that did may have been the people who had a better grasp of the material to begin with.

The geometry content, on the other hand, was much more challenging, and the participants had genuine difficulty with it. Fully one-half of them lost ground on the post-test at the end, but the jury is still out here because the school year has not gone by. Geometry involves much more than just numbers. It involves spatial thinking and arguing by contradiction, and the transformations that Wu used to prove the standard geometry theorems were brand new to many people. Visualizing the rotations, translations, and reflections was hard indeed.

It has been a mathematical pleasure and lots of work to assist in the teaching of such solid mathematics to teachers. Wu is dedicated to improving the teaching of mathematics in California and the U.S. I applaud his work and hope that others will take up the challenge.

3 Appendix

Teachers' End-of-Year Evaluations
Number Institute, May 2001

At the end of the institute's Saturday follow-up sessions, the following was asked of the teachers:

> **The primary purpose of the LHS Elementary Mathematics Professional Development Institute has been to deepen your understanding of the mathematics you teach so that you can deal more effectively with variations in the ways that**

your students learn. As a final evaluation, the staff would like you to reflect on the past year to let us know how your experience in the institute has affected you. The entire institute includes the summer institute and the five academic year meetings. Your feedback will help us shape the institutes that follow.
Here are some questions that you might want to think about and respond to in your reflections:

- **In what ways have your experiences in the institute affected your teaching of mathematics?**
- **How has your attitude toward mathematics changed?**
- **What changes have you noticed in your students' attitudes toward mathematics?**
- **What changes have you made in the way you explain the mathematics procedures that you teach?**
- **How much emphasis have you put on providing precise definitions for important concepts (e.g., fractions, multiplication of fractions, quotient and remainder in the division algorithm, etc.)?**

The following responses were received:

Participant #1

Now more than ever I really pay attention to my students' questions. I really think about what I am saying and how it might be misinterpreted or how it may not even be correct. I realize that there is no way that all children will understand, and so I constantly revise and perfect my lessons. My students ask better questions and they seem to appreciate that it's alright not to "get it" the first time. They are learning that there are benefits to doing problems various ways. I usually show the students, at least once, why any particular algorithm works and I even tell them that the reason I explain and illustrate is because I don't want them to just do it because I said so, but I want them to know why the procedures or shortcuts work. I am stressing the definitions 100% more. Before the institute, my definitions were very vague. Saturdays were hard because I really need the weekend. I also know that weekends scare off potential teacher participants. In any event, the institute was great and I highly recommend it to all who I speak with! Thanks!

Participant #2

Professor Wu's passion is catching. I found myself being much more excited about math. I can't wait to take the geometry institute.

Participant #3

Professor Wu is enthusiastic and he leaves all of us with the same energy and enthusiasm to take back to our students. As a result, I'm using more diagrams on the board, using the number line to explain relationships between numbers, concatenating line segments, etc. The excitement has spread to our students in great numbers. I now emphasize the definitions and laws in mathematics.

Participant #4

1. The passion/understanding of mathematical concepts transcends whatever grade level you may teach! Get excited, try it, you might like it!

2. Deepened my understanding of fractions, decimals, division, multiplication, laws of mathematics, and the way mathematicians look at the work and the teaching of mathematics in general.

3. Importance of understanding and defining what we teach.

4. I know I spend more time teaching mathematics that I did pre-institute and students are now recognizing some of the laws of mathematics for themselves and smiling when they recognize them. (e.g., $+$ and $-$ are opposites, $\times$ and $\div$ are opposites). They seem better able to risk trying, and ask WHY is it true?

5. I catch myself asking: What does 326 mean? And what $3 \times 6 = 18$ means. Why do you know it's true?

Participant #5

1. Math is more focused on mathematical principles. Math is focused on the connection between concrete and abstract.

2. More excited attitude.

3. Greater understanding. More connections between math strands.

4. I have more explanations for each procedure and a more standard vocabulary.

5. Definitions and vocabulary are the basis of concepts.

Participant #6

The institute's effect on my teaching and thinking has been immeasurable. I used to teach my students by rote this is the way you do it. I think I never even really considered that it was important to understand why. Now I first demand logical explanations of everything from myself which I can use with my students. I am no longer so frustrated and frightened by their questions. I am better able to break concepts down into small pieces and relate concepts to other concepts. I really feel like I have developed some mastery over my subject. Other teachers in my school have come to rely on me for explanations of the concepts they are teaching. I'm realizing now that I've gotten away from my earlier insistence on definitions, however I still see the powerful effects of my insistence earlier this year when we reviewed division and began fractions. I definitely have seen a change in many of my students' attitudes about math. They now believe that it is logical and can be explained. I think they've stopped feeling like it's magic.

Participant #7

My experiences in the institute have changed the entire way I teach math and the way I feel about math. Now that I (sort of) understand the background of the math I teach, I can give my students more ways of understanding what I'm teaching.

Providing precise definitions for important math concepts is the key to teaching.

Participant #8

1. It made me think from the students' perspectives. I guess I had forgotten that not all students have the foundation for pre-algebra. I am now more careful about assessing students' needs — preparing a lesson and reinforcing/making sure students are ready for the lesson. (Previous knowledge.)

2. I always felt that math could open/bring more opportunities to my students, and this class reinforced that belief.

3. A student once quoted me "We use math every single day" because I constantly try to connect it to real world situations. I feel that a lot of my

students are no longer afraid of math. In fact, some even like it!

4. I am more careful about selecting the definitions that textbooks provided.

Participant #9

I am able to visualize math concepts more readily. My thought processes are quicker and I can also respond more easily to everyday math. I see the relationship math has to other areas of academics, especially for elementary school age students.

Attending the institute has been a wonderful experience. The intense summer institute was hard and tiring. However, I was so fascinated by what I learned from Wu and my colleagues too. I know much more and have internalized some of the math concepts too.

Participant #10

Right now, after taking the assessment, I feel like I have a lot more to learn. For me, this past year has been a true learning experience in several ways. It has created an awareness of what I don't know, what I have learned, and what I have yet to know. It has been a great experience in professional development and how important and essential it is for teachers to continue to develop as professionals.

In my teaching of mathematics I have attempted to give students an understanding of what they do in math and what it means. I am not sure how effective I have been as I continue to learn along with my students. At my school we have flexible math groupings, which has made it difficult for me to develop my own continuity with my students because I don't know what the other teachers do and I have my own class 2 days a week. It has been very frustrating. But, I hope to integrate more of the institute into my teaching next year.

Participant #11

I truly cannot say enough about the institute. I really enjoyed it and I think that my teaching has certainly benefited. I think I am much more detailed in my explanations of all topics and have really made more of an effort to focus on why we can do certain things. I think my teaching will continue to improve as I see it as a "work in progress." I saw that some things work better than others and sometimes kids get a little turned off by my long-winded explanations. Many of them I can condense. I am very much looking forward to having textbooks that I will actually use. It will free up a lot of time standing spent Xeroxing on doing the meaningful work of developing lessons. If anything, the institute has made me more interested in learning higher level mathematics, which is something I rarely thought of before the institute. Watching Wu made me remember how much I like being challenged in mathematics and something I would like to pursue.

The Saturday sessions have been quite helpful in reminding me that I'm still not doing all I set out to accomplish in the fall. I'm still not as tight using definitions and I still am not teaching every child as effectively as possible. There is still much work to be done. . . one student at a time.

Participant #12

This math institute has made a big difference to me both personally and professionally: I feel my confidence in my ability to learn math has increased exponentially. I know I go further beyond the standards with my advanced students. My students generally love math and work at it. My remedial students have benefited from the fun. We do more math and spend more time on it.

I will spend even more time on math next year.

New for me this year, was integrating math more with science and social studies.

I also went further in depth with positive and negative numbers, choosing films like Eames Powers of 10 and not deciding they were too advanced. I found my nine- and ten-year-old students loved seeing the advanced science and math stuff. I learned they are more capable than I suspected. They love hearing the mathematical explanations.

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