Preservice professional development of mathematics teachers

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In June of 1998, the Conference Board of the Mathematical Sciences (CBMS) appointed a Steering Committee for the Mathematics Education of Teachers Project to begin work in coordinating university mathematics departments to upgrade the mathematical preparation ("preservice professional development") of prospective mathematics teachers. This would seem to be the first time such an effort has been made on a national level. In this article, I would like to make a contribution to this effort.

The present mathematics teacher crisis is precipitated in the main by two kinds of teachers:

(A) Those who teach mathematics without the proper training.
(B) Those who went through the standard credentialing program but did not acquire the requisite mathematics content knowledge.

The primary goal of this article is to address the issues raised by (B), but it would seem appropriate to first make a passing comment on (A). In California, the so-called emergency credentials for temporary teachers are responsible for the creation of the teachers in group (A). There, the number

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of mathematics teachers in middle schools and high schools without as much as a minor in mathematics is staggering: it is estimated to be around 50%. There is likely a similar problem in every state. Can we completely ignore a problem of this magnitude?

One justification for ignoring (A) is that it is really a social problem that is not within the control of the universities. Doing mathematics is a skill that is difficult to acquire, and because it is difficult and because the teaching profession does not pay well, there is a shortage of qualified mathematics teachers. All this may be true, but can we not at least consider proposals for remedial summer schools for the benefit of these teachers? On this subject, a tentative plan in California is to give intensive summer schools for the best teachers available, and train them to be "teacher-leaders" who will then go back to their districts to help teachers in category (A). Other proposals will undoubtedly surface once this problem is confronted honestly.

There are many ways to explain why so many mathematics teachers perform poorly in the classroom. One obvious reason is that they never learned the subject properly all the way from kindergarten through college. This then precludes the possibility of good teaching because one cannot teach what one does not know. Another explanation of teachers' poor performance is that their stilted, constricted, and rigid approach to the subject turns students off at the outset. In most cases, these teachers have never seen any kind of mathematics teaching other than their teachers' and professors' equally stilted, constricted, and rigid style all through school and college. Years and years of exposure to bad teaching naturally takes its toll. In theory, such flaws in one's teaching would be eliminated in classes on pedagogy in the school of education, but there are two reasons why this theory fails. The first is that undoing this kind of pedagogical problem requires a deep knowledge of the subject on the part of the prospective teachers and their instructors alike, as we shall see below. The second one is that one cannot undo the harmful effects of years and years of direct observations of bad teaching in a semester or two of gentle discussions of good manners in teaching. So a starting point of preservice professional development has to be better teaching across the board in college (and of course in the schools too). This sounds too vague, so I will go into some specifics.

Although my intention is to discuss the preservice professional development of teachers, the preceding paragraph necessitates the discussion of how to better educate the *teachers* of future teachers. This is more difficult because the scope is much broader. For one thing, the professors in college are usually those who got advanced degrees in mathematics and therefore went through only those mathematics classes which prepare them for graduate study. As we all know, these are not classes that go out of their way to emphasize pedagogy or motivation for the subject matter. More to the point, the extreme emphasis on technical competence in such courses—which is, incidentally, mostly justified—may in fact intentionally slight motivation or any pedagogical considerations. Since the main focus of these classes is usually on the main theorems of the subject, any discussions of how those theorems were arrived at are usually reserved for conversations between thesis advisors and their protégés. Given this academic tradition, one should not be surprised that a generic college professor in charge of prospective teachers would be unaware of the need to show these teachers that *the way mathematics is usually presented in lectures and written in textbooks is* not *the way mathematics is actually done*.

Content knowledge of mathematics includes the knowledge of how mathematics is usually done: the unending trials and errors, the need to search for concrete examples and counterexamples to guide one's intuition, and the need to make wild guesses as well as subject these guesses to logical scrutiny. This knowledge is indispensable to effective teaching because it directly impacts on class presentations, problem solving, and assessment of students' work. Without this knowledge, a teacher cannot help but make students believe that problems are solved by sitting down, meditating, and waiting for the perfect solution to ooze out of their pens like a tube of toothpaste when slightly squeezed. Out of this misconception about mathematics come classroom presentations that sound like textbook writing at the board. This is the kind of teaching that one finds, for example, in an article of Alan Schoenfeld (When good teaching leads to bad results; The disasters of 'well-taught' mathematics courses, *Educational Psychologist*, 23(1988), 145-166). It is perfectly clear to any working mathematician that the kind of teaching described therein is *bad* teaching—and those courses were poorly taught by any standard—and that the badness has little to do with failure of pedagogical technique and everything to do with the lack of mathematical content knowledge. I believe one achievement of the recent reform is to make professors realize that most students do not learn mathematics without being shown, step by step, how it is done in many concrete cases. A key emphasis in professional development, preservice or inservice, therefore has to be on exposing teachers to the process of doing mathematics.

A correction of this situation would require that even in courses designed for future mathematicians, the instructors make it a point of demonstrating to students that mathematics is mostly done in messy and tentative ways. This requires extra time: either the instructor donates it by doing extra sessions, or the course syllabus is reduced. Neither is attractive, though it can be done. An alternate solution may be for all instructors of courses designed for future teachers to undergo some pedagogical training which focusses on this need. This issue should receive more attention in current discussions of mathematics education.

Some people believe that what is needed in mathematics professional development at this juncture is a sequel to *A Call for Change* (James R. C. Leitzel, ed., Mathematical Association of Amererica, 1991) and the NCTM Teaching Standards (*Professional Standards for Teaching Mathematics* National Council of Teachers of Mathematics, 1991). If so, I would like to point out what seems to me a grievous damage done to the teaching profession by the latter document, and hope that the forthcoming CBMS volume would be able to counteract the ill effects the NCTM document has generated. In the 200 pages of the NCTM Teaching Standards, the critical importance of teacher's mathematical knowledge for effective teaching is described in all of 8 pages on pp. 132-140. Most likely few would ever get to p. 132. Moreover, if this knowledge is considered important, why do all the vignettes have to do with pedagogical techniques only? There seems to be a bit of a credibility problem here: does this document really believe in the importance of teachers' content knowledge? Lest I would seem to display an overwhelming bias against this document, I should hasten to add that, until recently, most of the other documents and articles on the subject of teaching that I have come across also suffer from a lack of proper emphasis on the need of basic mathematical competence.

It has been suggested that the lack of attention in the NCTM Teaching Standards on content knowledge was in fact due to the simultaneous appearance of the MAA volume *A Call for Change*. By tradition, MAA is supposed to deal with the content knowledge component of teaching, so that NCTM would defer to MAA on the issue of content and only take care of pedagogical techniques in any discussion of teaching. If this was indeed the case, then the NCTM document has misled its readers by not making an explicit statement to this effect anywhere. In the context of the mathematics education reform, such a reference to content knowledge is all the more relevant because the new pedagogical practices advocated by the reform—the discovery method and the spontaneous dialogue between students and teacher—require an uncommon mastery of the subject. Teachers already shaky in the mathematics of the traditional classroom would have their mathematical weakness mercilessly exposed in the new pedagogy.

There is perhaps no better way to demonstrate what a serious document on the teaching of mathematics can do in the way of underscoring the critical role of content knowledge in teaching than to cite one that does this surpassingly well. The recent volume by Liping Ma, *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States* (Lawrence Erlbaum Associates, Mahwah, NJ, 1999.), shows in a most convincing fashion that without a *profound understanding of fundamental mathematics*, it is impossible to be a competent mathematics teachers in K-5. On page after page, she gives concrete examples of how teachers with deficient understanding of some basic mathematical topics mangle their answers or explanations to innocuous questions that naturally arise in a classroom. Her main conclusion is that

> Having considered teachers' knowledge of school mathematics in depth, I suggest that to improve mathematics education for students, an important action that should be taken is improving the quality of their teachers' knowledge of school mathematics. (Chapter Seven, opening paragraph)

This point of view, I may add, has been advocated by an increasing number of educators, especially Deborah Loewenberg Ball. Consult the references in Ma's volume.

I cannot resist the temptation of inserting an observation here. Much has been made of the underachievment of our school students compared with those in other countries; for example, how our students' mathematical knowledge is generally a grade or two behind those of Japan or Singapore. Could this be directly related to our teachers' mathematical insecurity? It is a fact of human nature that teachers who are uncomfortable with the mathematics they teach are not the ones to push hard for excellence in the mathematical performance of their students, and most students—especially the young ones in K-5—do not make an effort to learn if they are not pushed hard. From this perspective, students' underachievement is altogether understandable.

I hope that if indeed there is going to be a sequel to *A Call for Change*, it will make a point of presenting vignettes on the teaching of basic concept or skills, which show on the one hand how a superficial knowledge of mathematics leads to a rote-learning type of teaching, and how a deeper understanding of the subject makes for lucid and vibrant classroom presentations. The Ma volume gives an ample supply of such examples.

This line of reasoning naturally leads to the question of whether the majority of those training prospective mathematics teachers are themselves mathematically qualified. When I started getting interested in education in 1992, the first thing I saw was that the mathematics education crisis was largely precipitated by mathematically unqualified teachers. I tried to raise this issue, but my knowledgeable colleagues immediately advised me to cease and desist because "the teachers should be part of the solution, not part of the problem". Now that we are in 1999, not only does the issue of "unqualified teachers" merit the attention of the American Mathematical Society, the Mathematical Associatioon of America, and CBMS, but I am even encouraged to express all my misgivings. With this in mind, I think I will risk some abuse this time around by raising this very legitimate issue of the qualification of the *teachers* of teachers. The main problem is that, in most cases, these people are being asked to do perhaps more than is realistically possible. We want them to be both good mathematicians and good teachers. In addition, we also expect them to have thought deeply about both the K-12 curriculum and the special needs of school teachers. It is therefore not surprising that in many states the professional development in mathematics has been less than effective, because there is a shortage of qualified people to do the job properly.

Would it be conceivable to convene a meeting of the professors engaged in producing teachers to open a dialogue on this issue? Could we even contemplate a summer school for these professors?

Some would consider such a suggestion about our colleagues on college campuses outrageous. If we take a step back, however, and look at what we are discussing here—how to rectify the situation regarding unqualified teachers—we would see that we are doing something that teachers would rightly regard as outrageous. Such being the case, why should we spare ourselves? I hope we do not wait another decade before this topic is legitimized and tackled.

Another unpleasant issue in this connection is the proposal of a standard credential examination for all new teachers. Because unqualified teachers do present a crisis in California, it occurred to me that the state-sponsored measures to meet this crisis, no matter how good, would matter little if new unqualified teachers continue to pour into the school system. What we need in this situation is a better filter. There are two standard objections to such a proposal. The first one is that such an examination does nothing to test whether a person can handle the many knotty teaching problems that arising in a classroom. A second one is that examinations are best left to colleges and universities; they should be the ones to uphold the minimum standards, not another bureaucracy. The response to the first one is that while such an examination can never pretend to be an instrument that generates good teachers, what it does, if it is done competently, is to eliminate the numerous absurd cases currently plaguing our school system. As to the second objection, the sad fact is that we are at this desperate juncture precisely because we have *misplaced* our trust in colleges and universities to do this job. I quite realize that emotions run high on this topic, but some kind of quality control is necessary and, until we have quality control (as physicians and lawyers do), the teaching profession will continue to be at the bottom of the salary scale of the professionals, which will insure the continued decline in the quality of teachers. . .

Incidentally, the current leadership of the American Federation of Teachers has come out in favor of such an examination for new teachers.

Finally, a few words about what mathematics should be taught to prospective teachers. The problem here is epitomized by the following personal experience of mine. Sometime ago, I was consulted by a colleague from a university in the west concerning the content of a course designed for prospective teachers pursuing an M.A. degree. I was told that his department had experimented on separate occasions with the teaching of set theory and the foundation of real numbers, but that the results of either effort were not good. At that point it so happened that Joseph Rotman's *Journey into Mathematics* (Prentice Hall, 1998) had just arrived. Its content is centered on induction, Pythagorean triples, the area of disks and the number π, and the problem of solution by radicals for polynomials. In short, topics that would be of vital interest to a high school teacher. Rotman stated in the preface that he gave special attention to proofs. He wrote the book with the hope that students would learn not only some new mathematics that is relevant to them but also how to prove theorems. I looked through the book a bit, thought that it was promising for exactly the kind of course my colleague had in mind, and recommended it accordingly. The answer came back that his department had misgivings about Rotman's book because it looks too much like a high school text, and would therefore very likely reject my recommendation.

There I think we have the generic problem in a nutshell: people who are engaged in preservice professional development for teachers are out of touch with what the latter really needs. They convince themselves about what these teachers *should* know, namely, "good mathematics", and proceed to devise curricula that fulfill their lofty vision. Thus prospective teachers should learn only what is worthy of a regular mathematics major, so they take courses which bypass or too quickly review high school topics, but which duly emphasize advanced materials. For example, how many mathematics courses for elementary school teachers linger over the multiplication algorithm and explain in detail that it is nothing but a shorthand for the distributive law? (Cf. the previously cited volume of Liping Ma) How many geometry-for-teacher courses pay special attention to enlarging teachers' knowledge and understanding of Euclidean geometry proper instead of devoting *the bulk of the time* to an abstract discussion of the subtleties of finite geometry and axiomatic systems? Is there any course that teaches prospective teachers how to approach fractions in schools, not as equivalent classes of ordered pairs of integers, but in ways accessible to, say, sixth graders? What we are witnessing is a certain disjunction between what is in the mathematics courses that prospective teachers are required to take and what these same teachers desperately need. There is an alarming *irrelevance* in the present preservice professional development in mathematics.

It would be wrong to infer from the preceeding paragraph that I favor the teaching of only those materials that can be immediately used in the classroom. On the contrary, I totally reject the kind of thinking that is responsible for making inservice professional development synonymous with the teaching of "classroom projects" sans mathematical substance. Between the two extremes of teaching what professors consider to be good mathematics and what teachers consider to be instantly useful, we must carve out a solid middle ground. Until objective conditions have changed, such as a marked improvement in teacher qualification overall, I would go out on a limb and declare that what is needed at present in the preservice development of teachers in grades 6-12 are courses which consolidate, *mathematically*, those topics which do not stray far from the high school mathematics curriculum.[1] In particular, they should revisit all the standard topics in high school

[1] In this article, I will lump together the professional development of junior high and high school teachers. This is as it should because teachers of junior high should know exactly what is needed in high school; for example, they cannot possibly teach Algebra I competently if they do not know thoroughly what is in store for their students in Algebra II. In practice, though, it may be necessary to separate these two groups of teachers

from an advanced standpoint, and enliven them with motivation, historical background, inter-connections and, above all, proofs. For example, in re-examining the quadratic formula, not only will its derivation be emphasized and its importance explained, but also the fact that the same proof works whether the coefficients a, b, and c of ax^2+bx+c are real numbers or merely elements of a field which has square roots. If a, b, and c are complex, then this would give a formula solving all complex quadratic equations. If a, b, and c are functions in y, then this would give an explicit expression of x in terms of y when x is defined implicitly by $ax^2+bx+c=0$. In addition, it can also be pointed out that the technique of completing squares used in deriving the formula is also used to analyze the graphs of quadratic polynomials in several variables without mixed terms.

Another example is the clarification of the concepts of length and area. It is safe to say that an overwhelming majority of teachers have no clear idea of what is meant by the area of a disk or the length of a circle. And this is in spite of the fact that πr^2 and $2\pi r$ are used routinely in classrooms of grades 6–12. One consequence of this gap in teachers' understanding is that students come out of high school without knowing what π is and, when told that it is the area of the unit disk, do not even know that this is a good definition because, to them, the concept of "area" is as mysterious as "π". A college course giving mathematically correct definitions of area and length—even if the techicalities are glossed over—would be more valuable to teachers than one on the Cayley-Hamilton theorem (a prime example of "good mathematics").

This discussion of length and area naturally leads to the question of why not just ask teachers to complete a course on introductory analysis. After all, when the Riemann integral is rigorously defined, the correct definitions of length and area for the most common curves and regions would follow. Would this not be sufficient? The answer to this question is in fact the key to the understanding of the special nature of preservice professional development. The point is that the definitions of length and area in terms of the Riemann integral are not of much use in school mathematics instruction. Giving a *correct* verbal explanation to a 6th grader about the area of a disk takes more than the writing down of an integral! The teacher must be able to give an intuitive definition of area that is stripped of technicalities but is

because the average junior high school teacher may not be motivated to go deeply into high school mathematics.

conceptually correct. But how many of our mathematics majors—regardless of whether they are prospective teachers or not—can extract from the definition of area as an integral a correct intuitive notion of area that makes sense to a young kid? The Utopian notion is that feeding our prospective teachers good mathematics would make them effective teachers because, *once the technicalities are digested*, they would be able to translate the abstract ideas into intuitive notions to enrich their teaching in the classrooms. The evidence all around us points to the fact that, at least in this country in 1999, this Utopian ideal has no contact with reality. It is therefore the duty of those in charge of preservice professional development to take it upon themselves to do the translation of the abstract mathematics into usable form for the use of prospective teachers. Because this translation becomes generically more difficult the further one ventures into the advanced parts of mathematics, for this reason, I advocate that preservice professional development should hover around the outskirt of the high school curriculum.

Let us take a cursory look at possible teaching materials for the professional development of teachers in grades 6-12. At the moment, scholarly texts along the line of Rotman's are few and far between. Because algebra seems to be the prevailing weakness of most teachers, the monograph by I. M. Gelfand and A. Shen, *Algebra* (Birkhaüser, Boston-Basel-Berlin, 1993) could be used to advantage for this purpose. This slim volume displays impeccable mathematical taste and, unlike most algebra books on this level, treats algebra as a *mathematical discipline* rather than a collection of symbols. It provides the perfect antidote to the usual misconception of algebra. A word of caution, however: as a textbook for a course, Gelfand-Shen suffers from excessive brevity and the presence of quite a few passages which are bound to mystify most classroom teachers. (For more details, see the review by the author, Reviews of three books by Gelfand, *The Math. Intelligencer* 17(1995), 68-75.) An instructor using Gelfand-Shen as a text should be prepared to provide plenty of supplementary comments and materials, e.g. exercises, applications, historical background, etc.

A recent volume by O. A. Ivanov, *As easy as* π (Springer-Verlag, 1999), was written expressly for Russian mathematics teachers, but the exposition may be too terse and the requisite mathematical maturity for its comprehension a bit too high for their American counterparts. One theorem, for example, is about the nonzero elements of a finite field being a cyclic multiplicative group, and the reader is assumed to be familiar with the concept of a finite field. But there are elementary materials in the book (elementary combinatorics, Pythagorean triples, arithmetic-geometric-mean inequality, Euler's formula for planar graphs, etc.), and a skillful expositor could conceivably make good use of it. There is an older book, *Excursions into Mathematics*, by A. Beck, M. N. Bleicher, and D. W. Crowe (Worth Publishers, NY, 1969), which is on roughly the same level as Rotman's. Although some may find it too discursive, it nonetheless deserves serious consideration as a text for preservice professional development. If one is willing to narrow the mathematical scope a bit, then the text *Elementary Number Theory* by Underwood Dudley (2nd ed., W. H. Freeman, 1978) is just right for this purpose. Would that there were more books like Dudley's in other areas of mathematics.

My overriding concern is that preservice professional development should be more realistic and should try to teach mathematical topics that are more elementary and more relevant to teachers' daily teaching duties. And make sure that they learn them! I could be blinded by my experience in California to overlook that the situation is really different elsewhere, but I have learned to be modest in my expectations of what a teacher should know. I once wrote a paper, *On the training of mathematics teachers,* (http://www.math.berkeley.edu/∼wu) which seems to contradict my present position. I seemed to be saying in that paper that I want all teachers to know Galois theory, fast Fourier transform and the Gauss-Bonnet theorem. Yes, I wish they knew all that, but that was because I was operating under the constraint of educating a mathematics major at Berkeley or similar universities to be a good teacher. If I were to design a specific curriculum for prospective junior high and high school teachers—such a curriculum exists in many institutions of higher learning—I would be much more realistic. The goal I set would be to require all teachers in grades 6-12 to be fluent in all of the following topics:

> the rational numbers and why the basic operations are valid ($\frac{a}{b}/\frac{c}{d} = ad/bc$, $(-\alpha)(-\beta) = \alpha\beta$, etc.); the equivalence of rational numbers with finite and repeating decimals (this includes the geometric series); real numbers as a complete ordered field and relation with the rationals (without undue emphasis on the completeness axiom); basic number theory (the fundamental theorem of arithmetic, the use of Euclidean algorithm to find *gcd*, the Chinese Remainder Theorem, and some $\bmod n$ arithmetic); complex numbers; polynomials (including the factor theorem, the Euclidean algorithm, and why polynomials with real coefficients

> must split into a product of linear or quadratic factors); solutions of cubics by radicals and the history of the general problem of solving polynomials by radicals; basic linear algebra (concept of a vector space, linear independence, basis, solution of linear systems by Gaussian elimination); basic coordinate geometry including conic sections in two dimensions and planes and lines in space; trigonometry including a good understanding of the sine-cosine addition theorems; basic combinatorics and a *good* understanding of the fundamental ideas of probability; elementary graph theory; the basic theorems of calculus (because many of them will be teaching AP calculus); basic statistics, including its limitations; Euclidean geometry (axiomatic systems, problems with Euclid's axioms, and especially advanced theorems); the concept of measurement in general, and length, area, and volume in particular; and finally, isometries of the plane and 3-space (their relation with the concept of congruence, and the structure of isometries as a composition of reflection, rotation and translation).

In addition to these mathematical topics, I would also require all teachers in grades 6-12 to have a general knowledge of the history of mathematics. They should certainly know something about Euclid's *Elements*, in particular the fact that it employs no algebraic notation so that even the "Euclidean algorithm" is presented geometrically. They should also know a little bit about the key figures and their contributions: Archimedes, Descartes, Fermat, Newton, Leibniz, Euler, Gauss, Riemann, etc. Above all, they must learn about the long and arduous road mankind had to travel before arriving at (what may be easily mistaken to be routine formalisms of) the decimal number system and algebraic symbolic notation. It is open to speculation whether those who advocate the deemphasis of symbolic computations in the current mathematics education reform are aware of the last bit of history.

I regret that my present vision does not include much of anything about groups, rings, Jordan normal form, or the *Theorema Egregium* (Gauss' own terminology for his fundamental discovery that the notion of Gaussian curvature is an intrinsic property of a surface). I will change my mind when things have improved.

In general terms, what I wrote in the above-quoted paper *On the training of mathematics teachers* about what I consider to be desirable characteristics of courses for teachers seems to be still valid, so let me just repeat the main points here:

(A) Only proofs of truly basic theorems are given, but whatever proofs are given should be complete and rigorous.
(B) In contrast with the normal courses which are relentlessly "forward-looking" (i.e., the far-better-things-to-come in graduate courses), considerable time should be devoted to "looking back".
(C) Keep the course on as concrete a level as possible, and introduce abstractions only when absolutely necessary.
(D) Ample historical background should be provided.
(E) Provide students with some perspective on each subject, including the presentation of surveys of advanced topics.
(F) Give motivation at every opportunity.

But what about teachers in K-5? It used to be thought that these teachers need to know very little mathematics, but works of Deborah Loewenberg Ball and the book of Liping Ma cited above, disabuse people of this illusion. In the best of all possible worlds, there would be mathematics specialists in K-5 and they too would know what secondary school teachers know. Since the world is far from best possible, however, I will have to propose a tentative list of topics for the preservice professional development of teachers in K-5. With the Liping Ma volume in mind, here is then the list:

> an understanding of the place value of integers in the decimal system; an understanding of the usual algorithms of the four arithmetic operations in terms of the associative, commutative and distributive laws; an understanding of division as the inverse of multiplication; the division algorithm with remainder and its application to finding the *gcd* of two integers; primes and the fundamental theorem of arithmetic; rudimentary $\bmod n$ arithmetic (e.g., the face of a clock); arithmetic and geometric progressions; arithmetic and geometric series; arithmetic to arbitrary base (with a careful study of at least one case, say to base 5), and relation with the decimal system; fractions as numbers on the number line and a proper understanding of the four arithmetic operations (including an understanding of why $\frac{a}{b}+\frac{c}{d}=(ad+bc)/bd$ is the *correct* definition of the addition of two fractions and why the usual one involving *lcm* should not be emphasized); relation of this definition of fractions as numbers ith the usual conception of fractions as operators ("one-third of a pie"); a general

> understanding of rational numbers and their position in the real number system; the definition of proportion and ratio as nothing more than the division of two numbers; expansion of rationals as decimals; equivalence of rationals with finite and repeating decimals; a qualitative understanding of real numbers as decimals; a general understanding of the concept of measurement; length and area; area formulas of standard rectilinear planar figures; the number π; the Pythagorean theorem and its proof by the use of area; rudiments of finite probablity.

Perhaps a point should be made about the teaching of *some* standard algorithms for the four arithmetic operations on whole numbers. Elsewhere (Basic skills versus conceptual understanding: A bogus dichotomy in mathematics education, *American Educator*, Fall 1999, 14–19, 50–52), I have presented some reasons why these algorithms should be taught, and taught well, because they embody substantive mathematics. On the other hand, a perceptive reader would notice the conspicuous absence of statistics in the preceding list. In California, for example, the teaching of statistics is supposed to begin in grade 1. It is not clear to me that devoting a sizable amount of time to the teaching of statistics in grades 1-5 is altogether advisable. Other than the most naive notion of statistical sampling, what is there to be taught at this level? For this reason, I hesitate to write the statistics requirement into the curriculum of preservice professional development.

One final comment: *if* the prospective teachers are well grounded in high school mathematics, everything in the preceeding list could be covered with ease in a one year course with 4 units per semester. A very interesting part of professional development is of course to decide what to do in a realistic situation when the teachers are far from being well prepared.