“Order of operations” and other oddities in school mathematics

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September 13, 2007

One of the flaws of the school mathematics curriculum is that it wastes time in fruitless exercises in notation, definitions, and conventions, when it should be spending the time on mathematics of substance. Such flaws manifest themselves in assessment items which assess, not whether students know real mathematics, but whether they could memorize arcane rules or senseless conventions whose raison-d'être they know nothing about. An example is the convention known as the *Rules for the Order of Operations*, introduced into the school curriculum in the fifth or sixth grade:[1]

(1) Evaluate all expressions with exponents.
(2) Multiply and divide in order from left to right.
(3) Add and subtract in order from left to right.

In short, these rules dictate that, to carry out the computations of an arithmetic expression, evaluate the exponents first, then multiplications and divisions, then additions and subtractions, and always from left to right. To sixth graders, these rules must appear random and therefore meaningless, and the meaninglessness of it all gives rise to the infamous *Please Excuse My Dear Aunt Sally* mnemonic device.

In this short article, I will first briefly discuss the mathematical background of these Rules in order to make sense of them, and then explain why it is unprofitable to pursue these rules vigorously in the arithmetic context the way it is

[0] **June 1, 2004**. I am grateful to Richard Askey for useful suggestions, and to Ralph Raimi for his criticisms of a preliminary draft which are as merciless as they are constructive.

[1] I am referring explicitly to the American curriculum. It may be different in other countries. See footnote 2 below.

done in most school classrooms. Finally, I will present an argument against using assessment items in standardized tests that assess nothing more than students' ability to commit certain definitions and conventions (like Rules for the Order of Operations) to memory.

The mathematical origin of Order of Operations

To understand how the Rules for the Order of Operations came about, one has to consider polynomial expressions in algebra. Look at the following polynomial function of degree 8 with coefficients 17, 2, 1, 0, 0, 6, 0, 3, 4:

$$17x^8 + 2x^7 + x^6 + 6x^3 + 3x + 4, \tag{1}$$

where x is any number. The notational simplicity of this expression must be obvious to one and all, and this simplicity is the result of a common agreement that what this expression *really* says is:

$$(17(x^8)) + (2(x^7)) + (x^6) + (6(x^3)) + (3x) + 4. \tag{2}$$

To be precise, the symbolic notation in (1) is free of the annoying parentheses in (2) because the **convention** of performing the operations in the order indicated by the parentheses in (2) is universally accepted, namely,

(A) exponents first, then multiplications, then additions.

To drive home the point, the sum $17x^8 + 2x^7$ does not mean

$$\{[(17x)^8 + 2]x\}^7,$$

which would be the case if the operations were uniformly performed from left to right, but rather

$$(17(x^8)) + (2(x^7))$$

as indicated in (2).

The rule of "multiplications before additions" may sound simple, but these three words contain more than meets the eye. Because we are in the realm of algebra, "division by a (nonzero) number c" is the same as "multiplication by $\frac{1}{c}$". Moreover, "minus c" is the same as "plus $(-c)$. Therefore if one rewrites what is in statement (A) above in the language of arithmetic, then one would have to expand it to:

(B) exponents first, then multiplications and divisions, then additions and subtractions.

Except for the stipulation about performing the operations "from left to right", (B) is seen to be exactly the same as the Rules for the Order of Operations. It is important to note that *this stipulation about "from left to right" is entirely extraneous*, because the associative laws of addition and multiplication ensure that it makes no difference whatsoever in what order the additions or multiplications are carried out. Everything that the Rules for the Order of Operations wish to achieve therefore has already been encoded in (B), and hence also in the succinct statement (A).

To recapitulate, the notational simplicity of expression (1) is worth preserving, and the convention — encapsulated in statement (A) — which ensures the preservation of this simplicity is itself disarmingly simple. This is why polynomials continue to be written as in expression (1).

Why Not in Arithmetic?

No convention is sacrosanct. Every convention is artificial, and as such, should be kept only if it continues to serve a purpose. We have seen clear evidence of why the convention in statement (A) deserves to be kept in algebra. What is under discussion is whether this convention, in the guise of Rules for the Order of Operations, should be taken seriously in the context of arithmetic. To this end, we digress to review the pedagogical aspect of these Rules in the elementary mathematics curriculum.

For young kids in the primary grades, arithmetic is essentially limited to a single arithmetic operation between two numbers, When they confront multiple additions or multiplications such as $8 + 4 + 9$ or $3 \times 7 \times 5$, it would be wise to let them add and multiply in any order. They would experience the associative and commutative laws in the process. To them, the need of accurate symbolic notation, so fundamental in mathematics when it has to be done in any level of depth, is far into the future. It is only around the fourth grade that the idea of each arithmetic operation being *binary* (i.e., each can operate on only two numbers at a time) is brought to their consciousness and parentheses are introduced to give explicit instructions as to which two numbers should be operated on at any point of a long succession of arithmetic operations.

The reason the Rules for the Order of Operations are taught around the sixth grade is twofold.[2] One is that, by the sixth grade, students are presumably already familiar with the use of parentheses so that some shortcut would be warranted. Familiarity always breeds the call for simplifications and abbreviations. It seems natural, for example, that students be allowed to write $15 + 6 \times 21$ in place of $15 + (6 \times 21)$. A second reason is that, anticipating the imminence of students' encounter with algebra, especially the symbolic expressions for polynomials as in (1) above, it is felt that a gentle introduction to some of the algebraic conventions would be appropriate. Thus, in addition to these Rules for the Order of Operations, students are also taught to use a dot in place of the multiplication symbol.

The introduction of these Rules in the fifth or sixth grade is therefore either a matter of minor convenience or a decision based on pedagogical considerations. In the context of the classroom, two comments immediately come to mind. The first is that by the sixth grade, most students already know about the associative and commutative laws of addition and multiplication and there is no reason whatsoever that students should be subjected to the stricture of performing arithmetic operations only from left to right. Moreover, they also know that subtraction is the adding of a negative number and division by c is multiplication by $\frac{1}{c}$. Under no circumstance, therefore, should the Rules be taught in the deadly three-step format quoted at the beginning of this article. It would make more sense to use it in the form (A) above, with ample explanation of its background and motivation. This would at least get rid of *Aunt Sally*.

To my knowledge, statement (A) is not commonly taught in sixth grade classrooms.

A second comment is that, once the reason for the introduction of these Rules in the sixth grade classroom is understood, it should be clear that it is not educationally worthwhile to pursue a policy of enforcing these Rules at all costs. There are in fact numerous situations where a pedantic insistence on these Rules

[2] I reiterate that this statement refers to the American curriculum as of 2004. My colleague Sasha Givental alerted me to the danger of a possible mis-interpretation of this statement as an argument for not teaching these Rules until the sixth grade, and then only teach them but do not insist that students know them! In the Russian curriculum, for example, third graders are already introduced to the symbolic notation and, for them, it would be better to know about the Rules sooner than later. Elsewhere, I have argued that if students are still ill-at-ease with the use of symbols in the sixth grade, then their chance of being successful in algebra in grade eight would be vanishingly small.

leads to bad results. Take, for example, the writing out of the expanded form of a number such as 52947. A blind adherence to the Rules would lead to the following expression:

$$5 \times 10000 + 2 \times 1000 + 9 \times 100 + 4 \times 10 + 7.$$

Does this expression serves the purpose of clearly displaying the positions (places) of the five digits 5, 2, 9, 4, 7? Of course not. One way to remedy the situation is to retain the rule of multiplications-before-additions but make a special effort to insert appropriate spaces between symbols, thus:

$$5 \times 10000 \quad + \quad 2 \times 1000 \quad + \quad 9 \times 100 \quad + \quad 4 \times 10 \quad + \quad 7.$$

This is not an efficient way to achieve clarity, however. It would be far simpler to just *ignore the Rules* and write instead,

$$(5 \times 10000) + (2 \times 1000) + (9 \times 100) + (4 \times 10) + 7.$$

A slightly different example of the silliness of the Rules is an expression such as the following:

$$24 \div 48 + 54 \div 18 \times 7 \div 14 + 2 \times 3 \div 18 \tag{3}$$

This kind of writing is obviously a nightmare to read because one cannot see clearly what computations have to be performed until one squints hard. By comparison, one gains an appreciation of the rule stated in (A) for symbolic computations when one observes how the absence of the $\times$ symbol in expression (1) makes all the multiplications stand out. Therefore, better to *forget the Rules* and replace "dividing by a" with "multiplying by $\frac{1}{a}$" to rewrite (3) as:

$$\left(24 \times \frac{1}{48}\right) + \left(54 \times \frac{1}{18} \times 7 \times \frac{1}{14}\right) + \left(2 \times 3 \times \frac{1}{18}\right).$$

The example in (3) is perhaps a bit extreme, and was concocted to make a point. However, the next one is taken from p. 207 of a school mathematics review book [1]:

Evaluate $4 + 5 \times 6 \div 10.$

Now one *never* gets a computation of this type in real life, for several reasons. In mathematics, the division symbol $\div$ basically disappears after grade 7. Once fractions are taught, it is almost automatic that $6 \div 10$ would be replaced by $6 \times \frac{1}{10}$. Moreover, if anyone wants you to compute $4 + 5 \times 6 \div 10$, he would certainly make sure that you do what he wants done, and would put parentheses around $5 \times 6 \div 10$ for emphasis. In a realistic context then, $4 + 5 \times 6 \div 10$ would have appeared either as $4 + (5 \times 6 \div 10)$, or $4 + (5 \times 6 \times \frac{1}{10})$. The original problem is therefore a kind of *Gotcha!* parlor game designed to trap an unsuspecting person by phrasing it in terms of a set of unreasonably convoluted rules.

School mathematics education should not engage in enforcing rules for its own sake, especially if the rules become indefensible. One may teach the Rules for the Order of Operations in arithmetic in the form (A), but make clear to students at the same time that these Rules should be applied judiciously and never at the expense of clarity.

Assessment and Related Matters

A sad consequence of the usual mathematics lessons given in an environment where things like the Rules for the Order of Operations are routinely taught is that, when it comes time to examine students on what they have learned, a disproportionate amount of space is given to questions concerning terminology and conventions, displacing questions about mathematics. Students are thereby encouraged to memorize things they never learn to use, and their teachers are also dragged into the game, because they know that sure-fire points on the examinations can be achieved by this useless memorization. What these test items ultimately succeed in doing is to legitimize teaching- and learning-by-rote

With this in mind, let us consider some standard assessment items.

Example 1.[3] If the expression $3 - 4^2 + \frac{6}{2}$ is evaluated, what would be done *last*?
(1) subtracting
(2) squaring

[3] Taken from New York State Mathematics A Regents Examination, June 17, 2003. I am using this item only because it comes in handy for the purpose of the discussion here. There was (July, 2003) a sizable controversy surrounding this state test, and I will not engage in this controversy.

(3) adding
(4) dividing

This example clearly has in mind the fact that students would resolutely follow the three-step Rules for the Order of Operations and end their computations by adding $\frac{6}{2}$. The correct answer would then be (3). But for reasons explained earlier, the order of addition or subtraction is irrelevant and one could very well end the computation by *subtracting* 16 from 6 ($= 3 + \frac{6}{2}$). This example therefore has no unique answer.

To understand how anything like the Rules for the Order of Operation manages to stay in the elementary classroom, we have to take a broader view of where mathematics education stands as of year 2004.

Mathematics is what mathematicians do.[4] Therefore when research mathematicians are excluded from mathematics education, as they have been in the recent past, the mathematics in school mathematics education becomes isolated from the mainstream of mathematics and, inevitably, evolves in ways that deviate from the mainstream. More than just a mindless invocation of the accepted Darwinian doctrine, this simple fact about school mathematics education has been amply borne out in the findings of research mathematicians when they took a serious look at school mathematics of the last decade. The observed phenomenon of teaching and assessing the Rules for the Order of Operations is but a small part of this deviation. Without going into an extended discussion of the more flagrant mathematical flaws in school mathematics (cf. [2]), let us just mention a few that are relevant to the present discussion.

The flaws we bring up fall into two categories, and both serve as reminders of what happens when one forgets what mathematics is about and gets fixated instead on its byways and superficial gloss. The flaws in the first category result from attaching too much importance to the *name* of a concept or the literal execution of a procedure, without attempting at the same time to learn the mathematical significance of the concept itself or the reason or spirit behind the procedure. The senseless pursuit of the Rules for the Order of Operations in arithmetic is an example of the latter. As to the former, there are perhaps

[4] This seems to be the only acceptable definition of mathematics to most mathematicians.

too many examples. Testing students on just knowing such terms as "additive inverse of a number", or "multiplicative identity", or the "inverse of a statement" illustrates how a vocabulary test can be passed off as a test of mathematical proficiency. In the context of mathematics learning, knowing that a number is called "the additive inverse " of another number is of zero value compared with knowing how to make use of the uniqueness of additive inverse to explain, for example, why $-(r+s) = -r-s$ and $(-r)(-s) = +rs$ for all rational numbers r and s. The quality of mathematics assessment would go up appreciably if a greater effort is spent on items of the latter type.

Now, no matter what one says about "additive inverse", it is at least part of the standard mathematics vocabulary. But when students are asked to learn what "the inverse of a statement" means, we pass from silliness to the land of Kafka. I have made an effort to ask a few research mathematicians informally whether they knew what the inverse of a theorem meant, and none had any idea. If this terminology seems not to matter to people who do mathematics for a living, should we insist that all students must know it?

Yet another example is to test students on their ability to recognize which of the associative, commutative, or distributive laws is used in a given situation, when they should be tested on the fundamental role these laws play in school mathematics. For example, why the multiplication algorithm for whole numbers is valid (the distributive law, mainly), or why the division of whole numbers can be given both the partitive and measurement interpretations (the commutative law of multiplication). Given that, at present, these laws already induce yawns among teachers and students alike, test items of the former kind would do nothing but further cement students' perception of mathematics as a meaningless ritual.

The following items taken from real tests[5] give substance to this discussion:

Example 2. What is the inverse of the statement "If Mike did his homework, then he will pass this test" ?
(1) If Mike passes this test, then he did his homework.
(2) If Mike does not pass this test, then he did not do his homework.
(3) If Mike does not pass this test, then he only did half his homework.
(4) If Mike did not do his homework, then he will not pass this test.

[5] New York State Mathematics Regents Examinations, January 27 and June 17, 2003.

Example 3. Which equation illustrates the multiplicative identity element?
(1) $x + 0 = x$
(2) $x - x = 0$
(3) $x \cdot \frac{1}{x} = 1$
(4) $x \cdot 1 = x$

Example 4. Tori computes the value of 8×95 in her head by thinking $8(100 - 5) = 8 \times 100 - 8 \times 5$. Which number property is she using?
(1) associative
(2) distributive
(3) commutative
(4) closure

To the second category belongs the undue emphasis placed on concepts and theorems that stray far from the center of mathematics. For example, most beginning algebra texts still devote pages to teaching the concepts of the *domain* and *range* of relations which are not functions, and teachers are known to proudly drill their students on learning these terms. But these concepts are not used in any part of mathematics except for rare circumstances such as the discussion of *correspondences* in algebraic geometry. There is not much to gain by spending valuable classroom time on this topic.

Another example is the enormous fuss put on the meaning of "variable", "independent variable" and "dependent variable" in almost all beginning algebra texts. The usual definition of a "variable" is that "it is a quantity that changes or that can have different values" (Cf. [1], Glossary). It is also part of the common wisdom in mathematics education that developing an understanding of variable over the grades is important (cf. e.g., [3], pp. 223 and 225). But "variable", so defined, is unknown to mathematics. How could any "understanding" be developed about "a quantity that changes" since any mathematical quantity, be it a number, or a point in Euclidean space, or a matrix, is just what it is. *It does not change.* Putting an elusive, non-mathematical concept of "a quantity that changes" front and center as one that beginning algebra students must come to grips with does irreparable damage to their mathematics learning. It distracts them from focussing on the key concept of a *function*, and all that time devoted to variables should be better spent on learning what a function is instead. They should get to know the rich variety of functions that show up in nature. They should also get used to looking at the totality of functional values $\{f(x)\}$ when x is allowed to be any point in the domain of f. More importantly, they should learn that the properties of a function $f(x)$ are always defined in terms of the behavior of f at each point x of the domain, *one x at a time* rather than for a "changing x."

This troubling encounter with variables need never happen, however. Typically, variable is introduced at the beginning of the discussion on polynomials, but the symbol x in a polynomial $p(x)$ can be taught more simply as an arbitrary *number*, in the same way that a criminal in an unsolved murder case is just an as-yet-unknown human being. There is no need for the extravagant discussion of a "variable" at all. The more sophisticated concept of an abstract polynomial, where x is an "indeterminate", can wait until a later course (e.g., Algebra II) after students have become more comfortable with algebra. Incidentally, mathematicians do use the word "variable" informally to refer to (for example) a point in Euclidean space for convenience, and "dependent variable" and "independent variable" are likewise used by applied mathematicians. These people can afford to do so because they have a thorough grasp of the concept of a function. This is analogous to NBA players engaging in a slam-dunk contest: they actually *take many steps holding the ball in their hands without dribbling it.* Beginners in basketball had better not try to do the same.

A final example of undue emphasis, though more in in-service professional development for teachers than in school mathematics, is the popularity of Pick's Theorem for the computation of the area of a polygon with vertices on integral lattice points in the coordinate plane. There is no evidence of its relevance to school or college mathematics at all.

There is a need to return school mathematics to an emphasis on substance over formalism. We want teaching mathematics to be more than teaching the mere trappings of mathematics, such as recognizing the names of concepts or blindly following rules without regard to reason or common sense.

References

[1] *Math on Call*, Great Source Education Group, 1998.

[2] H. Wu, What is so difficult about the preparation of mathematics teachers? 2001, http://math.berkeley.edu/~wu/

[3] *Principles and Standards for School Mathematics*, National Council of Teachers of Mathematics, Reston, 2000.