Invited Comments on the *NCTM Standards*

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Late in 1995, NCTM solicited comments from the mathematical community on the NCTM Standards in preparation for their Standards 2000 project. In response, I submitted these comments.

In response to Mary Lindqist's call for comments on the NCTM Standards, I would like to address the first question she raised, namely,

> Do you think the Standards might benefit from specific
> (a) clarifications, (b) expansions or additions, or
> (c) deletions?

Here are my suggestions (all references are to the *NCTM Curriculum and Evaluation Standards* unless stated otherwise):

(1) The Standards should be more careful in suggesting what topics to omit or de-emphasize, and even more careful in the exact phrasing of these suggestions. These suggestions are usually distorted to serve the reader's own agenda. Therefore extra care must be taken to prevent such misinterpretations. I will illustrate my point with some examples.

On p.127, it is suggested that "Two-column proofs should receive decreased attention". The phrasing carries the implication that there is something wrong with two-column proofs per se. Of course this is absolutely false: this is an excellent vehicle to guide the students' first steps in trying to write a proof. Two-column proofs get such a bad rap because most teachers do not understand proofs, with the consequence that they inevitably abuse two-column proofs and make them a liability in mathematics education. Thus to these people, the recommendation that "Two-column proofs should receive decreased attention" (without a carefully worded explanation to go with it) carries an automatic invitation to do without all proofs. Lest this statement be taken as an unwarranted exaggeration, may I point out the recent appearance of geometry texts which do essentially no proofs but only "experimental geometry". [I can send upon request a paper of mine that discusses this issue in detail.][1]

Also on p.127, it is suggested that "Word problems by type, such as coin, digit, and work should receive decreased attention". Pretty much the same comment as above again applies: there is nothing wrong with coin, digit or work problems. Some of these are very good problems. What is wrong is that in the hands of unqualified teachers, these problems become meaningless drills. The educational difficulties would not go away just by eliminating or de-emphasizing these problems: something else would take their place as an instrument of mindless drills as surely as the sun would rise from the east tomorrow if the teachers continue to be mathematically inadequate. What needs fixing is the teacher qualification problem. NCTM should find (diplomatic) ways to express this fact correctly. The present recommendation concerning "problems of type" is misleading at best.

As a final example, it is stated on p.96 that:

> This is not to suggest however that valuable time should be devoted to exercises like $\frac{17}{24}+\frac{5}{18}$ or $5\frac{3}{4}\times 4\frac{1}{4}$ which are much harder to visualize and unlikely to occur in real life situations. Division of fractions should be approached conceptually.

These are perfectly simple fractions which anyone who understands the

[1] The paper in question is: The role of Euclidean geometry in high school, *J. Math. Behavior* 15(1996), 221-237.

first thing about fractions would have no trouble handling.[2] Therefore the crux of the matter is that fractions are not taught properly in the lower grades, and NCTM should be addressing this major problem rather than trying to solve it by advocating this disastrous position. Indeed fractions are hard to teach, and I have given this matter a lot of thought. I have a paper that discusses a reasonable way to approach fractions in the 5th grade, and I have sent Gail Burrill a copy. All the same, I would be happy to send additional copies on request.[3] In light of these examples, I hope NCTM would carefully reconsider many of these recommendations.

(2) The Standards tries to fudge the issue of how to reach out to both the gifted students and those in the lower 50%. A dramatic exxample is furnished by the discussion in Standard 14 of Grades 9-12 which talks about proving theorems about groups and fields. Standard 14 is often cited by supporters of the Standards as an example of NCTM's serious interest in serious mathematics. **However**, if one looks at the over-abundance of topics put in the curriculum (discrete mathematics, probabliity, statistics, etc.) coupled with the general low level of suggested mathematics instruction in high school (as illustrated by the discussion of how to teach the cubic equation on pp.152-153), it would be ludicrous to regard Standard 14 as anything but window-dressing. A curriculum patterned after the one suggested in the Standards would never get to groups and fields until hell freezes over (as the saying goes).

[The cubic equation on pp.152-153 mentioned above, namely,

$$5x^3 - 12x^2 - 16x + 8 = 0$$

will be further discussed in **(5)** below.]

NCTM should be forthright in advocating at least two different tracks of mathematics education after the 10th grade. By this, I mean that the

[2] Added December 11, 1998: There is an illuminating discussion of the need of understanding of fractions by elementary school teachers in the forthcoming volume *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*, Mahwah, NJ: Lawrence Erlbaum Associates, 1999. See especially Chapter 3. It is refreshing to have an educator openly advocating the need of profound understanding of elementary mathematics, not just the so-called "conceptual understanding" but also the necessary technical skills.

[3] The paper in question is: Teaching fractions in Elementary School: A Manual for Teachers.

students should be given a choice as to which kind they want to pursue rather than be arbitrarily assigned by the school authorities to a particular track. According to Zal Usiskin, no school system in any of the developed nations practices what the Standards is perceived to be presently advocating: one kind of mathematics for all. If one points to two lines buried in p.130 of the NCTM Standards and claims that there has been a misunderstanding, one would be missing the point entirely. Such a central issue in the current education debate needs to be addressed in the most explicit way possible and, to put it mildly, the NCTM Standards has obviously not done that.

I believe this issue should receive top priority in the second edition of the Standards.

(3) A very striking aspect of the NCTM Standards is its over-emphasis on mathematics as a tools for earning a living in the high tech society. May I point out that American high schools have not yet been officially designated as mere vocational schools. Therefore a little emphasis in this direction already goes a long way. May I also point out that Japan and other countries are beginning to look at the NCTM Standards as our national statement on mathematics education. Such an overwhelming emphasis (if not obsession) with one single aspect of a compulsory education system would do nothing but reinforce other countries' stereotypical image of America as an cultural wasteland.

I happen to have written a review of Gelfand's high school math books[4] and so cannot help but recall what he wrote in the preface: "The most important thing a student can get from the study of mathematics is the attainment of a higher intellectual level." Should the NCTM Standards not place an equal stress on this spirit of intellectual enrichment since it claims to be a document on mathematics EDUCATION?

Needless to say, I am here suggesting an overhaul in the details of the prescribed curriculum rather than a harmless make-over of the facade. It is hardly necessary to go into the details because, to the extent that at each step of the writing of the Standards, a deliberate decision was made to emphasize the utility of mathematics in everyday matters at the expense of the internal structure of mathematics as an independent discipline, all that NCTM has to do is to retrace the steps and undo the earlier decisions. If I want to be gimmicky, I may facetiously suggest that NCTM should constantly ask itself this question: would a student coming out of the new curriculum have even a

[4] Reviews of three books by Gelfand, *The Math. Intelligencer* 17(1995), 68-75.

slight appreciation of what the recent excitement over Fermat's Last Theorem is all about? A student coming out of the present NCTM curriculum would in all probability have no inkling whatsoever of why one should bother with such a quaint statement as Fermat's Last Theorem.

To Gauss, mathematics is the queen of science and number theory is the queen of mathematics. The curriculum of NCTM Standards would instead make mathematics out to be the handmaiden of mundane run-of-the-mill phenomena, and something like number theory would have no place in a school curriculum.

A more moderate stance would be welcome.

(4) In an attempt to incorporate the calculator into mathematics education, NCTM has perhaps unwittingly inflated the importance of this tool in the students' learning process. Han Sah once likened putting a calculator in the hands of the unitiated to putting a gun in the hands of a child, and I think the analogy is apt. The calculator cannot be used as a substitute for understanding, but by not coming out openly for this point of view the NCTN Standards comes close to giving the impression that indeed it can.

Concommitant with, and ultimately related to this undue emphasis on the role of the calculator in math education is the lack of an explicit emphasis throughout the NCTM Standards on the importance of technical facility in mastering mathematics. The Standards set the tone of "understanding at the expense of algorithms and formulas" as a remedy for the ills of the traditional math education. The fact is that the algorithms and formulas are wonderful things to know, and the only missing ingredient is the requisite understanding. At the moment though, would anyone say that "adding understanding to technique" is the overwhelming message of the Standards?

On p.8, it is stated: "Contrary to the fear of many, the availability of calculators and computers has expanded students' capability of performing calculations." I do not know where did this conviction come from, but it appears to run contrary to facts. If by "calculation" numerical calculation is meant, then anecdotal accounts from all walks of like would contradict such optimism. If by "calculation" symbolic calculation is meant, then my personal observations in the classroom (which of course cannot be equated with robust research figures) would indicate that such a claim is spectacularly false.

Let me draw an analogy. Now that small linguistic translators are fast becoming as popular as calculators, do we see the various language departments advocate that from now on all linguistic instructions would be confined to grammar and syntax, and that there would no longer be any need for students to memorize new vocabulary? I think not, and rightly so. Yet someone with the mind set of the Standards would undoubtedly advocate that they do.

I very much hope that the NCTM would make very clear the fact that the fundamental concepts and basic techniques must be thoroughly mastered by the students before any calculator is placed in their hands. There is an obvious need of a balanced statement somewhat along this line concerning the role of technology in the classroom, and one would naturally expect NCTM to make it.

(5) It has been asserted many times that the NCTM Standards only sets the floor but not the ceiling of a national curriculum. My only wish is that the floor had not been set so low. I will illustrate my point by using the cubic equation in (2) as an example. Suppose instead of the recommendations for Levels 1-5 on pp.152-3, we recommend that the instruction should cover at least the following:

> State a pedestrian version of the Intermediate Value Theorem and ask students to graph and evaluate the values of this cubic polynomial $p(x)$ at the integers. From
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> $$\begin{aligned} p(-2) = -48 \quad &\text{and} \quad p(-1) = 7, \\ p(0) = 8 \quad &\text{and} \quad p(1) = -15, \\ p(3) = -13 \quad &\text{and} \quad p(4) = 72, \end{aligned}$$
>
> conclude that there is at least one zero in each of $[-2, -1]$, $[0, 1]$, and $[3, 4]$. Now teach them the bisection algorithm to get arbitrarily close approximations of the roots. For example, concentrate on $[0, 1]$: then the values
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> $$p(0) = 8 \quad \text{and} \quad p(1/2) = -19/8$$
>
> imply that there is at least one zero in $[0, 1/2]$. Once more, concentrate now on $[0, 1/2]$, and the fact that
>
> $$p(1/4) = \frac{213}{64} \quad \text{and} \quad p(1/2) = \frac{-19}{8}$$
>
> means there is at least one zero in $[1/4, 1/2]$, etc. *Explain* to students that bisection is a simple but effective proof-method

> useful not only here but in other parts of mathematics as well (To the teacher: cf. most discussions of the basic properties of the real number system, e.g., Heine-Borel, Bolzano-Weierstrass), so it is well-worth learning. Use the factor theorem to express this polynomial as a product of linear and quadratic polynomial. Now use the quadratic fomula to find the other roots. In general, when we find one root to a cubic equation, all other roots can be found by the same procedure.

This would seem to be a more reasonable “floor”.

Of course when we come to the curriculum for the 11th and 12th grades where there are two tracks, two different floors should be set. Since this bifurcation now allows for the inclusion of some real substance in the higher track, if the floor in this case is carefully set in the Standards, then there is hope that the next generation of scientists and engineers would get the education they need.

(6) If NCTM hopes to affect a real change in mathematics education, it may wish to affect a real change in the students' attitude towards mathematics in the first place. At the moment, the students do not realize what kind of hard work is really involved in learning mathematics, for the simple reason that nobody ever talks to them seriously about it. In addition, educators believe that the students are mortally afraid of hard work and therefore try all they can to lure the students into the false belief that mathematics is both fun and easy. Alas, these two facts feed on each other.

In a segment of “60 Minutes” on February 7, 1993, Andy Roony pointed to some elementary mathematics text and said: “Some of these books are attractive. They look good, but it is as if the schools were afraid to tell the kids that some of the things they should learn are hard and might even take some effort”. Moreover, if one also looks at some current activities such as the emergence of IMP and the implicit message in the TV documentary, “Good Morning, Ms. Toliver”, one gets the unmistakable message that there is a powerful campaign on foot to make everything in mathematics “FUN” and “RELEVANT”.

I don't think it is a controversial statement that not much substantive mathematics can be made fun and relevant, at least not in the sense that these terms are currently understood. NCTM is in the enviable position of being able to make a real contribution here by coming out and saying that learning mathematics requires very hard work. By telling the truth, NCTM would also be giving a tremendous lift to the many hard working teachers across the land who try to do an honest job of teaching mathematics. In this context, allow me to quote Gelfand one more time (I have no intention of pushing his books on NCTM, and in fact my review also contains critical comments; but one has to give credit where credit is due): “This book, along with the others in this series, is not intended for quick reading. Each section is designed to be studied carefully....And if it is difficult for you, come back to it and try to understand what made it hard for you.” This is the kind of general statement NCTM should be making in the Standards. The students must know the truth, and the teachers must be encouraged to tell the students the truth. I hope NCTM agrees and acts accordingly.

(7) The Standards should draw a clear distinction between heuristic reasoning and complete demonstrations (= proofs). On almost every page of the Standards, there seems to be a conscious effort to try to blur this distinction. In fact, one would do much less harm to the students if, instead of this intentional act of blurring, one simply gives heuristic reasoning everywhere in place of proofs, and *say so publicly.* The unique quality of a good mathematics education is that the students learn to face unflinchingly the sharp distinction between what is true and what is almost certainly true but just not good enough, and therefore false. It would seem that the education of the type advocated by the NCTM Standards would never allow a student to experience this distinction. Such a pessimistic conclusion is further confirmed by the grievous discussion on “reasoning, justification and proof” in Chapter 10 of GEOMETRY FROM MULTIPLE PERSPECTIVES:

> ...reasoning is the process of thinking about a mathematical question; a justification is a rationale or argument for some mathematical proposition; and a proof is a justification that is logically valid and based on initial assumptions, definitions, and proved results.” (p.61)

I sincerely hope that the Standards would rectify this situation the second time around.