The Rôle of Open-ended Problems in Mathematics Education[1]

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Open-ended problems have become a popular educational tool in mathematics education in recent years. Since mathematical research is nothing but a daily confrontation with open-ended problems, the introduction of this type of problems to the classroom brings mathematical education one step closer to real mathematics. The appearance of these problems in secondary education is therefore a welcome sight from a mathematical standpoint. More than this is true, however. While these problems may represent something of a pedagogical innovation to the professional educators, the fact is that many mathematicians have made use of them in their teaching all along and do not regard their presence in the classroom as any kind of departure in educational philosophy. For example, I myself have often given such problems in my homework assignments and exams.[2] Nevertheless, I have chosen to take up this topic for discussion here because, after having reviewed a limited amount of curricular materials for mathematics in the schools, I could not help but notice that they pose certains hazards in practice. These hazards include the possibility of misinforming the students about the very nature of mathematics itself. To avoid meaningless generalities, I have se-

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lected for discussion three such problems to illustrate these concerns of mine. They will serve as a basis for the detailed discussions and general comments about open-ended problems in this article. For reasons of authenticity, I have purposely transcribed these problems essentially verbatim from the original sources, including the use of double question marks and the proliferation of capital letters. Here is the first one:

PROBLEM I (7th Grade):
Little Arboreal gets a pot bellied pig for her birthday. Since it's ok in her city to have pot bellied pigs, she wants to build an enclosure for it in her back yard. The perimeter of her enclosure is 30 units.
WHAT MIGHT ITS AREA BE?? WHAT WOULD THE LARGEST AREA BE??
YOU HAVE TWO (2) WEEKS TO GET THIS DONE.
HINTS: OBVIOUSLY THERE IS MORE THAN ONE ANSWER AND SHAPE. THINK ABOUT **ALL** POSSIBLE SHAPES.

A sketch of the solution to this problem is given in Appendix 1. The key point is that the solution requires a knowledge of the isoperimetric inequality. Note also that in spite of the seemingly amorphous nature of the problem, its solution is totally unambiguous. However, the nature of the solution is such that an overwhelming majority of high school students, leave alone students from the 7th grade, would be unable to supply the correct solution to this problem. So what was the original intention of such a problem? I have talked to one of the teachers about this, and I was told that all that was expected of the students was for them to draw a few simple shapes for the enclosure and write down the area of each shape. Thus the idea seems to be to actively engage the students to think about perimeter and area and to explore the various possibilities on their own, and this is good. There are two worrisome aspects to this problem, however. One is that the students will come away from this problem without knowing its solution since most teachers would be unable to explain it, for the very good reason that the isoperimetric inequality is not part of the standard curriculum of a math major in college. Therefore the students would likely be misled into believing that there is no precise information on this topic whereas the exact opposite is true. The other is that, not knowing the answer to the second question (what would its largest area be?) the teacher may not be able to handle the students' guesses properly. Indeed, if the teacher does not know that there *is* a proof that the circle represents the maximum, would he or she be able to emphasize the fact that even guessing correctly ("it is the circle") is not enough, because whatever passes for mathematical knowledge must be proved to be correct?

It may be worthwhile to elaborate on the first point a bit. The fact that the students should be given the most complete information possible on any given topic should be a noncontroversial one, but just in case, let us consider two examples. If a teacher gives a brief history of this country without mentioning the Civil War, or if a brief description of the geography of this country omits mentioning the Rockies, no doubt eyebrows would be raised everywhere. Now, without making a parody of this, I believe one can argue quite conclusively that the rôle of the isoperimetric inequality in Problem I above is just as dominant as those of the Civil War and the Rockies in their respective situations. So why would otherwise sensible educators create a situation whereby the teacher is put in the awkward position of having to commit such a glaring omission? Moreover, when students come away from this problem without being told about this inequality, they would likely have their suspicion of mathematics confirmed, namely, that it is just a jumble of disjointed formulas.[3] But quite the opposite is true: one of the main concerns in mathematics is to discover general laws which govern seemingly disparate phenomena, and the isoperimetric inequality affords an excellent example of this fact. Problem I is so jarring to a mathematician precisely because it threatens to falsify this aspect of mathematics.

What was said above, that students should be given the " most complete information *possible*" on any topic, must not be misinterpreted to mean that the teacher is obligated to explain all the ins and outs of the isoperimetric inequality to the 7th graders. It suffices for her to clearly state the inequality, explain what it says and how it bears on this particular problem, and tell them that its proof will be accessible to them after they have learned more mathematics. The teaching of mathematics need not be *linearly ordered* in the sense of proceeding only in a strictly logical order. What is important is to make known to the students at each step whether something is proven or whether it is borrowed from the future with no risk of circular reasoning, and moreover, to make sure that they keep a clear distinction between these two kinds of information.[4] The isoperimetric inequality is an example of something that can be harmlessly borrowed from the future.

It is necessary to emphasize that one faults this problem not for its good intentions, but rather for its very real potential of being abused in a classroom situation. With a little more care, one could alter the problem so as to achieve essentially the same goals while minimizing this potential for abuse. Let us look at one such possibility. Suppose we change the question to the following:

> If the enclosure is in the shape of a rectangle, what might its area be? What would be its largest area?

While this problem has a much narrower scope than the original one, it would nevertheless serve the same educational purpose of encouraging the students to explore the relationship between perimeter and area. But by restricting the problem to this special case, we have gained the advantage of making its solution completely accessible to the students. Indeed, with the aid of a calculator, the students would be in a position to verify experimentally that the area must now satisfy $0 \leq \text{area} \leq (30/4)^2$, and the maximum occurs exactly when the rectangle is a square of side length $30/4$. Moreover, a few students may even be able to provide an *explanation* of this fact, since it suffices to apply the inequality

$$ab \leq (\frac{a+b}{2})(\frac{a+b}{2})$$

for nonnegative numbers a and b, which is equivalent to $(a-b)^2 \geq 0$. (Surprisingly, the same question with the rectangle replaced by a triangle at once becomes too difficult; the maximum in this case is of course an equilateral triangle, but the explanation would involve the so-called *Heron's formula* for the area of a triangle. See p. 25 of Kazarinoff [5].)

As a side remark, for students who have had some trigonometry and geometry, one can nudge them towards the discovery that the circle encloses the maximum area by posing the following modified version of Problem I:

> If the perimeter is in the shape of, respectively, an equilateral triangle, a square, a regular pentagon, and a regular hexagon, what are the areas of the enclosures? From this, what would you guess to be the shape of the enclosure of maximum area?

This problem allows the students to see for themselves that the area increases as the number of sides of the regular polygon increases. Thus the circle would be the logical answer for the maximum (although I must emphasize once more the need to inform the students that there is a proof of this fact).

PROBLEM II (9th Grade):
A farmer is taking her eggs to market in her cart, but she is hit by a trailer-truck. Though she herself is unhurt, every last egg is broken. So she goes to her insurance agent, who asks her how many eggs she had. She says she doesn't know, but she remembers from various ways she tried packing the eggs that when she put them into 2's, there was one left over. When she put them into 3's, there was one left over. When she put them into 4's, there was one left over, and the same for 5's and 6's. But when she put them into 7's, they came out even.
1. How many eggs did she have?
2. Is that the only answer possible?

A sketch of the solution is given in Appendix 2. Here again, there *is* a complete solution, but it would be accessible to very few, if any, of the 9th graders. The people responsible for this problem told me that they expected nothing more than getting the students to experiment with the integers and "come up with the number 301 by trial and error". Now experimentation is an integral part of doing mathematics, and some significant discoveries have been made by this process in the past. So getting the students into the spirit of experimentation is certainly a step forward in their educational development, mathematical or otherwise. But mathematics does not stop with experimentation; it is also concerned with the rational explanation of these experimental discoveries. Is there perhaps an extensive theory lurking behind the seemingly unrelated facts? How would these facts fit into the overall mathematical structure? For the problem at hand, while any student with enough patience would likely get to the first solution 301 by trial and error, the chance of his or her getting to 721 is more remote, and that of getting the whole set of solutions $\{301 + 420\,n\}$ is slim. Would the teacher be able to tell the students the complete solution to this problem? More importantly, would the teacher be able to tell them that all such (linear Diophantine) problems can be handled by a standard machinery? Since it would be unreasonable to expect the teacher to be conversant with the Chinese Remainder Theorem, the answers to these questions would likely be negative. So once again, we come face-to-face with the potential dangers of the Civil War and Rockies analogies above. In addition, having been encouraged to do experimentation in mathematics, the students must also be told about its limitations: discovery by experiment must not be treated as an end in itself, but rather as a first step towards a complete understanding of a given situation within a broad mathematical framework. One can hardly over-emphasize this point to a beginner. In the present context, the students' understanding of their (possible) numerical discoveries of the solutions 301 and 721 would be that much greater if the Chinese Remainder Theorem could be explained to them afterwards. The educational function of Problem II would then be completely fulfilled.

As before, with a little thought, it is possible to reformulate the problem in a way that would increase its educational value in a real-life classroom situation. For example, one can simplify the data so as to make the complete analysis of the problem within reach of the students on this level, as follows:

> When the eggs are put into 4's, there is one left over, and when put into 6's, there is also one left over.

One can begin with trial and error to guess at the first few solutions: the condition about the 4's tells us that the solutions must be among 1 , 5 , 9 , 13 , 17 , 21 , 25 , 29 , 33, ... Using the condition about the 6's, we know immediately that 1, 13 and 25 would do. How to get *all* the solutions? Well, the condition about the 6's tells us that if the solution is k, then $6|(k-1)$. (We use the common notation $6|(k-1)$ to denote "6 divides $k-1$".) Now for students in the 9th grade, it should not be difficult to convince them (without resorting to a proof of the Fundamental Theorem of Arithmetic) that $6|(k-1)$ is equivalent to $2|(k-1)$ and $3|(k-1)$. But we already know that $4|(k-1)$, because when the eggs are put into 4's, there is one left over. Therefore, the requirement of $2|(k-1)$ is redundant, and the conditions on k can be summarized as:

$$3|(k-1) \quad \text{and} \quad 4|(\mathrm{k}-1).$$

The condition $4|(k-1)$ tells us that the solution k must be of the form $k = 4n+1$, where n is any integer. But every integer n can be written in one and only one of the following ways: $n = 3m$, $n = 3m+1$, and $n = 3m+2$, where m is itself an integer. Substituting this into $k = 4n+1$, we see that:

If $n = 3m$, $\quad k = 12m+1$, which when divided by 3 has remainder 1.
If $n = 3m+1$, $\quad k = 12m+5$, which when divided by 3 has remainder 2.
If $n = 3m+2$, $\quad k = 12m+9$, which when divided by 3 has remainder 0.

It is now clear that the number of eggs has to be one of the integers $12m+1$, where (recalling now that the number of eggs cannot be negative)

$m = 0, 1, 2, \ldots$ and that there are no other possibilities. Observing also that the first three solutions from this list are 1, 13 and 25, exactly as found by trial and error above, we have now a concrete instance of "explaining" an experimental discovery by a general theorem.

Note that the simplification of the problem serves a dual purpose. On the one hand one can now see quite clearly how the number 12 came up: it is the *l.c.m.* of 4 and 6. From this perspective, the students would be able to appreciate the 420 in the solution of the original problem even without knowing the Chinese Remainder Theorem. On the other hand, the problem can now be solved *completely* by a process that can be made available, with some patience, to a 9th grader. More than this is true: the above solution is in essence one possible proof of the Chinese Remainder Theorem. Thus the students would get a chance to learn something useful as well.

PROBLEM III (10th Grade):

1. Using a sheet of construction paper, build the biggest box possible, i.e., the box with the biggest volume. By a box, we mean a container with four rectangular sides and a rectangular bottom. Your box should have a top.
2. Describe the box you think is the biggest. Try to come up with an intuitive explanation of why that box is bigger than any other box.
3. Using a second sheet of construction paper, make the biggest box possible *without* a top.

A complete solution of this problem is given in Appendix 3. Before any discussion we note that the phrasing of the problem is a bit confusing. When first presented with this problem, I was very concerned about how to optimally cut the construction paper because I was under the impression that it was a problem of cut and fold and discarding the leftover portions of the paper. Others also had the same impression. So there is a lesson to be learned here about trying to be too cute in mathematical writing. This said, let us reformulate this simply as a mathematical problem: Find the rectangular box with the biggest volume when the lateral area is held fixed. Next, how to do this problem? In spite of the fact that this problem is a routine exercise in calculus (see Appendix 3), one *can* obtain a solution within the limitations of 10th grade mathematics (ibid.). In reality, however, a 10th grader would be hard pressed to follow the rather intricate arguments of this solution, much less devise them. Referring to part 1 of this problem, I have been told that one teacher allows the students (**a**) to assume that the rectangular solid which has maximum volume necessarily has a square base, and (**b**) to guess with the help of a calculator (graphing the volume against the length of one side of the square base) that the cube is the answer. How did the teacher justify (**a**)? "Oh, they (meaning the students) know." The few teachers with whom I had the opportunity to discuss this problem more or less conceded that something along this line is about all that can be expected of the 10th graders. At the risk of being repetitious, I say once again that there is nothing wrong with the spirit behind (**b**), which is to enable the students to guess the correct solution with the aid of experimentation. However, after the students get the correct answer by making use of an unproven assertion and numerical experimentation, would they be firmly told that all they have so far is just a guess, but that a proof is still needed for its justification? In other words, will they come out of this problem knowing the clear distinction between what is, or is not, acceptable mathematical reasoning? It is said that, while not knowing something is bad, it is far worse to not know that one doesn't know it. If the students get used to making use of unproven assertions in solving problems, pretty soon they may not be able to tell the difference between *guessing* something and *proving* it anymore. And *that* may be worse than not knowing how to do the problem at all.

What conclusions can one draw from the three preceding problems and others similar to these? They point to a change in the perception, at least among some mathematical educators, of what constitutes a valid mathematical education. *These open-ended problems are symptomatic of only part of this change, but for the purpose of this article, we must limit our scope and simply address these problems alone.* It is said that the traditional problems which insist on one and only one correct answer are on the one hand too threatening to students, and on the other too rigid to allow them to to show what they know. Thus open-ended problems have been introduced so that all students can work on them at their own level. By not insisting on one correct answer, they give the students confidence to solve new problems. So the bottom line is that these problems are accessible to more students. In real-life, however, these open-ended problems have become in some cases synonomous with partial answers or unjustified guesses. The wording of the three problems cited above is certainly consistent with this perception. For example, the first question of Problem I invites the students to make up a shape to his or her liking and write down its area, and this is supposed to be the answer to the question; the second part, by the same token, certainly asks for no more than a wild guess. Again, Problem II makes it clear that if the students can come up with more than one number that satisfies the given conditions, the problem would be considered solved. In Problem III, the students are tacitly given clearance to make use of unwarranted assumptions to derive the solution.

It is impossible to disagree with this drive to open up mathematics by changing its facade. Mathematics *should* look more attractive and more hospitable than it has up to now. In the right context, none of the practices described above is objectionable, and I hope some of the preceding discussions have made this point abundantly clear. But just to be sure, let us retrace our steps a little. Suppose a teacher can adequately explain the isoperimetric inequality and show how it bears on Problem I to the students, then they would get to see how mathematics can extract order out of seeming chaos. Or suppose a teacher can show the students how their isolated numerical discoveries in Problem II can be explained once and for all by the Chinese Remainder Theorem, then they would learn firsthand the power of, and the need for abstract theorems, and perhaps also acquire the habit of always digging beneath the surface until they achieve a complete understanding of a given phenomenon. For Problem III, suppose the teacher tells the students that the whole problem is a routine one in the calculus of several variables but that it is too difficult for them at the moment. So all he or she wants them to do is to *get some feeling* that the answer ("a cube") is correct in the special case of a rectangular solid with a square base by plotting the graph of the volume against the length of one side of the square base. In this way, the students would harbor no illusions of having *proved* anything, and at the same time they would also develop some intuition about such maximization problems. By any measure, this would be perfectly valid, and even good mathematical education.

It would seem that the various methods of handling open-ended peoblems described in the preceding paragraph do not fall within the scope of of the original intentions of these problems. One cannot deny that giving students (open-ended) problems so that they can *work on them at their own level* is a very good idea, or that the same is true of making mathematical problems *accessible to more students*. Yet by the way these problems (and others like them) are formulated, it would appear that the fulfillment of these two objectives already defines to some educators a valid mathematical education. I would like to offer a differing opinion, however. I believe that a curriculum that allows students to blur the distinction between guessing and experimentation on the one hand, and valid logical reasoning on the other, misses one of the most critical and central features of mathematics. The trouble with open-ended problems such as Problems I-III may be that in going all out to achieve the two goals of humanizing mathematics and increasing its accessibility, they have also inadvertently misinformed the students about this important distinction.

It would not be out of place to point out that the philosophy underlying such an emphasis on accessibility also leads to other unexpected side-effects. For example, in discussing these three problems with some teachers, I was astounded to be told by one and all that they considered the first part of Problem I ("WHAT MIGHT ITS AREA BE??") to be a good problem because it allows the students to make up their own questions and answers, but that they thought the second part ("WHAT WOULD THE LARGEST AREA BE??") **was bad because it pins down the students to a single correct answer**. Since a good part of mathematics, pure or applied, is pre-occupied precisely with such maximization problems, we have here an example of an educational philosophy that has distorted the way a group of teachers think about the subject they are supposed to teach.[5] This should be a matter of grave concern.

Open-ended problems started off as a well-intentioned pedagogical device, but the preceding discussion points to a very real possibility of their being an educational liability instead. How did this come about? Whatever the direct cause, I believe ultimately it is because, during this frantic search for pedagogical improvements, the issue of mathematical substance got lost somewhere. Too often it is forgotten that the technical aspect of mathematics is a very important component of mathematical education. No mathematical education is of any value, regardless of its excellence in pedagogy, if it is not technically sound. The only reason such a truism is worth pointing out is that the technical soundness of any mathematical education can by no means be taken for granted;[6] the mathematical defects that show up in the three problems discussed above should erase all doubts about this seemingly outrageous assertion. Although the discussion in this short article is confined to open-ended problems, even a cursory acquaintance with some of the recent curriculum reforms is enough to make a mathematician wonder where mathematical education is headed. This points to the real need of a closer cooperation between mathematicians and educators in order to insure that our children will get an education that is both technically and pedagogically sound. Such a cooperation seems to be nearly nonexistent so far. Thirty or so years ago, we had the New Math debacle.[7] Let this be a reminder that a second debacle is a very distinct possibility unless both the educators and the working mathematicians continue to be vigilant.

Appendix 1: Solution of Problem I

WHAT MIGHT ITS AREA BE? WHAT WOULD THE LARGEST AREA BE?

Both questions are answered at once by the so-called *isoperimetric inequality* which asserts that if a closed curve of length L encloses a region with area A in the plane, then $4\pi A \leq L^2$, and the equality is attained exactly when the curve is a circle. Thus in our case, $L = 30$, and we see that the area A of the pot bellied pig enclosure must satisfy $0 \leq A \leq \dfrac{1}{4\pi}30^2$, and A attains the maximum value $\dfrac{1}{4\pi}30^2$ exactly when the pot bellied pig enclosure is a disk of radius $\frac{30}{2\pi}$. This is the complete answer to the problem because it is quite easy to show that given any number A', such that $0 \leq A' \leq \dfrac{1}{4\pi}30^2$, there is an enclosure with perimeter $= 30$ and area exactly $= A'$.

> The isoperimetric inequality is one of the most profound inequalities in mathematics, both in terms of its intrinsic significance and the impact it has exerted on the development of mathematics, past and present. For an elementary discussion without a proof of the inequality itself, see §5 of Kazarinoff [5]. Two proofs are offered on pp.105-108 of Chern [2]. A proof can be found also on p.186 of the classic work on inequalities by G.H. Hardy, J.E. Littlewood and G. Pólya [4]. There is a vast literature surrounding this inequality, and its many generalizations are still evolving with current research; consult for example the article of Osserman [6].

Appendix 2: Solution of Problem II

How many eggs did she have? Is that the only answer possible?

The problem calls for solving *in positive integers* the sequence of congruences:

$$
\begin{aligned}
x &\equiv 1 \pmod 2\\
x &\equiv 1 \pmod 3\\
x &\equiv 1 \pmod 4\\
x &\equiv 1 \pmod 5\\
x &\equiv 1 \pmod 6\\
x &\equiv 0 \pmod 7.
\end{aligned}
$$

By elementary considerations, this is equivalent to solving:

$$
\begin{aligned}
x &\equiv 1 \pmod 3\\
x &\equiv 1 \pmod 4\\
x &\equiv 1 \pmod 5\\
x &\equiv 0 \pmod 7.
\end{aligned}
$$

Applying the Chinese Remainder Theorem to the latter, we get immediately that all solutions are of the form $301+420\,n$, where n is any integer. Restricting n to nonnegative integers then gives *all* the possible number of broken eggs: 301, 721, 1141, etc.

> Solving simultaneous (linear) congruences is among the first things one learns in a first course on number theory. There are probably an infinite number of texts one can consult on this topic, but I highly recommend two of them: Dudley [3] and Stark [7]. Both are guaranteed to be not only informative, but entertaining as well.

Appendix 3: Solution of Problem III

First of all, the problem should be more clearly rephrased as follows: If you are required to construct a rectangular solid with a fixed lateral area, when will you get one of maximum volume? (Here, it is understood that if the rectangular solid has a top, then "lateral area" refers to the sum of the areas of all six faces; if however the solid has no top, then "lateral area" refers to the sum of the areas of the five faces, four on the side and one at the bottom.) If you know how to solve the problem with a top (i.e., part 1), then you already know how to do the one without a top (i.e., part 3): Indeed, let the given lateral area be A, and we want to find the box without a top which has the biggest volume. Get two such boxes, invert one of them (so that its "bottom" is now on top), and put this inverted box on top of the other (so that the two "topless" faces are now coincident). This creates a new rectangular solid *with a top*, to be called $\mathcal{R}$, which clearly maximizes the volume among all such with a fixed lateral area $2A$. If we anticipate the solution to part 1, then we know that $\mathcal{R}$ is a cube. Thus the solution to part 3 is a "half-cube", i.e., a rectangular solid whose bottom is a square and whose height is half the length of the edge of its bottom.

It remains to solve part 1. Let the sides of the box (with a top) be of lengths x, y and z. For simplicity, we let the given lateral area be $2A$ (rather than A). Thus we are given: $2xy + 2xz + 2yz = 2A$, which is the same as:

$$xy + xz + yz = A. \tag{1}$$

(If we had used A instead of $2A$, the right-hand side of equation (1) would have been $\frac{1}{2}A$ instead, and the subsequent computations would be more cumbersome.) The volume of the box is of course xyz. Thus our problem becomes one of finding the point (x, y, z) with *positive* coordinates x, y and z so that the function

$$f(x, y, z) = xyz$$

subject to the "constraint" in equation (1) achieves a maximum at this point. This is a routine problem in the use of *Lagrange multipliers* in the calculus of several variables. (Look up any book on advanced calculus, for example, Chapter 6 of Buck [1].) The solution is of course the point $(\sqrt{A/3}, \sqrt{A/3}, \sqrt{A/3})$, i.e., a cube with side length equal to $\sqrt{A/3}$. However, in the present context, we must present a solution without the use of calculus. This proceeds in two steps.

STEP 1. Given positive numbers x, y, z satisfying equation (1), if $x \neq y$, then there are three other positive numbers x_1, y_1, z_1 also satisfying (1), so that

$$x_1 = y_1, \text{ and } xyz < x_1 y_1 z_1.$$

(Geometrically, if you think of x and y as the sides of the bottom of the cube, this asserts that among all rectangular boxes with a fixed lateral area and a fixed area for the bottom, the one with a square bottom has the largest area.)

PROOF. We set $x_1 = y_1 = \sqrt{xy}$, then in order to satisfy equation (1), we must also set

$$z_1 = \frac{A - xy}{2\sqrt{xy}}.$$

We have to prove that $xyz < x_1y_1z_1$. By equation (1), $A - xy = z(x+y)$. Hence,

$$x_1y_1z_1 = xyz\,\frac{x+y}{2\sqrt{xy}}.$$

Thus it remains to prove that

$$\frac{x+y}{2\sqrt{xy}} > 1.$$

This is the same as proving $x+y > 2\sqrt{xy}$, or, since we are dealing with positi ve numbers x and y, the same as proving $(x+y)^2 > 4xy$, which in turn is equivalent to $x^2 - 2xy + y^2 > 0$. The latter is true because the left-hand side equals $(x-y)^2$ and by assumption, $x \neq y$. Q.E.D.

STEP 2. Let $x_0 = y_0 = z_0$ and let x_0, y_0, z_0 satisfy equation (1). Then for any positive numbers x, y, z satisfying (1),

$$x_0y_0z_0 \geq xyz.$$

PROOF. We first prove this for the special case where $x = y$. As before, since x, y, z satisfy equation (1),

$$z = \frac{A - x^2}{2x},$$

so that

$$xyz = x^2z = \frac{1}{2}\,x(A - x^2).$$

On the other hand, since x_0, y_0, z_0 satisfy equation (1) and $x_0 = y_0 = z_0$, we see that $x_0 = y_0 = z_0 = \sqrt{A/3}$, so that $x_0y_0z_0 = (A/3)^{3/2}$. Thus we must prove:

$$\frac{1}{2}\,x(A - x^2) \leq (\frac{A}{3})^{3/2}. \tag{2}$$

If $x = \sqrt{A/3}$, then (2) is an equality and there would be nothing more to prove. So assume $x \neq \sqrt{A/3}$, and we will verify (2). Let

$$x = \sqrt{\frac{A}{3}} + h, \tag{3}$$

where h is some nonzero number (positive or negative). Then:

$$\begin{aligned}
\tfrac{1}{2}\,x(A-x^2) &= \tfrac{1}{2}\left(\sqrt{\tfrac{A}{3}}+h\right)\left(A-(\sqrt{\tfrac{A}{3}}+h)^2\right)\\
&= \tfrac{1}{2}\left(\sqrt{\tfrac{A}{3}}+h\right)\left(2(\tfrac{A}{3})-2h\sqrt{\tfrac{A}{3}}-h^2\right)\\
&= \tfrac{1}{2}\left(2(\tfrac{A}{3})^{3/2}-2h^2\sqrt{\tfrac{A}{3}}-2h^3\right)\\
&= \left(\tfrac{A}{3}\right)^{3/2}-h^2\left(\sqrt{\tfrac{A}{3}}+h\right)\\
&= \left(\tfrac{A}{3}\right)^{3/2}-h^2x,
\end{aligned}$$

where the last equality uses equation (3). Since $x > 0$ and $h \neq 0$, we see that $-h^2x < 0$. Hence,

$$\frac{1}{2}\,x(A-x^2) = \left(\frac{A}{3}\right)^{3/2} - h^2x < \left(\frac{A}{3}\right)^{3/2},$$

which then proves equation (2).

We have just proved STEP 2 in the special case where $x = y$. In general, suppose $x \neq y$. By STEP 1, there is another triple, x_1, y_1, z_1 satisfying (1) so that $x_1 = y_1$ and $x_1y_1z_1 > xyz$. Now repeat the preceding argument with x_1, y_1, z_1 in place of x, y, z; then we get as above:

$$x_0y_0z_0 > x_1y_1z_1.$$

Coupled with $x_1y_1z_1 > xyz$, we get $x_0y_0z_0 > xyz$, as desired. Q.E.D.

Step 2 is the statement that, subject to the constraint (1), the function $f(x, y, z) = xyz$ achieves its maximum at the point (x, y, z) where $x = y = z = \sqrt{A/3}$.

Footnotes

1. A slightly expanded version of a lecture presented to the Bay Area Mathematics Project on July 27, 1992. The author is indebted to Serge Lang for a merciless critique of an early draft of this article, and to Alfred Manaster for many substantive corrections.

2. As far back as 1968, I posed the following question on the final of my undergraduate course on differential geometry: Let α be a curve and let M be its tangential developable. What can you say about M when α has zero torsion?

3. That is, one area formula for each shape.

4. This comment is particularly relevant in the context of a common failing in textbook writing, which is to pass off a heuristic argument as a proof without an explicit statement to the contrary.

5. One cannot help but notice the great irony in the fact that while these open-ended problems came into being because they were supposed to prevent the teachers from "looking for one correct response or one right answer" in the works of the students, each of the three problems above in fact admits only one correct answer. See Appendices 1, 2 and 3.

6. By comparison, it would be absurd for the same to be said about the teaching of English or history, say.

7. I do not wish to imply that *everything* connected with the New Math was bad. Zal Usiskin pointed out to me that one positive outcome of the New Math was the greater emphasis on proofs. On the whole, however, it was clearly a debacle. One day when the official obituary of the New Math is written, it will be noted that in addition to the excessive formalism of the new texts, the gravest mistake of the New Math movement might have been the over-emphasis on curriculum reform without an equal amount of effort devoted to the training of the teachers. Are the current curriculum reformers aware of this fact?

References

[1] R.C. Buck, *Advanced Calculus* (*3rd Ed.*), McGraw Hill, 1978.

[2] S.S. Chern (ed.), *Global Differential Geometry*, MAA Studies in Mathematics, Volume 27, Math. Assoc. Amer., 1989.

[3] U. Dudley, *Elementary Number Theory* (*2nd Ed.*), W.H. Freeman, 1978.

[4] G.H. Hardy, J.E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, 1952.

[5] N.D. Kazarinoff, *Analytic Inequalities*, Holt, Rinehart and Winston, 1961.

[6] R. Osserman, The isoperimetric inequality, *Bulletin Amer. Math. Soc.* 84(1978), 1182-1238.

[7] H.M. Stark, *An Introduction to Number Theory*, M.I.T. Press, 1978.