Professional Development: The Hard Work of Learning Mathematics[1]

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We start with the basic premise that the most urgent task in school mathematics education is to produce teachers who are mathematically well-informed (cf. Wu [7]). This talk addresses the question of how to accomplish this goal in professional development. It will not touch on the perhaps far more difficult issues of administrative support and social forces that shape career decisions.

With the exception of Part III, the comments that follow would be valid for both pre- and in-service professional developments. But I will concentrate for the most part on in-service professional development, for the simple reason that, in general, the pre-service situation presents immense difficulties. There are too many hoops to jump through in the pre-service case, such as bureaucratic decisions by universities and the generic lack of cooperation between math departments and schools of education.[2] Although the present climate

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[2]It is to be understood that sweeping statements of this nature always allow for a

in in-service professional development is, overall, not conducive to the teaching of mathematics either, as I will presently explain, the in-service route is nevertheless more amenable to individual initiatives. There is at least more flexibility in the in-service arena for each person to act on his/her new ideas to bring about change.

I will divide the discussion into three parts:

Part I Description of some of the obstacles.

Part II Suggestion on how to overcome the obstacles.

Part III An example of what can be done.

Part I One person's view of the *main obstacles* standing in the way of promoting the teaching of content in in-service professional development.

(i) Insufficient attention to the importance of content knowledge from the top down: from NSF-EHR to education officials in most states.

The consideration of content in the funding of proposals by NSF-EHR (the Education and Human Resources directorate of the National Science Foundation) appears to be of recent vintage. Its neglect of mathematical integrity has been commonplace for a very long time, and this neglect is likely still being replicated across the land. For example, in a recent survey by Tom Loveless, Alice Henriques, and Andrew Kelly of winning proposals among the state-administered Mathematics Science Partnership (MSP) grants from 41 states ([2]), it was found that while "Some of the MSPs appear to be offering sound professional development. Many, however, are vague in describing what teachers will learn". Typically, these "MSPs' professional development activities tip decisively towards pedagogy." I can also offer a personal anecdote.

small number of exceptions. For example, although an overwhelming majority of the colleges do not require *three* mathematics courses as part of the preparation of elementary teachers, there are some which have done that.

In year 2000, when the state of California convened a meeting with publishers on California's new criteria for the forthcoming math book adoption, the importance of correct mathematics was emphasized. After the meeting, a publisher representative approached me and confided that he had been to numerous state adoption meetings, but that it was the first time that he had heard *content* discussed.

(ii) Mistaken notion of what constitutes "content" in school mathematics.

Content is a word that is easily said, but its meaning in the context of school mathematics education has proven to be elusive. I will illustrate this elusiveness by way of two examples from both ends of the educational spectrum: university mathematicians who are sincere in their belief in the importance of mathematics, and educators who are equally sincere in their belief in the importance of *process* in education.

Exampe 1: A university mathematician once described how he had been presenting "fractions from the field axioms point of view" to algebra and pre-algebra teachers from grades 6–8.

Does this constitute appropriate professional development? Probably not, because this kind of mathematics is too abstract for use in the classroom of grades 6-8. More importantly, teachers in these grades are generally struggling to find ways to correctly teach fractions to their students, so learning another approach that they cannot use in their classrooms cannot be high on their agenda.

Most university mathematicians are not aware of the fundamental fact that

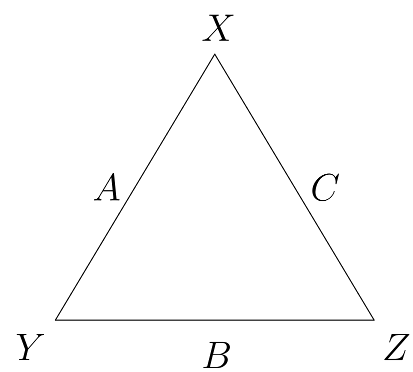
College mathematics $\neq$ School mathematics.

School mathematics is the customized version of college mathematics for the consumption of school classrooms (see the discussion in Wu [8]), just as a personal computer is the customized version of an IBM mainframe for use in ordinary households. In both cases, it is the added *engineering* process that makes the crucial difference.

Example 2: A university math educator illustrated how he approached the teaching of content to elementary teachers with a problem that he gave as a class project for open-ended investigations. Let X, Y, Z be the number of beans placed on the vertices of a triangle, and let nonzero whole numbers A, B, C be attached to the sides of the same triangle, as shown, so that

$$A = X + Y, \quad B = Y + Z, \quad C = X + Z \qquad (*)$$

The problem is: *If whole numbers A, B, C are given, would there be whole numbers X, Y, Z which satisfy $(*)$?*



Note that one can show, using mathematics that is understandable to a 6th grader, the following: *there is a unique solution* $\Longleftrightarrow$ $A+B+C$ *is even and the sum of any two of A, B, C is greater than the third.* By contrast, this educator let his pre-service teachers use the discovery method to carry out the investigations. He let them explore how various integral values of X, Y, Z lead to different values of A, B, C, and how a solution $\{X, Y, Z\}$ can be obtained by guess-and-check when certain whole numbers A, B, C are given. He also got them to look into other formulations of this problem (such as replacing the triangle by a quadrilateral). But *no mention was ever made of the necessary and sufficient conditions given above*.

Nevertheless, he believed that it was a wonderful learning experience for the pre-service teachers.

My concern is that this educator has confused "pre-mathematics" (in the sense of heuristic arguments, explorations, and other processes that precede the clear formulation of precise hypotheses, precise conclusions, together with the logical unfolding of the steps connecting the former to the latter) with mathematics itself (the clear formulation of precise hypotheses, precise conclusions, and the steps connecting them). Teaching should address both pre-mathematics and mathematics, there is no doubt of that. However, a common mistake in discussions of mathematics education in the past fifteen years has been to confer blessings on the replacement of mathematics with pre-mathematics. When professional development does likewise, it misleads teachers into teaching pre-mathematics *in place of* rather than *in addition to* mathematics, and students are the ultimate victims. In particular, these teachers will have no conception of *mathematical closure*, such as the enunciation of the necessary and sufficient conditions in the preceding problem and the explanation of why they lead to a precise mathematical understanding of the problem itself. How can these teachers be effective in the classroom?

(iii) Mistaken notion of what constitutes *correct mathematics* in existing professional development materials.

The thought that the mathematics in professional development materials could be defective is not something that comes naturally to mind, but in fact such defects are common. For example, a standard text for elementary teachers defines a *rational number* $\frac{a}{b}$ for integers a and b, as

the solution of the equation $bx = a$.

Then it sets up a table of the "several different ways in which we use rational numbers":

Use	Example
Division problem or solution to a multiplication problem	The solution to $2x = 3$ is $\frac{3}{2}$.
Partition, or part, of a whole	Joe received $\frac{1}{2}$ of Mary's salary each month for alimony.
Ratio	The ratio of Republicans to Democrats in the Senate is 3 to 5.[3]
Probability	When you toss a fair coin, the probability of getting heads is $\frac{1}{2}$

If you believe that the text goes on to explain why the solution to $bx = a$ would have these three other properties, i.e., partition, ratio, and probability, you are mistaken.

If you believe that the text goes on to make use of the solution to $bx = a$ to develop other properties of rational numbers, e.g., equivalent fractions, the addition and multiplication of fractions, then you are equally mistaken.

If, however, you believe that in this text, the *definition* of a rational number is irrelevant, then you are right.

Learning from a book like this would be like taking a tour in a zoo: you get to see each property of the rational numbers displayed like an animal in a cage, but not so much how these properties are interwoven into an organic whole. But

Mathematics is not a zoo. It is an organic entity.

Teachers have to be shown this organic entity.

This example also illustrates what might be called *the pro forma approach to content*. Having been told that definitions are important in mathematics, some textbooks choose to satisfy this requirement by putting forth precise definitions and then proceed to ignore them the rest of way.[4] Incidentally, the definition of rational numbers as solutions of $ax = b$ is inappropriate for the professional development of elementary teachers because it is pedagogically inappropriate to make use of such a definition in the elementary classroom.

(iv) Mistaken notion of what constitutes *correct mathematics* in existing education documents.

Again, I will simply illustrate with an example. On p. 26 of the CBMS volume on *The Mathematical Education of Teachers* ([1]), there is the following comment on the mathematics that middle school teachers need to learn.

> Proportional reasoning is psychologically and mathematically a sophisticated form of reasoning based on intuitive pre-school experiences and developed in school through appropriate experiences.

It should be made absolutely clear that what is called *proportional reasoning* in middle schol mathematics is nothing other than mathematical reasoning based on the concept of a *linear function without constant term*. Briefly, it requires the recognition that a situation is completely described by a function of the form

$$f(x) = cx \quad \text{for some constant } c.$$

Without formally introducing the concept of a "linear function", one can nevertheless talk about proportional reasoning by making explicit that $\frac{f(x)}{x} = c$ for *all* nonzero x. For example, walking at a constant speed of 2.7 mph means, **by definition**,

$$\frac{\text{the number of miles walked in } t \text{ hours}}{t \text{ hours}} = 2.7$$

[4]The practice in calculus books of the sixties and seventies to define a function as a set of ordered pairs at the beginning, and then never mention ordered pairs again in the remaining hundreds of page, readily comes to mind.

no matter what t may be. (See pp.46-50 of Milgram-Wu [3] for related discussions). Knowing $\frac{f(x)}{x} = c$ for all x, proportional reasoning is the statement that for any two distinct nonzero values x_1 and x_2,

$$\frac{f(x_1)}{x_1} = \frac{f(x_2)}{x_2} ,$$

and the reason is that both are equal to c. In the language of school mathematics, the four numbers $f(x_1)$, x_1, $f(x_2)$, and x_2 *form a proportion.*

There is thus no mystery to forming a proportion once the presence of the linear function $f(x) = cx$ is recognized. Because pre-algebra school mathematics does not have a tradition of making explicit the presence of, or bringing out, the underlying linear function without constant term, both teachers and students have a hard time understanding why there is a proportion. It is perhaps in this mathematical vacuum that some educators would bring psychology into the fabric of mathematics to "explain" the mystery of forming a proportion.

In every kind of learning or creation, psychology must play a role. Learning or creating mathematics is no exception. But mathematics itself is completely **WYSIWYG**, *what you see is what you get*; everything must be on the table. As teachers, we are required to make every concept and every skill logically self-contained: all the requisite information, such as whether or not a linear function is at work, must be made available to students before talking about the psychological factors that may affect the way they internalize the concepts or skills. If a piece of mathematics cannot be learned without appealing to psychology or some ineffable pre-school experiences, then it is bogus mathematics. The reason is simple: students' pre-school experiences vary, so a piece of mathematics that relies on a special kind of pre-school experience for its mastery automatically excludes some segment of the students. But mathematics is an open book, and if our students do not buy into this fact, what incentive could there be for them to learn?

Professional development must strive to make our teachers aware that mathematics is an *open book*, because they are the instrument to spread this message to students. It would not do to mislead our teachers into believing that psychology is an integral part of mathematics.

To drive home the point that mathematics must be psychology-independent, consider the following problem (cf. the discussion in Wu [6], Section 10):

> A group of 8 people are going camping for three days and need to carry their own water. They read in a guide book that 12.5 liters are needed for a party of 5 persons for 1 day. How much water should they carry? (NCTM Standards, [4] p. 83)

It is obvious that this problem cannot be solved without the assurance that *each person drinks the same amount of water each day,* and such an assurance is not forthcoming in the problem as it stands. If we only want to achieve mathematical correctness, we could of course add this pedantic assumption outright to the problem. A more effective formulation of the problem[5] would be to add flexibility to the wording so that students realize the need to make an "on average" calculation":

> A group of 8 people are going camping for three days and need to carry their own water. They read in a guide book that roughly 12.5 liters are needed for a party of 5 persons for 1 day. What does this suggest as the quantity of water they should take with them?

In any case, what is at issue here is that, in the original formulation, **students are expected to take this fact, drinking the same amount of water each day, for granted**, presumably because they should have the psychological maturity to see that this must be so. The minimal requirement of mathematics is, however, that such a (*nonobvious*) fact must be made explicit as part of the given data of the problem. Yet, problems of this nature, where a crucial assumption is hidden from students, are routinely given to assess students' mathematical proficiency.

Three consequences of the practice of using such problems for assessment are worth noting. On the one hand, when students fail to "set up the correct proportion" to solve such problems in standardized tests, students are

[5]Suggested to me by Tony Gardiner.

blamed for a lack of conceptual understanding of proportional reasoning. Of course we know that the cause of this failure is not students' lack of understanding but that *they have not been taught correct mathematics*. A second consequence is more pernicious. Some students manage to learn from their exposure to these problems that, when a problem is not solvable as stated, they simply make up extra assumptions in order to get a solution. But when the perception becomes ingrained in these students that *mathematics always carries a hidden agenda* in the form of these extra assumptions, and that guessing that agenda is part of doing mathematics, then in a real sense, they cease being mathematics learners. Instead, they would serve as Exhibit A of the collateral damage of a failed mathematics education.

A final consequence of the common use of such problems is that many students are forced to cope by setting up a proportion at all costs. This is of course the same as assuming that every function under the sun is linear. A few colleagues have expressed frustrations about the impossibility of convincing some college students that not all functions are linear. We don't need to look far for an explanation.

We want our teachers to know that this kind of problem is not acceptable mathematics.

Incidentally, it is possible to profit from the use of such problems-with-hidden-assumptions in a classroom. Hand out such a problem and ask students what additional assumptions are needed to make it solvable. In the hands of a knowledgeable teacher, this could be an enriching educational experience.

Part II Suggestions on how to meet teachers' mathematical needs in the face of these obstacles.

If professional development is to help our teachers learn the mathematics they need, then we must face the facts unflinchingly: Most teachers have been immersed in this culture of imprecise, incorrect mathematics throughout their 17 years as students as well as all through their careers as teachers. It is therefore to be expected that many of them would have little or no conception of the need for precise reasoning or mathematical closure, have no understanding of why precise definitions are indispensable in mathematics, and also see no need to teach mathematics as an open book with no hidden agendas. One cannot overcome such cumulative misconceptions of mathematics by teaching just a few mathematical skills or concepts, nor by offering a few one- or two-day workshops. A reasonable guess would be that one must make an intensive and sustained effort to systematically revamp their mathematical knowledge if one has any hope of revamping their perception of what mathematics is really about. To this end, there are at least four major areas that are deserving of special emphasis.

(i) Precise definitions *which are grade-level appropriate* are the cornerstone of school mathematics; they must be made an *integral part* of the instruction in order to demonstrate their importance.

To get a sense of to what extent definitions are neglected in school mathematics, I will make a partial list of the basic concepts whose precise definitions are usually missing from the school classroom:

the remainder in the division of whole numbers;

fraction;

decimal;

the sum, product, and quotient of two fractions;

constancy of speed;

constancy of the rate of work;

graph of an equation;

half-plane;

graph of an inequality;

circumference;

area of a planar region;

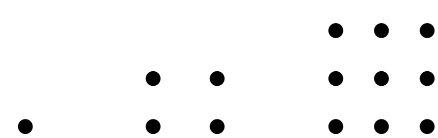
volume of a solid.

congruent figures

similar figures

(ii) Mathematics is WYSIWYG; there is no hidden agenda anywhere, especially in its assessment items.

I have discussed the problem about eight people on a camping trip as well as "proportional reasoning" problems in general. In the same breath, I should also mention the standard pattern problems in elementary school designed to promote so-called algebraic thinking. A typical such problem asks for the number of dots in the 18-th figure in a sequence whose first three figures are shown:



The expected answer is of course 18^2, but this answer assumes that each figure of the sequence is a square with one more dot on each side than the preceding square. This implicit assumption is then the *hidden agenda*, because there is no reason why the sequence *must* progress as described, e.g., these three figures could be repeated forever, in which case, the correct answer would be 3^2. If we wish to test students' ability to count correctly in a pattern, one might try to reformulate the problem by prefacing the three figures with words to the effect that, "If dots are arranged in successively larger square arrays, starting as shown here, how many dots would there be in the 18th figure starting with four dots?"

(iii) Every assertion in mathematics is supported by reasoning; we have to teach students to reason logically so that they can solve problems.

Unexplained statements are scattered all over the K-12 curriculum, and unfortunately it falls on the teachers to clean up this mess, at least in the short term. It is ironic that, at a time when problem-solving has become synonymous with mathematics education, the far more fundamental need for logical reasoning does not make the headlines. Logical reasoning is the backbone of problem solving. Our students' purported inability to solve problems is inextricably linked to the fact that logical reasoning is not part of the routine in the mathematics classroom. Until students are constantly exposed to logical reasoning, their performance in problem solving is unlikely to improve. To make such exposure a reality, our teachers have to develop the conditioned reflex that to every mathematical assertion there is an explanation. This is unfortunately not an easy task for teachers brought up by the present kind of pre-service professional development, because it would require a fundamental re-orientation of their mindset.

Again, to give an idea of the magnitude of the problem, here are some major topics of school mathematics for which, generally, little or no reasoning is given:

the standard algorithms for whole numbers and decimals;

invert-and-multiply for fraction division;

solutions of rate problems;

multiplication and division facts for rational numbers;

the relationship between linear equations and straight lines;

the relationship between linear inequalities and half-planes;

the relationship between roots of quadratic polynomials and their factorization.

(iv) The mathematics across the grades is tightly-structured and coherent;[6] mathematics teaching at all levels benefits from knowing this structure and coherence.

Often a teacher's knowledge is limited to the mathematics of one or two grades. When a teacher is unaware of the longitudinal coherence of the mathematics across K–12, the teaching inevitably suffers. Consider, for example, the following strands of development among standard topics:

> whole numbers $\longrightarrow$ fractions $\longrightarrow$ rational numbers $\longrightarrow$ polynomials and rational expressions;
>
> proportional reasoning $\longrightarrow$ linear functions;
>
> similar triangles $\longrightarrow$ graphs of linear functions;
>
> the concept of congruence $\longrightarrow$ concepts of length, area, and volume.

I will now illustrate the relevance of these strands to the present discussion with a trivial example. It is often said that children come to the study of fractions with the fixation that multiplication makes things bigger and division makes things smaller, and that teachers of fractions have a hard time undoing the misconceptions. Such misconceptions are attributed to the fact that, among whole numbers, multiplication magnifies and division shrinks. This is only the partial truth. Such misconceptions are also the result of teachers in the lower grades being totally oblivious to the needs of more advanced grades, in particular, to the future instruction in fractions where such phenomena of magnification and shrinking are no longer valid. If a primary teacher is aware of the longitudinal coherence of the development from whole numbers to fractions and to rational numbers, she would, at each step, caution students against possible pitfalls so that incorrect fixations do not take hold in students' thinking. Point out, for instance, that if we multiply 3 by 0, we get 0, and if we multiply it by 1, we get 3. So in neither case does multiplication lead to

[6] But one has to add that, once we get to the traditional sequence of Algebra I, Geometry, Algebra II, etc., one has to look past the articificial boundaries created by these courses to see the coherence.

a bigger number. If this message is repeated sufficiently often, even young kids would develop a healthy respect for the complexities of multiplication, and whatever misconception they might secretly harbor would be less likely to calcify. Of course if the primary teacher also explicitly mentions the possibility that multiplication by some numbers students have yet to learn may led to *smaller* numbers, so much the better.

One reason many, if not most, primary teachers pay no attention to the connection between whole numbers and fractions is that they are taught that "fractions are such different numbers". They are never told that fractions are nothing more than natural extensions of whole numbers (see e.g., Wu [8] for a more elaborate discussion of this point).

The importance of teachers' awareness of the *global structure* of school mathematics cannot be over-emphasized.

Part III The foregoing ideas guided my own work in professional development since year 2000. Perhaps there is some virtue in giving a brief description of the in-service summer institutes I have conducted since then, if for no other reason than to provide a point of reference for discussion.

The basic format of the institutes is as follows (cf. Wu [5]):

(i) They meet for three consecutive weeks in the summer, from 8:30 am to 4 pm everyday.[7]

(ii) There are also five follow-up Saturdays sessions in the succeeding school year devoted to reviews, some new topics, and especially pedagogical issues.

(iii) The daily schedule is as follows. Before 2 pm there are three hours of lectures *on mathematics*, each hour followed by twenty minutes of seat work and break. There is a one-hour break for lunch. The period from 2:30 pm to 4 pm is devoted to small group discussions of homework, mathematics of the day, and pedagogy.

[7] I have also conducted one week institutes on focussed topics, e.g., length and area.

(iv) There are daily homework assignments.

Why three weeks? The glib answer "because few would come for four weeks" is actually the truth. As the discussion in Part II indicates, the more time one can spend with teachers, the better chance one has to re-orient their thinking about mathematics. Experience indicates, however, that three weeks may be the maximum that most teachers are willing to sacrifice in the summer. They need time to recharge their batteries.

Three additional comments may be relevant. One is that teachers in these institutes are paid 100 dollars a day for participation. A second one is that daily homework assignments are essential for learning. Finally, the Saturday follow-up sessions are an indispensable part of the professional development. Teachers cannot possibly learn all they are taught in three weeks; this is just a reflection of the usual time lag between being exposed to an idea and actually learning it. By coming back to some of the topics in a more leisurely fashion in the ensuing nine months, and often from a different perspective, we greatly increase the chances that learning would take place. Equally important is the opportunity for the teachers in the follow-up sessions to begin thinking about how to use what they learn in their lessons. Such pedagogical attempts of course enhance the learning of mathematics as well.

To successfully convey to teachers the basic structure of mathematics, twenty days of contact time is not enough. For this reason, not much time during the summer can be explicitly spent on pedagogy. To compensate for that, one has to make sure that every piece of mathematics taught in the institute is inspired by concerns for the school classrooms and, more importantly, the mathematics is taught at a level as close as possible to what takes place in a school classroom. For example, fractions will not be taught as equivalence classes of ordered pairs of whole numbers, *no matter how attractive the explanation may be*. It is therefore appropriate that I conclude with a brief description of the contents of the four institutes that I have given. I hope the brevity does not preclude a glimpse into the pedagogical decisions.

Numbers and Operations: whole numbers, one-sided number line, standard algorithms, elementary number theory (prime factorization, GCD, LCM, Euclidean algorithm), definition of fractions as certain points on the number line and their arithmetic, definition of decimals as special fractions and their arithmetic, definition of rational numbers as points on the two-sided number line and the arithmetic of rational numbers.

Geometry: definitions of basic geometric concepts (polygons, polyhedra, perpendicularity, parallelism, angles), rigid motions in the plane in terms of hands-on activities and the definition of congruence, dilation as a hands-on activity and the definition of similarity, definition of area and area formulas (including area of the circle and definition of π), definition of volume and volume formulas, the Pythagorean theorem (two proofs).

Pre-Algebra: definitions of fractions, decimals, rational numbers and their arithmetic; rigid motions in the plane and congruence, dilation and similarity, simple geometric proofs, the Fundamental Theorem of Similarity, basic criteria for similar triangles, definition of the slope of a line, definition of parallelism and perpendicularity and theorems thereof in terms of slope.

Algebra (Wu [6]): algebra as generalized arithmetic, symbolic manipulations, transcription of verbal information into symbolic language, geometry of linear equations in two variables with special emphasis on the relationship between a linear equation and its graph, simultaneous equations, definition of half-planes and graphs of linear inequalities, linear programming, functions and their graphs, linear functions and proportional reasoning, graphs of quadratic functions and the quadratic formula.

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