Chapter 11
Inservice Mathematics Professional Development: The Hard Work of Learning Mathematics

Hung-Hsi Wu

11.1 Introduction

We all want to improve school mathematics education, but before making any recommendations on how to take it to the next level, we would do well to first find out where we stand. The answer: not in a good place. For the past five decades or so, the mathematics we teach in school has been mostly flawed and unlearnable.[1] For example, the fractions that students have to compute extensively with from grade 5 to grade 12 are supposed to be thought of as pieces of pizza. The resulting fraction phobia has been something of a national pastime for decades (see, e.g., https://www.gocomics.com/peanuts/1966/04/21). Another example: we do not make any effort to teach students proofs (reasoning) in the K-12 curriculum outside the high school geometry course, and yet in that one geometry course alone, students are suddenly called upon to *prove everything*—no matter how trivial or boring—on the basis of a collection of new objects called "axioms." The situation would not be as bleak if we had educated our mathematics teachers properly so that they could help smooth students' learning path along such a rugged obstacle course, but we haven't. Since teachers are only equipped with this body of flawed and unlearnable mathematical knowledge, they inevitably inflict the same flawed and unlearnable mathematical knowledge on their students. So the vicious cycle continues to this day.

[1] We use "unlearnable" in this article to mean "unlearnable by a majority of students." We note that, in this case, learning *mathematics* includes learning how to reason; see Sect. 11.3 below for the fundamental principles of mathematics.

H.-H. Wu (✉)
Department of Mathematics, University of California at Berkeley, Berkeley, CA, USA
e-mail: wu@berkeley.edu

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Obviously, two things must be in place before there can be any improvement in school mathematics education: a correct and learnable version of mathematics is made available in textbooks to students and a corps of teachers who are capable of teaching the same. These are the tasks before us, and they have recently acquired increased urgency because of the advent of the CCSSM (Common Core State Standards for Mathematics, [CCSSM]). The mathematics advocated by the CCSSM represents the first—but major—step towards meeting the goal of being correct and learnable, so the need for better school textbooks and teachers who are mathematically more knowledgeable can no longer be put off to the distant future.

To achieve the first goal of getting better school textbooks, some recent developments have given us hope, but in any case, there is now a detailed mathematical guide on what constitutes correct and learnable school mathematics. To achieve the second goal about a corps of mathematically competent teachers, we need a serious commitment to content-based PD (professional development) to meet this problem head-on. However, two large-scale impact studies of PD for teachers carried out in the past decade by IES (the Institute of Education Sciences) have raised serious doubts about the ability of content-intensive PD for inservice *mathematics* teachers to improve student learning (see Garet et al. 2011 and Garet et al. 2016). We are therefore forced to take a close look at this claim by the two IES studies. In our view, the claim is not supported by the available evidence, and we will make some effort to explain why not. Along the way, our explanation will also suggest the kind of PD that may be more likely to produce mathematically knowledgeable teachers who can improve student learning.

This article will expand on the preceding rather cryptic statements. A brief outline follows. Section 11.2 gives a description of the flawed and unlearnable body of knowledge—what we call TSM (*Textbook School Mathematics*)—that has dominated school mathematics education for the past half century. Section 11.3 introduces the *Fundamental Principles of Mathematics*, which are the *sine qua non* of mathematics. School mathematics that respects these fundamental principles will be called PBM (*principle-based mathematics*, see Poon 2014), and we will explain why PBM, because of its transparency, is *learnable*. In Sect. 11.4, we briefly discuss the situation regarding school textbooks that respect PBM. In Sect. 11.5, we give a fairly detailed discussion of the kind of PD needed to produce inservice teachers who can teach PBM *and* of the obstacles that stand in the way of implementing such PD. Section 11.6 presents an in-depth analysis of the aforementioned 2011 IES impact study and explains why its PD could not have produced teachers capable of teaching PBM: the PD did not help them overcome the handicap of knowing only TSM but not correct mathematics. The last section offers a variety of comments, including the need for "mathematics teachers" in elementary school and what may be preventing effective *preservice* PD from becoming a reality on university campuses.

11.2 Mathematical Engineering and TSM

To understand the kind of "mathematics" that has dominated school mathematics education for the past 50 years or so after the demise of the New Math around 1970, we need to step back to get some perspective on the nature of school mathematics and the overall state of school math education.

School mathematics is not part of *mathematics* proper—the mathematics we teach in universities and use in science and mathematics research—but is, rather, a particular version of mathematics that has been customized for consumption by K–12 students (see Wu 2006). This is analogous to the case of electrical engineering, which is not part of physics but is a customized version of it for the purpose of creating electrical and electronic products to meet humans' everyday needs. It is in this sense that school mathematics is a product of *mathematical engineering*, and a good part of school mathematics education is just mathematical engineering (Wu, *loc. cit.*). Of the need to customize university mathematics for consumption in K–12, there can be no doubt. After all, we do not introduce fractions to elementary students as the positive elements in the quotient field of the ring of integers. Rather, we directly develop fractions from whole numbers using the number line (Jensen 2003; Wu 1998, 1999a, and 2011a); this will be discussed in Sect. 11.3.2. Similarly, in K–12, a line in the plane is not a linear map from $\mathbf{R}$ to $\mathbf{R}^2$ but the unique curve joining any two of its points as specified by Euclid's first postulate.[2] And so on.

This engineering takes many forms. Sometimes it recasts the whole concept in a different but equivalent setting, as in the case of fractions and rational numbers. Sometimes it makes use of advanced theorems without any proof (so long as there is no circular reasoning), such as the fundamental theorem of algebra, the Jordan curve theorem for polygons, or the existence of the exponential function e^x. At other times it simply leaves out topics that are too conceptually sophisticated, such as the structure of the real numbers[3] or the concept of continuity. But regardless of the engineering decisions, there will always be good and bad engineering. In the same way that bad engineering in electrical engineering produces electronic gadgets that are hazardous to users, bad mathematical engineering produces a body of mathematical knowledge for K-12 that is unlearnable, basically because it is often *wrong as mathematics*. The mathematical knowledge that has dominated school mathematics education for the past five decades is unfortunately one example of what bad mathematical engineering has wrought. We call it **TSM**, *Textbook School Mathematics*, because its most complete realization resides in all the standard school mathematics textbooks and almost all the textbooks for mathematics teachers'

[2]Nevertheless, some recent publications have done just that: defining a line in the plane as the graph of an equation $y = mx + b$. See, e.g., page 711 of Billstein-Liebeskind-Lott 2007.

[3]This particular engineering decision makes a tremendous impact on the school curriculum because, as a result, the principal number system in K–12 is actually the rational numbers $\mathbf{Q}$ and *not* the real numbers $\mathbf{R}$. This is the reason why fractions are so important in K–12.

professional development (see Askey 2018; Baldridge 2013; Douglas 2015; Cuoco-McCallum 2018; Wu 2011c and 2018).

Although TSM looks superficially like mathematics, it differs from mathematics in important ways, especially in its lack of precise definitions and reasoning. TSM is not concerned with students understanding concepts or developing a capacity for reasoning, but instead focuses on *getting right answers to problems that TSM sees fit to pose.* To this end, TSM offers students a set of procedures, which, when followed conscientiously, lead to the right answers to these problems. To make the procedures more attractive to students, TSM uses only intuitive language to describe the concepts to make students believe that they "get it." The absence of precise definitions—and the attendant absence of reasoning—in TSM is therefore part of the design.

We must confront TSM directly because of its tenacious and pervasive hold on school mathematics education. It is the mathematics used by teachers and education researchers in their work, and its omnipresence can be easily explained. Teachers and educators[4] learned TSM in their K-12 years, and when they were students in institutions of higher learning, they learned mainly about the *pedagogical* issues of the K–12 curriculum. On the rare occasion that they got to take a course on school mathematics, almost all the textbooks for such courses—as mentioned above—consisted of little more than polished presentations of TSM. Once teachers and educators begin their professional lives, the mathematics they deal with is once again TSM. This is especially true for teachers because textbooks are "the authority on knowledge and the guide to learning . . . many teachers see their job as just 'covering the text' " (Romberg and Carpenter 1985). We therefore have a vicious cycle that reinforces the dominance of TSM in American school mathematics education, including education research. Thanks to this well-established recycling program, it would be fair to say that TSM *is now part and parcel of the mathematics education literature.* The article of Armstrong and Bezuk (1995) illustrates this point very well. These authors discuss the difficulty teachers have trying to teach the multiplication and division of fractions in middle school. They observe that teachers teach these concepts procedurally (without reasoning) not because they intentionally want to "withhold conceptual understanding from their students," but because

> It is quite possible that the teachers do not know that a conceptual base for multiplication and division of fractions even exists. Nothing in their mathematics learning experiences would have provided a hint of that existence. (*loc. cit.*, page 91)

From our perspective, what this says is that most teachers have been denied the opportunity to learn a correct approach to the multiplication and division of fractions simply because all they have access to is TSM.

[4]We use the term "educators" to refer to university faculty in schools of education.

11.2.1 Some Examples of TSM

Some examples will clarify why we object to TSM.

Example 11.1 TSM explains equivalent fractions by using what is often called the *Giant One*. For example, to show $\frac{3}{2} = \frac{12}{8}$, TSM reasons as follows:

$$\frac{3}{2} = \frac{3}{2} \times \mathbf{1} = \frac{3}{2} \times \boxed{\frac{4}{4}} = \frac{3 \times 4}{2 \times 4} = \frac{12}{8} \qquad (11.1)$$

This "reasoning" is probably too well-known to require any comments. Formally, the starting point of this "reasoning" is that $\frac{3}{2}$ and $\frac{12}{8}$ *are fractions*, and the conclusion is that *the two fractions are equal*.

From the outset, this "reasoning" faces two insurmountable obstacles: *TSM has no precise definition of a fraction* and, therefore, *it is unclear what it means for two fractions to be equal.* So TSM begins with a vague hypothesis and arrives at a conclusion that is equally vague. Hardly an ideal setting for doing mathematics. Yet, the greater obstacle is the use of fraction multiplication in this attempted "proof." Since the concept of equivalent fractions appears almost as soon as fractions are introduced, *before* students get to know how to add or multiply them, fractions are not ready to be multiplied in this argument. In this light, the transgression implicit in the first step, $\frac{3}{2} = \frac{3}{2} \times 1,$ seems relatively harmless because "1 times anything is the thing itself." The key step in Eq. (11.1) that

$$\frac{3}{2} \times \boxed{\frac{4}{4}} = \frac{3 \times 4}{2 \times 4}$$

is, however, totally out of place because the validity of the product formula that says $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ (for any positive integers a, b, c, and d) itself depends on the use of equivalent fractions (see pp. 62-63 of Wu 2016a). Therefore, this "proof" in TSM is guilty of circular reasoning at the very least.

TSM's inability to define what it means for two fractions to be equal also plays a role in the next example.

Example 11.2 The article of Otten et al. (2010) tries to give a demonstration of the *cross-multiplication algorithm* (**CMA**): If two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal, then $ad = bc$. Because the authors were working within TSM, they had no precise definition of a fraction at their disposal, so they made up an *ad hoc* definition of *equality for fractions* by saying that $\frac{a}{b} = \frac{c}{d}$ means that there is a nonzero whole number k so that $c = ka$ and $d = kb$. Then they used this definition of "equal fractions" to prove the theorem. The fact that this definition of *equality* is incorrect (e.g., $\frac{6}{9} = \frac{14}{21}$, but there is no whole number k so that $14 = k \times 6$ and $21 = k \times 9$) and that such a hypothesis trivializes the theorem is almost beside the point here. What is striking is that we get to witness the struggle the authors were going through in trying to break free from TSM, and how TSM ultimately defeated them.

Incidentally, CMA should be taught in grade 5, not long after the theorem on equivalent fractions has been proved, and it (together with its various extensions) belongs in the survival kit of every student and every teacher in K–12.

Example 11.3 What does it mean to add two fractions such as $\frac{3}{8} + \frac{5}{6}$, and what is the sum? TSM provides no answer to the first question; for the second, it prescribes the following procedure: Get the least common denominator (**LCD**) 24 of 8 and 6 and observe that $24 = 3 \times 8,\ 24 = 4 \times 6$. Then add as follows:

$$\frac{3}{8} + \frac{5}{6} = \frac{3 \times 3}{3 \times 8} + \frac{4 \times 5}{4 \times 6} = \frac{9 + 20}{24}$$

In terms of students' mathematics learning, one has to take note of the fact that when elementary students encounter the addition of fractions for the first time, they expect that it will be more or less the same as the addition of whole numbers, i.e., addition is "putting things together." However, not only is there no indication in TSM that adding fractions has anything to do with "putting things together," but there is also nothing in the preceding procedure—LCD and all—to suggest any connection with "putting things together." TSM makes learning how to add any two fractions more complicated and difficult than it needs to be.

Example 11.4 What does it mean to multiply two fractions such as $\frac{2}{3} \times \frac{5}{8}$ and what is the product? Again TSM has nothing to say about the first question, and it answers the second by declaring that fractions are multiplied by the following rule: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ for any whole numbers a, b, c, and d (with the understanding that $bd \neq 0$). It then follows easily that the preceding product is equal to $\frac{10}{24}$. No explanation is given for this rule, but there is usually some effort to make this rule seem reasonable by discussing the special case where $b = 1$ (i.e., a whole number multiplies a fraction) and also the special case where $d = 1$ and b divides c (i.e., a fraction multiplies a whole number which is a multiple of the denominator of the fraction). How then do we use multiplication in word problems? Again, do it by rote: when the word "of" appears, it means "multiply" (see Moynahan 1996).

Example 11.5 TSM introduces the concept of a *mixed number* right after the definition of a fraction—but before the addition of fractions is discussed. Thus, $2\,\frac{3}{4}$ is, by definition, "2 *and* $\frac{3}{4}$." TSM also explains the conversion of mixed numbers to improper fractions by rote, e.g.,

$$2\,\frac{3}{4} = \frac{(2 \times 4) + 3}{4} = \frac{11}{4}$$

This procedure has to be done by rote because of TSM's refusal to *define* a mixed number as the sum of a positive integer and a *proper fraction*, and that, for example, $2\,\frac{3}{4}$ is the shorthand notation for $2 + \frac{3}{4}$. Notice that the word "and" has been purposely used to hide the fact that the addition of fractions is involved, an inexcusably bad piece of mathematical engineering. If mixed numbers were introduced *after* the addition of fractions, they would be a perfectly simple topic to learn.

Example 11.6 TSM considers finite decimals to be a different kind of number from fractions and it teaches finite decimals on a parallel track, independent of fractions. For example, a finite decimal such as 2.307 is defined to be "2 and 3 tenths and 7 thousandths." Once again, the word "and" is purposely used to hide the fact that the addition of fractions is involved, so that 2.307 is actually the following sum of fractions,

$$2 + \frac{3}{10} + \frac{0}{100} + \frac{7}{1000}$$

So TSM knows that a finite decimal *is* a fraction, but nonetheless tries to hide it. Bad mathematical engineering again. Such an approach to the teaching of finite decimals has produced misconceptions that are legendary (see, for example, https://tinyurl.com/y6k59uqp).

These examples are cited for their relevance to our discussion, but we must emphasize that they do not come close to exhausting the sins of TSM. A few other examples are the obsession in TSM with the so-called *order of operations*, which elevates a notational convention to a major topic in middle school mathematics, or the use of FOIL in TSM to expand the product of two linear polynomials, or the convention in TSM geometry that precludes a square from being a rectangle, an equilateral triangle from being an isosceles triangle, a parallelogram from being a trapezoid, etc. There is another glaring defect that should not be overlooked: the cavalier way TSM handles real numbers. In middle school, irrational numbers begin to encroach on many mathematical discussions because numbers such as π and square roots of whole numbers can no longer be avoided. Real numbers are not the province of K–12 mathematics, granted, but when students are asked to believe—*without a word of explanation*—that $\frac{\sqrt{2}\cdot\sqrt{3}}{\sqrt{2}} = \sqrt{3}$ because of the "usual" cancellation law for fractions (whose numerators and denominators are *whole numbers*), things clearly have gotten out of hand.

In summary, TSM represents a major transgression against what is acceptable in mathematics.

11.2.2 *The Neglect of Definitions in TSM*

To get an idea of the scope of TSM's devastation of school mathematics, it may be of some interest to see at least a *partial* list of the fundamental mathematical concepts in K-12 that are either not defined or defined incorrectly in TSM:

- the *remainder* in the division-with-remainder of whole numbers;
- fraction; *equality* of fractions; one fraction being *bigger* or *smaller* than another; addition of fractions; multiplication of fractions; division of fractions;
- finite decimal; *equality* of decimals; one decimal being *bigger* or *smaller* than another; addition of decimals; multiplication of decimals; division of decimals;
- ratio; percent; rate; *constant* rate;
- expression, equation; *graph* of an equation; *graph* of an inequality; half-plane;
- *slope* of a line;
- the 0-th power of a number, negative power of a number;
- polygon, regular polygon, parabola;
- *congruent* figures; *similar* figures; scale-drawing;
- length of a curve; area of a plane region; volume of a solid.

Because reasoning is impossible without definitions, TSM has to teach the skills related to all the concepts on this list entirely *by rote*. For example, because there is no definition for either the "graph of a linear inequality in two variables" or a "half-plane," there is no explanation in TSM for the fact that the graph of a linear inequality in two variables is a half-plane. The absence of precise definitions for fraction, decimal, ratio, percent, and rate will be particularly pertinent to the discussion of the PD program of the 2011 IES impact study (mentioned in the Introduction) in Sect. 11.6.2 below.

Beyond its failure to define key concepts, TSM also does great harm to mathematics learning by introducing spurious concepts, notably "variable" and "proportional reasoning." It is not difficult to see that neither can be defined in a way that makes any sense as mathematics, but if a fuller explanation is needed for why these are not *mathematical* concepts, see Section 3.2 of Wu (2018). (One can find a more detailed discussion of "variable" in pp. 2–3, 28–29, 38–39 of Wu 2016b, and of "proportional reasoning" in Section 7.2 of Wu 2016b.)

11.3 Fundamental Principles of Mathematics and PBM

Thus far, we have criticized TSM for its many mathematical flaws, and we have referred vaguely to the need for *correct and learnable* mathematics in the classroom. Now it is time to explain in greater detail what "correct and learnable" school mathematics is.

11.3.1 Fundamental Principles of Mathematics

First, consider the following five **Fundamental Principles of Mathematics**:

(I) Every concept has a precise definition.
(II) Every statement is supported by reasoning.

(III) Precision attends every statement.
(IV) The progression from topics to topics is coherent.
(V) The progression from topics to topics is purposeful.

It will be clear from the following discussion that all five overlap each other and that the first three form a close-knit unit. The examples of the last section illustrate the fact that TSM violates every one of these principles, but we will provide more of such examples below.

These principles form a minimal set of characteristic properties of mathematics, and any mathematical exposition of that violates any one of these principles is not a faithful representation of mathematics. For our present purpose, we call *school mathematics* that respects these five fundamental principles **PBM** (*principle-based mathematics;* this term was coined by Poon 2014). Thus PBM is a body of knowledge that is consonant with both the progression of the K-12 school mathematics curriculum and the fundamental principles of mathematics. Henceforth, we will use PBM as a shorthand for *correct and learnable* mathematics.

We now explain the preceding fundamental principles from the specific vantage point of learning *school mathematics*.

(I) The need for precise definitions stems from the fact that the learning of mathematics involves the learning of many new concepts. A *precise definition* of a concept tells students what it is (a number? a pair of numbers? a function? an equality? a geometric figure? etc.), and what properties it is assumed to possess. From a pedagogical perspective, the purpose of having precise definitions is to lighten students' cognitive load by clearly setting forth—for the purpose of learning—*everything they need to know about the concept in question.* A precise definition of a concept eliminates second-guessing: it assures students that they are already in possession of all they need to know for any reasoning involving this concept, no more and no less. *This is how mathematics works.* There is no need for students to wonder whether the textbook and the teacher have something up their sleeve that is not being shared with them.

To understand what this means, suppose a fraction is defined to be like a piece of pizza. But every student knows that the metaphorical piece of pizza will inevitably turn into something else at a moment's notice. After all, if TSM asks them how long it will take a faucet to fill a tub of $57\frac{1}{2}$ gallons given that the rate of the water flow is a constant $14\frac{2}{3}$ gallons per minute, their common sense would tell them to forget whatever has been taught about fractions-as-pizzas and, instead, concentrate on their rote skills. This illustrates how students in TSM are put in a state of constant distrust. How can real learning take place under the circumstances? Worse, if a concept such as the *division of fractions* is taught without a definition, students are left to cope with problems about fraction division *without knowing what they are doing.* This is why we have "ours is not to reason why, just invert and multiply" and the attendant fraction phobias.

Having a precise definition of a concept—and consistently basing any reasoning involving this concept only on what is in the definition—is therefore a necessary first step to build trust and make it possibly for students to learn about reasoning with the concept. The precise definition eliminates any need for students to constantly look over their shoulders and try to guess what additional information about the concept may be coming their way.

Two further comments about definitions will round out the picture. The first is that, insofar as a definition is supposed to inform students of everything they need to know about a concept, the need for *simplicity* in a definition should be obvious. For example, consider the following "definition" of a *right triangle*: *it is a triangle so that one of its angles is* 90° *and so that if* a, b, *and* c *are the lengths of its sides and* c *is the largest, then* $c^2 = a^2 + b^2$. Such a "definition" is not wrong in a formal sense, but it clearly fails to be informative because students would wonder whether there are any "right triangles" in this world that can meet both requirements. After all, what is the equality $c^2 = a^2 + b^2$ all about? If students' first reaction to this definition is one of disbelief, how to convince them to learn about right triangles? Therefore, we have to pare such a definition down to "*a right triangle is a triangle so that one of its angles is* 90°" and then show how to use reasoning *on the basis of this definition* to prove the equality $c^2 = a^2 + b^2$.

The second general comment about definitions is that the connection between *precise definitions* and *reasoning*—to the effect that any reasoning about a concept must be based *only* on what is contained in the definition—seems to have stayed under the radar in the mathematics education literature for the past few decades. This could be because of the dominance of TSM, which considers "definitions" to be largely superfluous and completely separate from the many rote-learning rules that make up TSM.

(II) We have just seen that having precise definitions is not an end in itself but, rather, the means to an end, the end being to make reasoning possible. Reasoning is the lifeblood of mathematics; there is no difference between reasoning and what is called *problem solving* in the education literature[5] when the latter is correctly interpreted. However, for the purpose of mathematics learning, reasoning plays the pivotal role of serving as the glue that connects concepts and skills. It is well-known that such connections make mathematics more learnable than a collection of concepts and skills that are memorized by rote (see pp. 118–120 of National Research Council 2001 for the large body of research evidence supporting this claim). Another way that reasoning helps to make school mathematics learnable is that it empowers all students to decide *for themselves* whether what they are doing is correct or not without having to submit themselves to the authority of their teacher or textbook. Learning how to reason therefore enhances students' self-confidence and their disposition[6] toward learning, which will in turn generate more learning.

[5]Problem solving is currently the main goal of school mathematics education in certain circles. It is well to note that there is no way to get students ready for problem solving (i.e., reasoning, which is the second fundamental principle of mathematics) without the help of the other four fundamental principles.

[6]Compare the fifth strand of *mathematical proficiency*—productive disposition—in Chapter 4 of National Research Council (2001).

(III) It is a truism that precision minimizes misunderstanding in teaching and learning. In the case of mathematics, however, we can be more specific: *without precision, learning about reasoning becomes well-nigh impossible.* For example, a typical definition of the division of fractions in TSM is the following:[7]

> Division and multiplication are **inverse operations**. Inverse operations are operations that undo each other.

These sentences sound plausible, but ultimately make no sense because multiplication sends *two* numbers, e.g., 2 and 3, to the third number ($2 \times 3 = 6$ in this case). Similarly, division sends *two* numbers, e.g., 6 and 3, to the third number ($6 \div 3 = 2$ in this case). So start with 2 and 3 (let us say), multiplication sends them to 6. Now how to "undo" 6 to send it back to 2 and 3 by division? If the definition does not make sense, how can we teach students to reason about fraction division using the definition?

A little bit more attention to precision would likely have averted this travesty by rephrasing the preceding "definition" as follows: *if a fraction* $\frac{a}{b}$ *is fixed, then dividing it by a nonzero* $\frac{c}{d}$ *yields a fraction so that, when the latter is multiplied by* $\frac{c}{d}$, *we get back* $\frac{a}{b}$.

Perhaps a more telling example of the need for precision is the way CMA (cross-multiplication algorithm) is used in TSM. Let x be a number that satisfies a proportion:

$$\frac{4.6}{13\frac{4}{5}} = \frac{x}{8\frac{1}{2}} \tag{11.2}$$

Then, a standard procedure to solve for x is to use CMA to get $13\frac{4}{5} \cdot x = 4.6 \cdot (8\frac{1}{2})$, thereby obtaining $x = 2\frac{5}{6}$. This solution method has unquestioned authority until we stop to ask: why is CMA applicable to Eq. (11.2)? Here, we need the fact that $\frac{a}{b} = \frac{c}{d}$ implies $ad = bc$. In TSM, CMA is either not proved (see Example 11.2 in Sect. 11.2.1) or proved only for *fractions* $\frac{a}{b}$ and $\frac{c}{d}$, in which case, $a, \ldots, d$ are *whole numbers*. The numerators and denominators in (11.2) are definitely not whole numbers, and it is shocking to realize that TSM never proves the CMA when the numbers involved are not whole numbers! How then can we inspire students to learn how to reason when they consistently bear witness to the fact that TSM plays fast and loose with results obtained by reasoning? What is the point of reasoning?

Such imprecision also has a pernicious side effect: it implicitly invites students not to take what they read literally, because *anything they read is likely to be correct in a wider context.* Consequently, students who are taught that

> if A, B, and C are nonzero fractions, then $A < B$ implies $CA < CB$

[7] This is taken directly from a textbook.

have every right to believe that this must also be true when A, B, and C are *any numbers*. Reports that the author heard consistently from teachers in the field is that many students are dismayed by the fact that

> if A, B, and C are rational numbers and $C < 0$, then $A < B$ implies $CA > CB$.

Such imprecision puts students in a difficult position: how to decide when to believe—or not to believe—what they are taught?

Moral For mathematical learning to take place, precision must be the rule so that students know at each step *exactly* what is true and what is false.

(IV) Roughly, the *coherence* of mathematics means that mathematics, far from being a mere random collection of facts, is a tapestry in which all the concepts and skills are logically interwoven to form a single fabric. Mathematics unfolds logically, from basic assumptions (axioms) and definitions to theorems, and from theorems and other definitions to more theorems. Because of this logical progression, different parts of mathematics, even when far apart, often echo each other or are interconnected. It is this interconnectedness that comes from the unfailingly logical development of mathematics that we call *coherence*.[8]

The impact of coherence on learning can be seen in the learning of the arithmetic operations on whole numbers, fractions, rational numbers, and eventually real numbers. These operations are *conceptually the same* across the various number systems. (This fact is a main emphasis in Wu 2011a.) As a consequence of this coherence, if these operations on whole numbers are taught correctly, then the learning of these operations on fractions becomes streamlined and the popular perception in TSM that "fractions are such *different* numbers from whole numbers"[9] will be banished forever from school mathematics education. (Again, see Wu 2011a.)

The impact of coherence on learning can also be seen in a smaller scale in the most mundane of all school mathematics topics: the standard algorithms for whole numbers. When taught as rote skills, these algorithms are the embodiment of mindless tedium. But they are in fact held together by a single leitmotif:

> A knowledge of the addition, subtraction, multiplication, and division of *single-digit numbers* empowers us to perform all arithmetic operations with ease on any whole numbers, no matter how large. (See Chapter 3 of Wu 2011a.)

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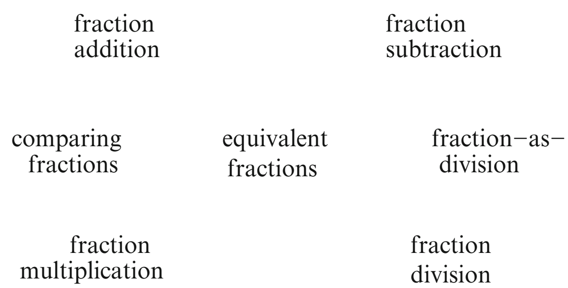
[8] According to Cuoco-McCallum (2018), what we have just defined is the *coherence of content*. The Cuoco-McCallum article is, in their terminology, concerned with the *curricular* coherence of the school mathematics curriculum.

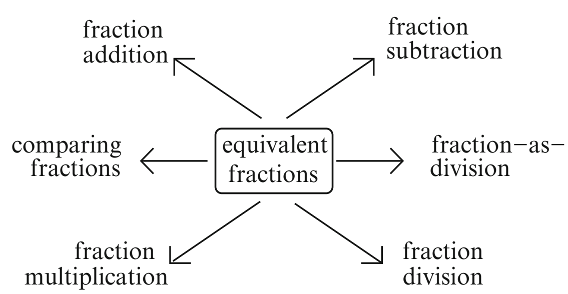
[9] This is a direct quote of what one parent told the author. Such a popular perception is in fact a reflection of not only what transpires in a TSM classroom (compare, e.g., Examples 11.3 and 11.4 in Sect. 11.2.1 above), but also what is in the education literature. For example, "Such difficulty with fractions is often attributed to the fundamental differences between whole numbers and fractions." (Namkung and Fuchs 2016). Or, "Children must adopt new rules for fractions that often conflict with well-established ideas about whole numbers." (Bezuk and Kramer 1989).

The phrase "with ease" refers to the fact that any one of these operations on large numbers can be indescribably tedious (e.g., 2573×496 means adding *2573* copies of 496), but when reduced to single-digit computations (which is what the standard algorithms do) it becomes relatively simple. If this leitmotif is made known to elementary students—and of course if the algorithms are explained to them too—they are more likely to learn the algorithms and especially the multiplication table with greater enthusiasm and, more importantly, they will also learn *a substantial amount of valuable mathematics* because these four algorithms bring to light a recurrent theme in all of mathematics: reducing the complex to the simple.

Yet another example of how coherence can impact learning is in the teaching of fractions. Here are the seven most basic topics in fractions:



It is difficult to make sense of them when they are presented starkly as seven rote skills. But when reasoning is introduced into the discussion, a clear picture emerges: the other six topics are now seen to follow from the one central fact on equivalent fractions (see, e.g., Chapters 13-18 of Wu 2011a). From this perspective, we can make sense of all seven topics, and *fractions begin to be learnable.*



Incidentally, this is analogous to the phenomenon that while it is impossible to commit to memory the contents of even one page from a telephone book,[10] a thousand-page book like *Don Quixote* is quite memorable.

(V) *Purposefulness* refers to the fact that everything in mathematics is done with a purpose; this fact is of vital importance for the purpose of doing and learning mathematics but is unfortunately not something that is brought out in most books in mathematics education, least of all in TSM.

It is easy to explain the important role of purposefulness in school mathematics. Many skills and concepts have competed to stay in the (more or less) universally accepted school curriculum for more than a century, if not longer, and those that have survived to stay in the present day curriculum are the winners of many rounds of elimination. The reason these skills and concepts are still here could only be because *they serve a vital purpose.* If we can bring out this purpose to make students see why these skills and concepts are worth learning, students will be more motivated to learn them and student achievement will improve as a result. For example, we have already alluded to the likelihood that emphasizing the purpose (the leitmotif) of teaching the standard algorithms will increase student learning.

There is probably no better illustration of how *purposefulness* can impact student learning than the topic of *rounding whole numbers.* During my many years of doing inservice PD, I was once asked by a teacher why we bother to teach *rounding*, a skill that she considered to be meaningless. She said her students had no idea why they should learn it. Subsequently, other teachers concurred. Their complaint was entirely justified because TSM never explains that, quite often, one wants to round off a number because *precision is not wanted or is simply unattainable, or both.*

For example, the Census Bureau's estimated population of Houston was 2,303,482 in 2016. If a visitor from afar asks you how many people live in Houston, are you going to say "2,303,482"? *You had better not,* because you would sound ridiculous. Such precision is not the intent of the question. Your visitor probably only wants to know, roughly, how Houston compares with New York (population approx. 8,540,000) or San Francisco (population approx. 870,000). In other words, you are probably only expected to say whether the Houston population is closer to nine million or nine hundred thousand. With this in mind, you would likely *round* 2,303,482 *to the nearest million* to get two million. Then you look at your visitor in the eye and say with great confidence, "about two million."

One can also point to another kind of purpose for rounding: *when precision is not attainable.* Consider the 2016 estimate of Houston's population again. The Census Bureau probably had to release the figure of 2,303,482 for bureaucratic reasons, but such precision clearly makes no sense given the instability of a major city's population due to the unending cycle of births and deaths, the presence of a large transient population, and its ever-changing homeless population. Therefore, the most conservative estimate of Houston's population in 2016 is that the last three

[10] If a reader doesn't know what a "telephone book" is, please ask anyone over 60 or email the author!

digits, 482, are completely meaningless. We can de-emphasize them by *rounding to the nearest thousand* and list 2,303,000 as Houston's population in 2016. But if you round it to the nearest hundred thousand and list Houston's population as 2.3 million, I doubt that eyebrows would be raised.

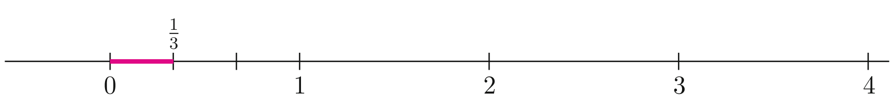
If TSM would take the trouble to explain the purpose of rounding, many of our teachers probably would cease being exasperated by having to teach it. Students too would likely approach the learning of this skill with greater enthusiasm.

Of course, there is no end of examples to illustrate how the teaching of a concept or skill would be enhanced by bringing out the purpose of introducing said concept or skill. In addition to the four standard algorithms, think of *place value* (no, it is not due to a decree from on high that the 3 in 35 must be 30; see Chapter 1 of Wu 2011a), the introduction of *negative numbers* (see, for example, Chapter 26 of Wu 2011a), the introduction of *absolute value* (see Section 31.3 in Wu 2011a), etc.

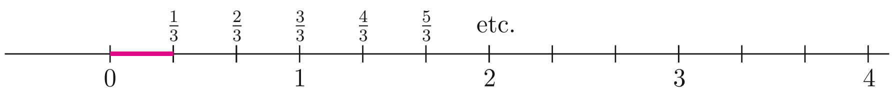
11.3.2 PBM vs. TSM

Thus far, we have discussed in general terms some special features of TSM and PBM. Because the overriding theme of this article is to help teachers get rid of their knowledge of TSM and replace it with PBM, we will now revisit the six examples in Sect. 11.2.1 from the perspective of PBM. Because all these examples are about fractions, we will begin with a brief presentation of the definition of a fraction using the number line (first presented in Wu 1998, but see Wu 2002 and Chapter 1 in Wu 2016a). We will try to be brief, except that the discussions of Example 11.3 (adding fractions) and Example 11.4 (multiplying fractions) will be intentionally detailed because we want to illustrate explicitly how to use definitions in reasoning (see the discussion on *learning about definitions* in Sect. 11.5.1 below).

We begin with a (horizontal) number line on which a sequence of equidistant points marching to the right are labeled by the whole numbers. Now proceed as follows to get the **fractions with denominator 3** (and by extension any and all of the other fractions). Partition the **unit segment** [0,1] into three **equal parts** (= three segments of *equal length*). The part adjoining 0 is the third. Denote its right endpoint by $\frac{1}{3}$.

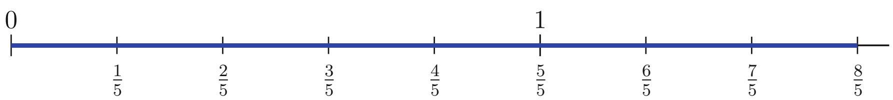


Fix the distance between 0 and $\frac{1}{3}$. Marking off *equidistant* points to the right of $\frac{1}{3}$ as we have done with whole numbers, we obtain a sequence of points, denoted by $\frac{2}{3}$, $\frac{3}{3}$, $\frac{4}{3}$, etc.



The segment $[0, \frac{1}{3}]$, by convention, is identified with its right endpoint, $\frac{1}{3}$. Similarly, the segment $[0, \frac{2}{3}]$ is identified with its right endpoint $\frac{2}{3}$, the segment $[0, \frac{5}{3}]$ with its right endpoint $\frac{5}{3}$, etc. Call these **the sequence of thirds**. Also call $\frac{n}{3}$ the **length** of the segment $[0, \frac{n}{3}]$ for any nonzero whole number n.[11]

Similarly, the nonzero fractions with denominator 5 are the **sequence of fifths**, determined by the partition of $[0, 1]$ into 5 equal parts and by repeating the construction as in the sequence of thirds. For example, $\frac{8}{5}$ is the last point on the right:



Then we call $\frac{8}{5}$ the **length** of the segment $[0, \frac{8}{5}]$, etc. (See Wu 1998; Chapter 12 of Wu 2011a has more details.)

If n is any nonzero whole number, then we obtain the **sequence of *n*ths** by partitioning the unit segment $[0, 1]$ into n equal parts, denoting the right endpoint of the part adjoining 0 by $\frac{1}{n}$, and marking off equidistant points to the right of $\frac{1}{n}$. The union of 0 and the collection of all the sequences of $n$ths for $n = 1, 2, 3, \ldots$ is what we call the **fractions**.

Now that we know what a fraction is, we may ask if this definition amounts to anything. First of all, a fraction is an abstract concept and there is no point in hiding this fact[12] because introducing students gradually to abstractions is an integral part of school mathematics education. Defining a fraction as a certain point on the number is therefore nothing more than an honest acknowledgement of the abstract nature of the fraction concept. A teacher can mention to elementary students the fact that "$\frac{2}{3}$" is an abstraction the same way "5" is an abstraction.[13] But if fractions are just abstractions, i.e., points on the number line, how do they get involved in describing so many things that seem to have nothing to do with the number line? Furthermore, does the definition shed light on the addition and multiplication of fractions?

Let us answer the first question first. The key is the meaning we assign to the unit 1: it is the meaning of the unit that connects the number line to *every possible real-world situation* involving fractions. Consider, for example, the following problem:

[11]By convention, we also define $\frac{0}{n}$ to be 0 for every nonzero n.

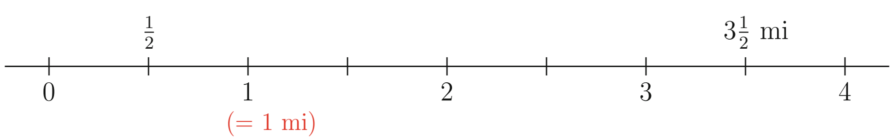
[12]But there is also no need to emphasize it in elementary school either.

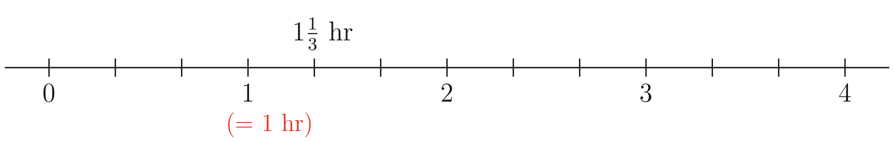
[13]It is all too easy to forget that the symbol "5" *is emphatically* an abstraction.

if $\frac{1}{4}$ *of a bucket of water is added to a* $\frac{2}{3}$ *of a bucket of water, how much water is now in the bucket?* To do this problem, we let the unit 1 on the number line be *the volume* of this bucket of water. The length of the unit segment [0, 1] now has to be interpreted as *the volume* of one bucket of water. So $\frac{1}{4}$ of a bucket of water—which is one part when the bucket of water is divided into 4 equal parts *by volume*—will be represented on the number line by one segment when the length of unit segment (= the volume of one bucket of water) is divided into 4 equal parts (= 4 segments of *equal length*). Therefore $\frac{1}{4}$ of a bucket of water is represented on this particular number line by the point $\frac{1}{4}$ (= the first point to the right of 0 in the sequence of fourths). In a similar way, $\frac{2}{3}$ of a bucket of water is represented by the fraction $\frac{2}{3}$ on this number line (= the second point to the right of 0 in the sequence of thirds). The total volume of water obtained by adding $\frac{1}{4}$ buckets of water to $\frac{2}{3}$ buckets of water is therefore what we normally call "$(\frac{1}{4}+\frac{2}{3})$ buckets of water." We will explain this sum in the discussion of Example 11.3.

Notice that the unit segment [0, 1] is what TSM calls "the whole," and that is a blatant error. The unit 1 has to be *the volume* of one bucket of water, but *not* "one bucket of water." The latter would leave open the question of whether we are dividing the bucket of water into "equal parts" by height, weight, or volume, or in fact, by another kind of measurement. Mathematics has no room for such ambiguity. It is sobering to realize that, in TSM, even the meaning of "the whole" is not correct.

There is another way the number line makes contact with other real-world situations, and we should touch on this briefly. Consider the following problem: *if Helena walks* $3\frac{1}{2}$ *miles in* 1 *hour and* 20 *minutes, what is her average speed in this walk?* Here we have to deal with two number lines: one whose unit is 1 mile, and another whose unit is 1 h. Since 1 h and 20 min is $1\frac{1}{3}$ h, we have the following two number lines:

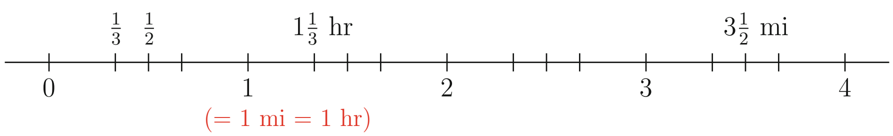




Now the average speed of Helena's walk is, by definition, the division

$$\text{average speed of walk} = \left(\frac{\text{distance traveled}}{\text{time duration}}\right) = \frac{3\frac{1}{2}}{1\frac{1}{3}} \tag{11.3}$$

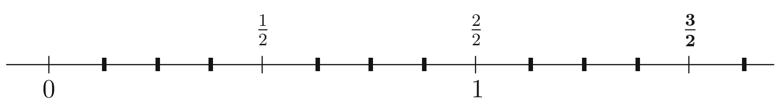
Since division takes place only between two numbers on the same number line (see Wu 2011a, Chapter 18), Eq. (11.3) does not make sense as it stands. We rectify the situation by *identifying* the two number lines,[14] i.e., by identifying the two units, and obtain this picture:



The division in Eq. (11.3) can now take place.

It is time to return to the six examples in Sect. 11.2.1.

Example 11.1 Revisited We can show $\frac{3}{2} = \frac{12}{8}$ as follows. $\frac{3}{2}$ is the third point (to the right of 0) in the sequence of halves.



Now divide each of the segments $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[1, \frac{3}{2}]$, etc., into 4 equal parts. Then together with the sequence of halves, these new division points become the sequence of eighths. The point $\frac{3}{2}$ now becomes the 12th point in the sequence of eighths, and it follows from the definition of fractions that $\frac{3}{2} = \frac{12}{8}$.

The reasoning for showing equivalent fractions in general, $\frac{ca}{cb} = \frac{a}{b}$ for all fractions $\frac{a}{b}$ and nonzero whole numbers c, is entirely similar (see page 29 of Wu 2016a).

Example 11.2 Revisited We will prove CMA, i.e.,

$$\frac{a}{b} = \frac{c}{d} \quad \text{implies} \quad ad = bc \tag{11.4}$$

by making use of equivalent fractions (see Example 11.1), but without making use of the multiplication of fractions. We have $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$ by equivalent fractions. Therefore the hypothesis means $\frac{ad}{bd} = \frac{bc}{bd}$. Thus in the sequence of bd-ths, the ad-th point coincides with the bc-th point. This can happen only if $ad = bc$.

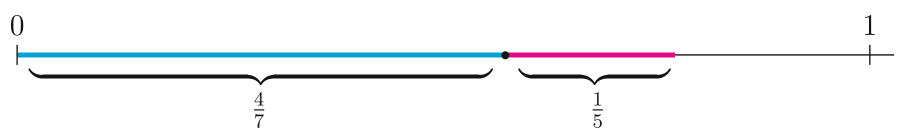
It may be mentioned that in the mathematics education literature, CMA is regarded as an algorithm that is "rote and without meaning" (see page 348 of Billstein-Liebeskind-Lott 2007, for example). This is a piece of misinformation

[14]By *combining* the two number lines into one, if one prefers to think of it this way.

that begs to be corrected. As we already remarked at the end of Example 11.2 in Sect. 11.2.1, CMA *is a basic skill in K–12 that should be in the repertoire of every student and every teacher.* In addition, the fact that CMA in the form of (11.4) continues to hold for *rational numbers* a, b, c, and d is given on page 180 of Wu (2016a). The extension to *real numbers* a, b, c, and d is guaranteed by what is called FASM (Fundamental Assumption of School Mathematics); see Wu (2016a), Section 2.7 (the proof of FASM is given in Section 2.1 of the third volume of Wu 2020).

Example 11.3 Revisited To compute $\frac{3}{8} + \frac{5}{6}$, we begin by *defining* the **addition of fractions**. To this end, we look to whole numbers for guidance because, as points on the number line, whole numbers and fractions are on an equal footing. For whole numbers, addition holds no mystery: $4 + 3$, for example, is the total length obtained by combining segments of lengths 4 and 3, respectively. Precisely, consider the **concatenation** of the two segments of lengths 4 and 3, which is the segment obtained by placing these segments end-to-end on the number line:

Now suppose we are given two fractions $\frac{4}{7}$ and $\frac{1}{5}$ (for example). By definition (Wu 1998; also Section 14.1 of Wu 2011a), the **fraction addition** $\frac{4}{7} + \frac{1}{5}$ is the length of the concatenation of the two segments of lengths $\frac{4}{7}$ and $\frac{1}{5}$:



This definition of fraction addition immediately shows that addition—even for fractions—is still just *putting things together*. (See the discussion of the *coherence* of mathematics in (IV) of Sect. 11.3.1.)

Now that we know what we are asked to do regarding $\frac{3}{8} + \frac{5}{6}$, we can try to compute it, i.e., obtain a formula for $\frac{3}{8} + \frac{5}{6}$. First, observe that the addition of fractions with the same denominator becomes very simple. For example,

$$\frac{4}{7} + \frac{6}{7} = \frac{4+6}{7} \tag{11.5}$$

because $\frac{4}{7}$ is the total length of 4 segments of length $\frac{1}{7}$ and $\frac{6}{7}$ is the total length of 6 segments of length $\frac{1}{7}$, so that by the definition of fraction addition, the left side of Eq. (11.5) is the total length of $(4+6)$ segments of length $\frac{1}{7}$ and is therefore equal to the right side of (11.5). Observe that, conceptually, there is no difference between

$\frac{4}{7}+\frac{6}{7}$ and $4+6$. In general, $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$ for all fractions $\frac{a}{b}$ and $\frac{c}{b}$, for the same reason.

Although $\frac{3}{8}$ and $\frac{5}{6}$ do not have the same denominator, we can "make them have the same denominator" by appealing to equivalent fractions (see **Example 11.1 Revisited** above). Thus, both fractions $\frac{3}{8}$ and $\frac{5}{6}$ belong to the sequence of 48-ths ($48 = 6 \times 8$) because

$$\frac{3}{8} = \frac{18}{48} \text{ and } \frac{5}{6} = \frac{40}{48} \tag{11.6}$$

Therefore

$$\frac{3}{8}+\frac{5}{6} = \frac{18}{48}+\frac{40}{48} \tag{11.7}$$

By the preceding observation (see (11.5)), we have

$$\frac{18}{48}+\frac{40}{48} = \frac{18+40}{48} \tag{11.8}$$

Putting equations (11.7) and (11.8) together, we obtain

$$\frac{3}{8}+\frac{5}{6} = \frac{58}{48} \tag{11.9}$$

More generally, if we retrace our steps and do not multiply out everything, then what this computation shows is actually that

$$\frac{3}{8}+\frac{5}{6} = \frac{(3\times 6)+(5\times 8)}{6\times 8}$$

Now if we introduce symbolic notation, the same reasoning shows in general that for all fractions $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}$$

(Whether or not this symbolic formula should be *proved* in a 5-th grade classroom will depend on the teacher's judgment of the quality of the students. In general, a symbolic proof may be too much of a good thing for the average fifth grader.)

Critical Observations The computation in Eq. (11.9) is the culmination of steps (11.6)–(11.8), and each of which is based *strictly* on the definition of what a fraction is, the definition of what fraction addition means, the *prior* established facts on equivalent fractions and (11.5), and standard logical deduction. There is nothing about some abstruse higher-order "conceptual understanding" that students are supposed to "get" but often don't, and nothing that students have never seen before.

So it is learnable. Furthermore, this reasoning only requires that the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ be changed to two fractions with the same denominator (as in (11.6)), and it doesn't matter what that denominator is. Therefore, any thoughts about the *least* common denominator would be extraneous to this reasoning. Apparently, this approach to the addition of fractions (Wu 1999a) has been implemented in school classrooms with some success (Bingea undated).

Example 11.4 Revisited To compute $\frac{2}{3} \times \frac{5}{8}$, once again, we have to first find out what these symbols mean. So we need a *definition* of the multiplication of fractions: $\frac{2}{3} \times \frac{5}{8}$ means the total length of *2* of the parts if we partition the length of the segment $[0, \frac{5}{8}]$ between 0 and $\frac{5}{8}$ into *3* parts of equal length.

Now, how to partition the segment $[0, \frac{5}{8}]$ into *3* parts of equal length? For this purpose, we call on equivalent fractions to rewrite $\frac{5}{8}$ as

$$\frac{5}{8} = \frac{3 \times 5}{3 \times 8} \tag{11.10}$$

The motivation for doing this is that the numerator of the right side, 3×5 now exhibits an obvious partition into 3 equal parts, namely $3 \times 5 = 5 + 5 + 5$. This then leads to the following simple fact (since we are doing fraction multiplication, of course *the addition of fractions is already an established skill*):

$$\frac{5}{8} = \frac{5+5+5}{3 \times 8} = \frac{5}{24} + \frac{5}{24} + \frac{5}{24} \tag{11.11}$$

According to the definition of fraction addition (see the preceding **Example 11.3 Revisited**), the right side of (11.11)—being a concatenation of 3 segments each of length $\frac{5}{25}$—exhibits a partition of $[0, \frac{5}{8}]$ into three parts of equal length, with each part having length $\frac{5}{24}$. Therefore, using "part" as an abbreviation for "one of the parts when $[0, \frac{5}{8}]$ is partitioned into 3 parts of equal length," we obtain

$$\text{total length of 2 parts} = \frac{5}{24} + \frac{5}{24} = \frac{10}{24} \tag{11.12}$$

In view of the definition of $\frac{2}{3} \times \frac{5}{8}$, (11.12) implies that

$$\frac{2}{3} \times \frac{5}{8} = \frac{10}{24} \tag{11.13}$$

Once again, if we retrace our steps and do not multiply out everything, what this reasoning demonstrates is the fact that

$$\frac{2}{3} \times \frac{5}{8} = \frac{2 \times 5}{3 \times 8}$$

Critical Observations As in the case of adding fractions, the conclusion in Eq. (11.13) is reached via steps (11.10)–(11.12), and each of the latter is based on either a definition (e.g., fraction addition, fraction multiplication) or an established fact (e.g., equivalent fractions, how to add fractions), or both, and the use of logic. This kind of reasoning, i.e., the ability to envision a rough sketch of the intermediate steps (11.10)–(11.12) together with the argument supporting each step, does not come easily to most people, especially beginners. It takes plenty of exposure *and* practice to learn it, and we have to convince students that it is worth learning because *this process of reasoning is the basic methodology of mathematics.*[15] Of course, beginners learn by imitation (as do we all, including professional mathematicians) during their halting first steps towards proficiency, so a classroom teacher can ask students, right after showing this piece of reasoning, to go to the board to explain something like $\frac{2}{3} \times \frac{11}{7} = \frac{22}{21}$ or $\frac{4}{5} \times \frac{5}{8} = \frac{20}{40}$. Then, perhaps, also $\frac{5}{8} \times \frac{2}{3} = \frac{10}{24}$. In due course, the teacher can point out the obvious, namely the fact that if the preceding reasoning is written out in greater detail, then it actually proves that $\frac{2}{3} \times \frac{5}{8} = \frac{2 \times 5}{3 \times 8}$, that $\frac{2}{3} \times \frac{11}{7} = \frac{2 \times 11}{3 \times 7}$, etc., so that in general,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

for all fractions $\frac{a}{b}$ and $\frac{c}{d}$. (The symbolic statement likely will not be appropriate for all classrooms.) There is no end of variations on this pedagogical theme, and each teacher will find his or her own preferred method of delivery.

In the two preceding examples, the method of logical inference used is standard and therefore available to all, and the established facts (such as equivalent fractions) are also available to all. If precise definitions are also routinely given in textbooks, then the whole process of reasoning will become an open book that is available to all. *This is the necessary first step that will make mathematics learnable to one and all.* Therefore having precise definitions for all concepts is a critical ingredient in making reasoning—and hence mathematics itself—learnable.

At this point, it should be clear that we insist on having precise definitions and reasoning in school mathematics education, not because they are mathematicians' professional fixations, but because, as we said earlier, *school mathematics is not learnable without them*. Concepts and skills not connected by reasoning become isolated factoids that can only be learned by brute force memorization. Therefore one may speculate that, as students go up through the grades, such concepts and skills pile up in TSM and, at some point, they overwhelm students' memory banks[16] by sheer volume and TSM ceases to be learnable even by memorization. This speculation about the effects of TSM on student learning is consistent with the performance of US students on TIMSS in 1995 (TIMSS 1995 Results, 1995). However, when *reasoning* is there to connect the concepts and skills, it includes

[15]To a large extent, this is the basic methodology of science as well.

[16]In the terminology of computers, not enough RAM.

them in a story line that makes sense of them; it renders them learnable (also see pp. 118-120 of National Research Council 2001).

Example 11.5 Revisited As noted in Sect. 11.2.1, if mixed numbers are introduced *after* the addition of fractions, then the mixed number $7\frac{2}{3}$ would be defined as the abbreviation for $7 + \frac{2}{3}$, so that

$$7\,\frac{2}{3} \;=\; 7 + \frac{2}{3} \;=\; \frac{7 \times 3}{3} + \frac{2}{3} \;=\; \frac{23}{3}$$

and no memorization would be necessary.

Example 11.6 Revisited The correct definition of a finite decimal is that it is a fraction whose denominator is 10^n for some whole number n. For example, 2.307 is the fraction

$$\frac{2307}{1000}$$

Once we know how to add fractions, then the expanded form of 2307 being $2307 = 2000 + 300 + 7$, we get

$$\frac{2307}{1000} \;=\; \frac{2000 + 300 + 7}{1000} \;=\; \frac{2000}{1000} + \frac{300}{1000} + \frac{7}{1000} \;=\; 2 + 0.3 + 0.007$$

Hence, the fact that 2.307 *is* “2 and 3 tenths and 7 thousandths” becomes a *provable* theorem if students are taught about finite decimals *after* fractions.

11.4 Textbooks

We can now return to the first of our two main concerns: how to give students access to PBM rather than TSM.

A main thrust of this article is about how to repair the damage inflicted on teachers and students by TSM, *Textbook* School Mathematics. An obvious question is why we are wasting our time here talking about damage control instead of directly going to the source and writing better school mathematics textbooks. The simple answer is that most of the school textbooks come from major publishers, and there are no ready-made tools to combat the bottom-line mentality of big business (in this regard, the article Keeghan 2012 is very informative). For this reason, most of the nation’s schools are still dependent on TSM textbooks from the major publishers. It is also the case that the publishing industry is not under any kind of federal or state control and is free to produce any textbooks it can afford to put out. From a publisher’s standpoint, so long as its products are welcomed by enough teachers, there is little incentive to change anything, TSM and all. Since there are many teachers out there who were brought up by TSM and are therefore comfortable teaching TSM as of 2019, there is still a ready-made market for the publishing industry to exploit. It therefore seems likely that, until the majority of teachers reject TSM-infested textbooks, TSM will live on in school classrooms. This then adds urgency to our second topic of concern: how to produce inservice teachers who are capable of teaching PBM. Getting better-informed teachers who reject TSM out of hand would seem to be the best hope of breaking the vicious cycle of TSM.

Since the release of CCSSM in 2010, there have been several attempts to write curricula according to the CCSSM by exploiting the internet using online publishing. A few of the new curricula show promise, according to some reports. However, since so few in the world of education seem to be at all concerned with mathematical content or aware of the continued menace of TSM, many of the textbook evaluation agencies should be approached with a great deal of caution. Overall, much remains to be done in the arena of curricular evaluation.

Common Core was quite aware of the inadequacy of existing textbooks for the implementation of CCSSM. It has published two documents for the benefit of publishers: a 24-page document (Common Core 2012) on the K–8 curriculum and a 20-page document (Common Core 2013) on the high school curriculum. They exhort publishers to meet the goals of focus, coherence, and rigor in their textbooks. Neither document mentions the phenomenon of TSM, however.

A more ambitious undertaking is a six-volume, 2500-page project from this author that gives a complete exposition[17] of the K-12 mathematics curriculum according to PBM (Wu 2011a, 2016a,b, and 2020). There are presumably many ways to present the school mathematics curriculum in accordance with the fundamental principles of mathematics, but for now we can make use of what we have got. These six volumes are not student textbooks; they are textbooks for teachers' PD. Given the level of detail in these 2500 pages, however, it should not be difficult to create student texts out of them with the help of some standard pedagogical embellishments. In any case, an eighth-grade student textbook based on these volumes will be offered online (https://math.berkeley.edu/~wu/) in the near future. Since the drafts of some of these six volumes have served as blueprints for a good many standards in CCSSM, there is no fear that any curricular materials based on these volumes will be out-of-date anytime soon.

11.5 Professional Development

Everything we have said so far points to the urgency of replacing our teachers' knowledge of TSM by PBM. This will certainly tax our ability to do effective PD. Let us be clear about what we expect the PD to accomplish. It will not be about

[17] Strictly speaking, these six volumes do not cover geometry in grades K-5 because the file on this topic promised in Wu (2011a), has not yet been made available. However, Chapters 4 and 5 in Wu (2016a), serve to fill this gap to a large extent.

tweaking teachers' content knowledge here and there but, rather, about a revamping of their knowledge of mathematics from the ground up. Because of our systemic negligence, teachers have never been exposed to anything resembling PBM (see, e.g., Wu 2011b), yet we want them to master PBM in short order and turn around to teach it to their students. This is not going to be easy.

In the first subsection, we will go into some detail to explain the kind of hard work that is involved. We will focus on PD for *inservice* teachers[18] because the current implementation of CCSSM (Common Core 2010) requires teachers who can teach PBM. For example, CCSSM asks teachers to teach mathematics in a way that is "coherent," "stresses conceptual understanding of key ideas," helps students to "reason abstractly and quantitatively," encourages students to "construct viable arguments and critique the reasoning of others" and "attend to precision," etc. (pp. 3-7 in Common Core 2010). The long-term neglect of the mathematical education of teachers leads us to believe that most teachers may not be able to rise to this lofty challenge and that their need for content-based inservice PD will be considerable. Although there is apparently no hard data as yet to substantiate this belief, the available anecdotal evidence (cf. Education Week 2014; Loewus 2016, 2017, and Sawchuk 2016) does point in this direction. In addition, what the author has personally learned from teachers and math coaches in several states—including California—is also consistent with this belief. Our proposed PD therefore cannot be the routine variety and its parameters must be carefully prescribed. This is what we will try to do in the second subsection. In the third subsection, we will describe—for the sake of providing a point of reference—one PD program that has been tried with some success to teach teachers PBM.

There is a jarring note hidden behind this optimistic discussion of PD, however. In the last decade, two studies by IES on the impact of content-based PD on student learning have appeared, Garet (2011) and Garet (2016). They seem to shut the door on any hope that PD can help teachers raise student achievement. If there is any validity to the IES studies, the present article on what "good" PD is and how to implement it would simply be a waste of everybody's time. For this reason, we must make an effort to examine these studies *critically*. This will be carried out in the next section.

11.5.1 The Hard Work of Learning PBM

For inservice mathematics teachers trying to learn PBM, a useful analogy may be learning the second language.[19] The immense difference between PBM and TSM

[18]We will also make some comments about PD for preservice teachers in the last section of this article (Sect. 11.7).

[19]One should not push this analogy too far, however. No language has anything like the logical coherence of mathematics.

dwarfs what little they happen to have in common: the topics and the skills, for instance. Since PBM asks teachers to repackage these topics and sometimes even to teach them in a different order (e.g., define mixed numbers only after fraction addition has been discussed; see **Example 11.5 Revisited** in Sect. 11.3.2), the prospect of learning PBM will be daunting to most. There is also a paradoxical aspect to the attempt by inservice teachers to learn PBM, and it is the fact that while we find fault with TSM for oversimplifying school mathematics to a few sound bites, it is actually easier to just "teach" sound bites! Some teachers who have gotten used to "teaching" the sound bites of TSM *may* find teaching PBM with its many attendant cognitive complexities to be a very big stretch. What is good for the learners may not always be easy for the teachers! Let this be a warning. What follows is a more detailed explanation of the hard work involved in learning PBM.

Learning About Reasoning

Learning how to reason is painstaking work under the best of circumstances.[20] Except for the most rudimentary, one-step variety that we inherit from our ancestors on the African savanna tens of thousands years ago, such as "fright → flight," reasoning is not an inborn skill like speech or running. For most teachers who have been immersed in TSM all their lives, learning how to reason about basic tasks that they used to teach by rote with ease is difficult enough. Having to also learn how to *explain* the reasoning process to students makes it doubly difficult, and trying to empower students with the fundamentals reasoning skills is trebly difficult.

Take the case of adding fractions (see Example 11.3 in Sect. 11.2.1). We can complain all we want about the use of LCD and the absence of any explanation of what "addition" means in TSM, but to most inservice teachers, this rote skill has probably become second nature. This LCD skill is simple to teach by rote! The procedure is short, and all a teacher has to do is give students lots of drills. Now PBM changes all that: a teacher has to explain what it *means* to add two fractions, use equivalent fractions to put the two given fractions into the same sequence of $n$ths for some n, and remind students of the *meaning* of adding whole numbers (see **Example 11.3 Revisited** in Sect. 11.3.2). We know how some students hate to be reminded of anything other than what is right in front of them! It is definitely a lot more work than teaching the rote skill using LCD.

The case of multiplying fractions (see Example 11.4 in Sect. 11.2.1) is somewhat similar. Even some mathematicians mistakenly consider fraction multiplication to be a pleasure to teach because it is procedurally so simple (see, e.g., Aharoni 2015): just multiply across the top and the bottom. By contrast, look at **Example 11.4**

[20] It has been suggested that there is an apparent contradiction between this statement and the earlier one made in **Example 11.4 Revisited**, to the effect that PBM will make reasoning learnable to one and all. But there isn't. Such a misunderstanding would arise only if one erroneously equates "learning" with "learning without effort." Learning anything worthwhile in life requires effort, e.g., learning how to read require the strenuous effort of memorizing the alphabet and a continuous influx of new vocabulary. What is at issue is whether unnecessary roadblocks are thrown in the learner's path. TSM throws such roadblocks—too many to count—but PBM doesn't.

Revisited in Sect. 11.3.2: the definition of fraction multiplication is among the longest and most complex in elementary mathematics, and the reasoning in the teaching of this so-called simple skill becomes quite delicate according to PBM (see Eqs. (11.10)–(11.12) therein). Real effort is now required for its mastery.

We should mention another telling example about reasoning: the teaching of speed problems and the related so-called *rate* problems. Consider the following:

> Luis usually walks the 1.5 miles to his school in 25 minutes. However, due to road repair, he has to take a 1.7-mile route today. If he walks at his usual speed, how much time will it take him to get to his school? (Siegler et al. 2010, page 38.)

In TSM, the phrase "at his usual speed" (or "at this speed") is code for *setting up a proportion*. In other words, given that Luis walks 1.5 miles in 25 min, if Luis walks 1.7 miles in x minutes "at his usual speed," TSM instructs us to invoke what is known as *proportional reasoning* to set up a proportion:

$$\frac{1.5}{25} = \frac{1.7}{x} \tag{11.14}$$

Now use CMA[21] to get $1.5x = 25 \times 1.7$. So $x = 28\frac{1}{3}$ min.

In the present context of getting teachers to learn about reasoning, something almost leaps off the page: the simple solution involves a rote skill but *no* reasoning! But how can one arrive at (11.14) by the use of reasoning?

The fact is that, as is, the problem cannot be solved because since we have no idea how Luis normally walks to school,[22] we know nothing about how he walks "at his usual speed." Consequently, we have no information about how he walks to school during the road repair either. *We have not been given sufficient information to know how to proceed.*

If we want to base mathematics on reasoning, then we will have to add some precise assumptions about the way he walks. Here is a standard one: let Luis walk a total distance of $f(t)$ miles after t minutes, then we assume that $f(t)$ is a *linear function of t without constant term*, i.e., $f(t) = vt$ for a fixed constant v. This assumption would justify equation (11.14) because all it says is that

$$\frac{f(25)}{25} = \frac{f(x)}{x}$$

Indeed, both are equal to v in this case.

Now, because such problems are usually introduced into the curriculum before students learn about linear functions, we will describe another way—suitable for

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[21] As noted at the end of **Example 11.2 Revisited** in Sect. 11.3.2, what is being used here is actually not CMA but the extension of CMA to rational numbers.

[22] Does he run the first mile in 10 min and slowly stroll to school in the remaining 15 min?

use in the 6th or 7th grade—to deal with Eq. (11.14). For an object in motion, we introduce the concept of its **average speed over the time interval from t_1 to t_2**, ($t_1 < t_2$), as

$$\frac{\text{total distance traveled from time } t_1 \text{ to } t_2}{t_2 - t_1} \tag{11.15}$$

In terms of *average speed*, the Luis problem can be properly reformulated as follows:

> Luis usually walks the 1.5 miles to his school in 25 minutes. However, due to road repair, he has to take a 1.7-mile route today. If his two trips *have the same average speed,* how much time will it take him to get to his school?

Now equation (11.14) is correct because it is based on the assumption that the two average speeds are the same. We are using the *definition* of "average speed"!

The more common way—and a more nuanced way—of handling the Luis problem is to formulate it in terms of *constant speed*. By definition, a motion **has constant speed v** if its average speed over *any* time interval is always equal to v. Then a correct formulation of the preceding problem in terms of constant speed is the following:[23]

> Luis usually walks the 1.5 miles to his school in 25 minutes. However, due to road repair, he has to take a 1.7-mile route today. If he always walks at the same constant speed, how much time will it take him to get to his school?

Equation (11.14) is now justified by recognizing the fact that its left side is Luis' average speed over the time interval it takes him to walk the normal 1.5 miles, and the right side is the average speed over the time interval it takes him to walk the 1.7 miles. The assumption that he walks at the *same* constant speed then implies that these two average speeds are equal, which is Eq. (11.14).

In all three cases, we get to witness one of the basic characteristics of reasoning: *make explicit use of precise definitions* to draw conclusions. The goal of PBM is to get students used to the habit of analyzing each problem on its own merits by the use of explicit assumptions, explicit definitions, and reasoning.

One may object that the amount of reasoning used to solve the preceding problem formulated in terms of constant speed is too little to be cause for celebration. Granted, but look at the alternative of appealing to "proportional reasoning": the latter is rote learning, plain and simple.

Let us not forget that our teachers were brought up in TSM and are used to setting up proportions. If they want to teach PBM, then they must know this background information about speed problems to be able to answer students' questions, e.g., "why don't we just set up a proportion?" Above all else, teachers who want to

[23] It can be shown that the assumption of constant speed is equivalent to the earlier assumption that the distance function describing Luis' distance from his starting point is a linear function without constant term. See Theorem 7.1 on page 138 of Wu (2016b).

promote PBM have to know why "proportional reasoning" is not *reasoning* at all. For example, let us do the following problem by proportional reasoning:

> A free-falling stone is dropped from 600 ft. It drops 64 ft in 2 seconds. How far does it drop in 3 seconds?

Obviously, proportional reasoning yields an answer of 96 ft, whereas physics gives the correct answer of 144 ft. The reason is that *the falling stone does not move at constant speed*. So unless teachers insist on having precise definitions for all the relevant concepts—so that *constant speed* gets defined—and unless teachers insist on solving problems by reasoning, they cannot even explain to students *why* the free-falling stone problem cannot be done by setting up a proportion.

Altogether, we see that we are imposing a heavy cognitive load on teachers in trying to get them to embrace PBM. Nevertheless, we must do all we can to help teachers acquire the reasoning skill because we have no choice. Students must learn to reason for their survival in year 2019, and if teachers cannot learn to reason mathematically, how can we hope that their students will? So we must try harder.

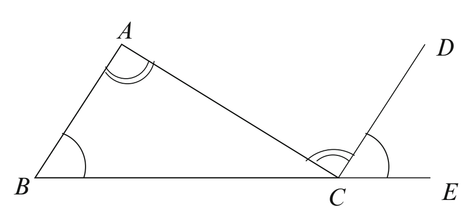
It remains to make some general remarks on the need for flexibility in teaching reasoning in the school classroom. When we say reasoning should attend every statement in mathematics, we actually mean *grade-appropriate* reasoning. For example, in **Example 11.3 Revisited** of Sect. 11.3.2, we mentioned that although the reasoning suffices to prove the general formula for addition,

$$\frac{a}{b} + \frac{c}{d} \;=\; \frac{ad + bc}{bd}$$

it may not be appropriate to give this general reasoning in a typical fifth grade classroom. A more reasonable alternative is to prove the formula only for many specific values of a, b, c, and d. We also made a similar remark about the product formula for arbitrary fractions in **Example 11.4 Revisited** of the same subsection:

$$\frac{a}{b} \times \frac{c}{d} \;=\; \frac{a \times c}{b \times d}$$

To firm up this message, we will use an example from middle and high school geometry: the teaching of the theorem that the angle sum of a triangle is 180°. There is the standard, intuitive proof obtained by drawing a line parallel to side AB and passing through the vertex C of the following triangle:



Then there is a *correct* proof that fills in all the nonintuitive gaps of this standard intuitive proof. The latter is unfortunately very subtle as well as very boring. For *middle school* students, the correct proof is not worth the investment of time and effort. Thus, on page 316 of Wu (2016a) (which was written for mathematics educators and middle school teachers), the standard, intuitive proof is given, followed by the remark on page 317, (*loc. cit.*) to the effect that the steps in the given proof are

> essentially correct. Nevertheless, from a strictly *mathematical* standpoint, one can find fault with them for certain omissions in the details.

Then it goes on to say that insofar as

> our main purpose here is to acquire geometric intuition and make the first step towards the mastery of geometric proofs,

the intuitive proof will serve. Needless to say, a correct proof should be given when the occasion calls for it in the high school course on geometry (see Section 6.5 of the second volume, *Algebra and Geometry*, of Wu 2020).

Teaching is, among many things, the result of negotiations between what is correct and what is possible in the face of the reality in a classroom The purpose of our effort to get teachers ready to teach PBM is to provide them with the needed mathematical information so that they have the freedom to decide what is possible in a given classroom.

Learning About Definitions

It should be clear from the discussion up to this point that definitions and reasoning are essentially intertwined, but there are a few things about definitions that deserve to be discussed separately.

For teachers brought up in TSM, perhaps the most difficult thing to accept about PBM is that the definition of a concept ceases being something to be memorized for standardized tests and then cast aside, but is now the foundation for any reasoning about the concept. TSM has no reasoning to speak of, so definitions play no role in its version of "mathematics learning." Teachers with a TSM background therefore have difficulty getting used to the fact that PBM puts every definition to use for the purpose of reasoning. We have seen how the definition of a fraction and the definition of the addition of fractions are used, *literally,* to derive the formula for the addition of fractions (see **Example 11.3 Revisited** in Sect. 11.3.2), how the analogous derivation happens with fraction multiplication (see **Example 11.4 Revisited** in Sect. 11.3.2), and how Luis' walking problem in the early part of this subsection can be solved simply by the use of reasoning once a precise definition of *constant speed* is given. This point is stressed throughout all six volumes of Wu (2011a, 2016a,b), and 2020, but if personal experience is any guide, it still does not come easily to teachers.

One may speculate that things would go more smoothly in the ongoing effort to convince teachers about the importance of precise definitions if the education literature would also make such an advocacy. This is not happening, however, because TSM has held sway over many educators as well.[24] Consequently, some educators downplay the importance of precise definitions. In discussing ratio and rate, for example, Susan Lamon makes the following statement:

> Even if we could precisely define ratios and rates and the difference between them, definitions do not discharge the full meaning of the idea being defined. The nature and meaning of rates and ratios come from problem situations. (Lamon 1999, page 165.)

There is an obvious misunderstanding here about the role of definitions in mathematics: the mathematical definition of a concept is not required to "discharge the full meaning" of the concept so defined. All that it is obligated to do is furnish *all* the information that is needed for any reasoning regarding that concept, no more and no less. In this connection, what Polya has to say is very much to the point: "The mathematician is not concerned with the current meaning of his technical term.... The mathematical definition creates the mathematical meaning" (Polya 1957, page 86). As we mentioned in Sect. 11.3.1, the main virtue of presenting a precise definition of a concept is to tell the whole *mathematical* truth from the beginning to facilitate mathematics learning.[25] This no-hidden-agenda feature of mathematics is essential to making mathematics learnable because, above all else, it establishes a sense of trust between the learner and the mathematics. It tells the learner that all the cards are now on the table, so just look closely at what you have got! The failure of TSM to develop mathematics according to precise definitions has so far resulted in gnawing suspicions and distrust from learners at the outset. This is no way to make mathematics learnable.

Having argued for the need to make teachers see the importance of precise definitions in PBM, we also want to supplement the argument with the remark that by no means are we advocating for the unmotivated (highhanded) presentation of definitions that we sometimes see in advanced mathematics. Any PD must also pay attention to the art of persuasion in giving definitions. This is why the definition of a fraction in terms of the number line in Wu (2016a), is preceded by eight pages explaining why a precise definition of a fraction is necessary (pp. 3-10, loc. cit.). Likewise, the definition of slope on page 66 of Wu (2016b) is preceded by almost five pages of discussion about the intuitive meaning of the concept of slope and how this intuition may be captured in a precise definition. There are many other such suggestions about how to present a precise definition in those two volumes as well as in Wu (2011a). Teachers should be aware of the many pedagogical flexibilities in making precise definitions an integral part of their teaching.

Finally, let us consider one more objection to the recommendation that there be precise definitions for every concept in school mathematics, to the effect that

[24]When all is said and done, mathematicians have to be held mainly accountable for the deterioration of the content component in mathematics education (see Wu 2011b).

[25]This view of the role of definitions in mathematics in general is a product of the twentieth century, as a result of many trials and errors to make mathematics more transparent and less prone to obscurantism. The attitude towards definitions of the mathematicians in centuries past was actually remarkably close to that of Lamon's (cf. e.g., Quinn 2012).

students should not be *told* the definition of a concept because a definition is something that one formulates at the end of an exploration. So the theory is that we should let students explore a new concept until they themselves come up with something resembling a correct definition. This viewpoint about how to teach mathematics in general, and teach definitions in particular, is part of an old pedagogical debate about the Moore Method vs. direct instruction. It will not be productive to wade into this debate here except to point out that, however, the exploration approach is done, it is very time-consuming and one must take the availability of instructional time into account (see Section 3 of Wu 1999c). Moreover, after a precise definition has been formulated at the end of the exploration, one must bring mathematical closure by *retracing the steps of the exploration to let students see how the relevant mathematics can be developed on the basis of the precise definition*. This will further decrease the available instructional time. Thus far, discussions in the education literature seem to be oblivious to the need for retracing the steps to show students how mathematics is developed using definitions and the need for additional instructional time to make room for the retracing. This is but one manifestation of the tendency in the education literature to make an advocacy without also meticulously enumerating the possible detrimental side effects (see Section 4 of Wu 1999c for a fairly comprehensive discussion).

Other Issues

For teachers transitioning from TSM to PBM, a minor—though a significant—issue to be confronted is that they will have to learn how to teach certain topics in a different logical order. We have already remarked in Sect. 11.3.2 that the concept of mixed numbers can no longer be taught right after fractions are introduced (as in TSM) but must wait until after the discussion of the addition of fractions (**Example 11.5 Revisited**). In the same subsection, we also remarked that finite decimals can no longer be taught in a separate track independent of fractions but must be taught as a special kind of fractions (**Example 11.6 Revisited**); this profoundly changes the teaching of finite decimals because we can now *explain* the algorithms for decimal addition, subtraction, and multiplication (for division, see pp. 81-86 in Wu 2016a). Perhaps the most prominent change of this kind is the teaching of the slope of a line in middle school. In PBM, the definition of the slope depends on having available the angle-angle criterion for similar triangles so that slope has to be taken up *after* a serious discussion of the concept of similar triangles (Wu 2016b, Section 4.3). Now, although similar figures in TSM are those "with the same shape but not necessarily the same size," PBM will insist on a precise definition of similarity. This, in turns, requires that the school curriculum pave the way for such a precise definition. Teachers must therefore be prepared for these massive changes in their internal conception of school mathematics. Since such a change in the teaching of slope is also part of CCSSM, I received an email from an indignant teacher in the state of Washington right after the release of CCSSM in June of 2010. He wrote:

> After 13 years of teaching high school algebra, I wonder why you see similarity as critically important to Algebra I mastery—that certainly never occurred to me as a teacher of algebra.

> . . . What makes you say that a student needs to understand similar triangles in order to write the equation of a straight line between two points?

Clearly, teachers have to be willing to keep an open mind to learn PBM.

Among the fundamental principles of mathematics, the longitudinal coherence of school mathematics may be one of the hardest things for teachers to appreciate. The only way they can learn it is to be exposed to several grades' worth of PBM over a long stretch of time. For elementary school mathematics, we can be more specific. Take, for instance, the coherence of the four standard algorithms for whole numbers. We already pointed out that all four revolve around a single idea: if we can compute with single-digit numbers, then we can compute with any numbers no matter how large (see Sect. 11.3.1). It would be impossible to see such coherence unless one knows how to prove this fact for each of $+$, $-$, $\times$, and $\div$, and this may not be so easy especially for the long division algorithm. (Is that algorithm even a theorem? And if it is, what does it say in the first place? see Sections 7.3–7.5 of Wu 2011a.) Thus a PD session devoted to making elementary teachers see such coherence has to first give the *detailed* proofs of four separate theorems, and then has to give them time to digest them so that they can step back and take note of the similarity in these proofs. It should take no less than three full days. Teaching elementary teachers about fractions in a way that enables them to see the coherence between the four arithmetic operations on whole numbers and fractions should take at least another five full days. And so on.

It takes time to learn PBM.

11.5.2 The Inservice PD We Need

Our tentative conclusion is that, for real improvement in school mathematics education to materialize, we will need a massive investment in long-term, content-based, inservice PD to get our teachers ready to teach PBM. But can we put our trust in PD to get this done? The answer is unfortunately not straightforward.

First of all, we have to provide more details about this proposed long-term, content-based, inservice PD. By *long-term*, we have in mind a long stretch of time of 1 week to 3 weeks during the summer. One may believe that, for example, instead of 1 week in the summer, we can parcel out the 40 h of PD into 20 two-hour sessions during the school year, with one session per week. The problem with breaking up 1 week in the summer into 20 sessions in the school year is that teachers have too many obligations during the school year to remember what they learn from week to week. If we have any design on impressing the coherence of PBM on teachers, these 2 h sessions will not be the answer. Moreover, learning mathematics requires serious mental concentration. Given how teachers already have to multi-task all through the school year, summer may be the only time they can summon this kind of concentration necessary for learning. Creating a learning environment for teachers by holding the PD in consecutive days in the summer is thus the only viable option. Holding the PD in consecutive days also tends to yield the pleasant dividend of promoting collaboration among teachers. Beyond the summer session, we should also help them retain—or remind them to apply—the new knowledge in the following school year. To this end, some Saturday review sessions throughout the school year would also be advisable.

About the *content* of the content-based PD, we hope no argument is necessary at this point that this content refers to the content of PBM and not TSM. We note in passing that, because most professional developers were brought up in TSM, getting and vetting competent providers for the needed PD will be a nontrivial problem.

"Long-term" and "content-based" are certainly not qualities one normally attributes to most of the PD currently provided by school districts. Thirty years ago, Judith Warren Little wrote about the PD system in California and showed that, instead of providing learning opportunities for teachers, PD had too often devolved into a series of uncoordinated rituals (Little 1989). To those working in the trenches with the firsthand knowledge of mathematics PD in the past decades, not much has changed since the appearance of Little's article (see, e.g., Wu 1999b and U. S. Department of Education 2009, p. 95). This is by way of saying that if we are committed to using PD to help teachers learn PBM, we must be ready to fight for *long-term* and *content-based* as nonnegotiable requirements. By a happy coincidence, the 2017 publication of *Effective Professional Development* (Darling-Hammond et al. 2017) also comes to a similar conclusion. The work of Darling-Hammond et al. set out to discover the common features of effective PD in all fields and is not exclusively about mathematics, so it has little overlap with the present article. Nevertheless, it too concludes that effective PD must be "content focused" and "of sustained duration."[26]

In addition to these intellectual concerns, there is also a practical matter that is no less important. Learning mathematics is almost never a fun activity in the everyday sense of "fun"; it is *hard work*, and the PD cannot succeed without teachers' willingness to work hard. The only way to ensure that the teachers will put in the hard work is to pay them generously for their daily attendance. This therefore means that the desired PD will not only be of sustained duration and PBM-based, but also *expensive*.[27]

We should also address the role of pedagogy in PD. Let it be noted that our concept of "content" already takes pedagogy into account because PBM addresses not mathematics but *school mathematics*, i.e., how mathematics should be taught in schools. In addition, while specific pedagogical issues will inevitably arise in any PD, we must be careful to keep the amount of purely pedagogical discussions in the PD in check. I took note of this fact for the first time when I had the opportunity to observe other people's PD in California back in the late 1990s (Wu 1999b). What I found was that when content knowledge was only one of many topics of concern in

[26]We also agree with Darling-Hammond et al. on the need to "support collaboration," as we shall see in the next subsection.

[27]Though a generous stipend by itself will not guarantee hard work or success.

PD, it would not get the attention it deserved, *nor would it inspire the needed effort on the part of teachers to make foundational changes in their content knowledge.*

It remains to point out that while we have been talking about PBM as if it is a common entity that is easily accessible, the fact is that it is not. The PD providers for the PD under discussion will have to create their own materials because the PD literature is almost completely immersed in TSM. If people want to take a look at PBM, however, they can always look up the six volumes of Wu (2011a, 2016a,b), and (2020).

Having pinned down the general parameters of the PD we need, we are now ready to go to work—except that, as we mentioned at the beginning of Sect. 11.5, there is one more hurdle to overcome: recent research by the Institute of Education Sciences (**IES**) asserts that even "teachers who received the best of the best PD" are unlikely to "see large, lasting improvements in their practice, knowledge, or student learning" (Hasiotis 2015). More precisely, the two studies by Garet et al., one in 2011 (conducted over a 2-year period in 2007–2009) and another in 2016 (conducted in 2013–2014), raised serious doubts about the ability of content-intensive PD for inservice *mathematics* teachers to raise student achievement. Fortunately, a closer examination of these two studies—which will be given in the next section, Sect. 11.6—reveals serious flaws in the design of their PD that might have led them astray. Thus the jury on the alleged *in*effectiveness of PD is still out, and we all have every reason to proceed with the PD that has just been carefully outlined above. The next subsection gives a slight nudge in this direction.

11.5.3 An Inservice PD Program

The kind of PD proposed in the preceding subsection did not come out of the blue. It is based on the author's personal experience and it serves the modest goal of providing *one* data point that attests to the possible validity of such an approach to PD.

Each summer, from 2000 to 2013, I gave 3-week PD institutes for mathematics teachers of K–8, mostly in Berkeley, CA, and sometimes more than once a year. The goal of these institutes was for inservice teachers not only to learn PBM, but to also achieve long-term retention of the new knowledge. The institutes met 5 days a week (M–F), about 8 h a day (including lunch), with homework assignments every day. I would lecture to the whole group for 4–5 h each day, and the day would always end in small group meetings lasting 60–90 min, led by my three assistants. Each PD institute was followed by five Saturday follow-up sessions (one every 2 months) in the following school year to review the new content knowledge and to discuss the progress teachers were making putting it into practice in their classrooms.

Each year we put the word out about these institutes and asked teachers to apply to participate. Every participating teacher who attended all 3 weeks of the institute received a stipend of \$1500, i.e., \$100 a day.[28] For every follow-up Saturday session, each participant received a stipend of \$100. We actively encouraged group applications by teachers from schools in the same district because we believed that being able to consult with colleagues about mathematics would ensure better learning as well as better retention of PBM. It is gratifying to report that we did witness many cases of collaboration during and after the summer institutes that have continued to this day.

There were three kinds of institutes:

(1) **Elementary Institute:** whole numbers (4 days), elementary number theory (2 days), fractions—including decimals—and their arithmetic (6 days), percent, ratio, and rate (3 days). (Reference: Wu 2011a.)
(2) **Pre-Algebra Institute:** Review of fractions, percent, ratio, and rate (4 days), rational numbers (3 days), experimental geometry (3 days), geometric vocabulary, congruence, and similarity (4 days), length and area (1 day). (Reference: Wu 2016a.)
(3) **Algebra Institute:** use of symbols (2 days), linear equations in one and two variables, including a correct definition of the slope of a line (3 days), simultaneous linear equations (1 day), laws of exponents, exponential functions and their graphs (4 days), quadratic functions and their graphs (3 days). (Reference: Wu 2016b.)

Although these institutes were unapologetically devoted to the dissemination of PBM, pedagogy also played a role. In the first few years, I arranged for a short session of about an hour on pedagogical discussions at the end of each day, but it turned out that almost all the teachers were so absorbed in learning the mathematics that essentially *nobody wanted to broach the subject of pedagogy.* So I stopped making time for the discussion of pedagogy in the institutes thereafter. Instead, I made sure that pedagogy and content were given equal emphasis in the Saturday follow-up sessions. Teachers were asked to share personal stories about their own attempts to integrate the newly acquired knowledge into their classrooms, and their pedagogical strategies were then openly discussed and critiqued. It seemed quite clear that the teachers began to understand the mathematics on a deeper level when they tried to build their pedagogy on a foundation of correct school mathematics. Year after year, teachers would tell me that it was usually not until the fifth (and last) Saturday follow-up session, 9 months after their first exposure to the new content, that they *began* to feel that they owned the material. Apparently, it took the combination of the *intensive three-week immersion in content* (120 h) and the leisurely 9 months of gestation and fledgling attempts at classroom applications for them to begin making *foundational changes* in their content knowledge.

Naturally, I was interested in whether these institutes had any effect on the participating teachers' performance. Together with a school district's Director of Instruction, we applied twice to federal agencies, in 2012 and 2013, for a grant to teach fractions to elementary teachers and then (1) videotape the PD sessions and make them available online, and (2) follow each participating teacher's student scores for 3 years to examine their value-added measurements. However, our

[28] In the last six or so institutes, I tried to raise the daily stipend to \$125, but there was insufficient funding to do it.

applications were rejected. This explains why I have no hard data to report. Nevertheless, a few individual teachers have privately contacted me to share their personal successes, and these are recorded in Section 2.4 of Wu (2018). The anonymous evaluations by the teachers of the PD institutes from 2009 to 2013 are also available on request: please write wu@berkeley.edu.

11.6 The IES Impact Studies

In this section, we will closely examine the study by (Garet et al. 2011) on the impact of content-intensive PD on inservice *mathematics* teachers' ability to raise student achievement. Some attention will also be given to the impact study of Garet et al. (2016).

First of all, these two studies differ from earlier ones in their built-in methodological credibility: they used a rigorous experimental design and, in addition, had large meaningful sample sizes: the 2011 study involved some 150 teachers and the 2016 study 221 teachers. The bleak conclusion of the 2011 study was that

> after two years of implementation, the PD program did not have a statistically significant impact on teacher knowledge or on student achievement in rational numbers (page 53 of Garet et al. 2011).

The outcome of the 2016 study was that "the PD did not have a positive impact on student achievement" but "the PD had a positive impact on teacher knowledge" (page 40 and page 35, respectively, of Garet et al. 2016).

Together with an earlier impact study on *early reading instruction* (Garet et al. 2008), the 2011 impact study of Garet et al. led to the 68-page report published by TNTP, *The Mirage* (TNTP 2015), on the ineffectiveness of inservice PD in general, not just in mathematics or reading. The main conclusion of *The Mirage* was:

> In short, we bombard teachers with help, but most of it is not helpful—to teachers as professionals or to schools seeking better instruction. We are not the first to say this: In the last decade, two federally funded experimental studies of sustained, content-focused and job-embedded professional development have found that these interventions did not result in long-lasting, significant changes in teacher practice or student outcomes. (TNTP 2015, page 2)

The two "federally funded experimental studies" referred to above are the 2008 and 2011 studies of Garet et al. These studies, together with *The Mirage* and the 2016 impact study of Garet et al., were written up in the popular press (Layton 2015, and Loewus 2016), and the perception began to take hold that inservice PD as a means of achieving education improvement is a blind alley (see, e.g., Dynarski 2018). Were the conclusions of these impact studies valid, we would be wasting our time here talking about the use of PD to help teachers learn PBM.

For all these reasons, a critical examination of (at least) the 2011 impact studies on mathematics is overdue. Although the 2016 study is about fourth grade teachers (rather than seventh grade teachers as in the 2011 study), the flaws of the two studies are essentially the same from a broader perspective—both ignore the crippling effect of TSM on school mathematics in their PD design. For our purpose here, we will simply concentrate on the 2011 study.

11.6.1 The PD Program of the 2011 Impact Study

To better understand the PD program of Garet et al. (2011), we begin with a brief description of its design. It was a 2-year program for 7th grade mathematics teachers on the following topics: "fractions, decimals, percent, ratio, rate, and proportion." In the first year, teachers were given:

a 3-day summer institute on content instruction (18 hours)
5 one-day follow-up seminars during the school year (30 hours)
10 days of coaching (20 hours)

Of the *8* days of content instruction and seminars, 4 were devoted to *fractions and decimals* and the other 4 to ratio, rate, proportion, and percent. In the second year, teachers were given:

a 2-day summer institute on content instruction (12 hours)
3 one-day follow-up seminars during the school year (18 hours)
8 days of coaching (16 hours)

Of the *5* days of content instruction and seminars, 4 were devoted to ratio, rate, proportion, and percent, and 1 day to fractions and decimals. (By the second year, the PD organizers realized that teachers were having *real* trouble with ratio, rate, proportion, and percent, and they adjusted accordingly.)

The report states that the PD provided by the impact study made use of the number line for the discussion of fractions and emphasized precise definitions, but the report also states that *it was not designed to improve teachers' content knowledge* (page 21 of Garet et al. 2011). Rather, the focus of the PD was on pedagogical enhancements such as developing their ability to

> identify and address persistent student misconceptions. . . The pedagogical techniques that received the most attention were eliciting and responding to student thinking, using charts to keep track of particular student misconceptions. . . (page 21 of Garet et al. 2011)

Since students' most serious misconceptions regarding ratio, rate, proportion, and percent stem from TSM itself—which we will demonstrate in the next subsection—it is not clear how this PD could "address" these misconceptions without identifying and uprooting TSM and replacing it with PBM. We are not aware of any pedagogical strategy that can transform TSM into PBM.

In greater detail, this impact study refers to "the knowledge of topics in rational numbers that students should ideally have after completing the seventh grade" as **CK** (*common knowledge*), and to "the additional knowledge of rational numbers that may be useful for teaching rational number topics" as **SK** (*specialized knowledge for teaching*). Keep in mind that the "CK" as stated consists of nothing but TSM.

Of the total number of 13 (= 8 + 5) days devoted to content instruction and seminar in the PD program,

> the focus of the presentation in both years was on SK, and instruction in common knowledge of mathematics content CK was mainly implicit. . . . the PD was not presented to teachers as an opportunity to improve their understanding of rational number content. (p. 21 of Garet et al. 2011)

Overall, we may summarize the PD program of the impact study as maintaining teachers' knowledge of TSM at the level of students' grade 7 textbooks and empowering them with better pedagogical techniques.

We note that such a PD program, favoring SK over CK, is not compatible with a recommendation from the National Mathematics Advisory Panel, to the effect that "teachers be given ample opportunities to learn mathematics for teaching. That is, teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching and the connections of that content to other important mathematics, both prior to and beyond the level they are assigned to teach" (Recommendation 19 on page xxi of National Mathematics Advisory Panel 2008). In the next subsection, Sect. 11.6.2, we will explain why *without a far better content knowledge than the grade-level TSM they possess at present, teachers cannot hope to become more effective in teaching ratio, rate, proportion, and percent.* Granting this, the conclusion of the impact study would have been more accurately described as follows:

> The study results are consistent with the expectation, as of 2019, that *a mathematics PD program that does not replace teachers' content knowledge of TSM with PBM will not be likely to have a statistically significant impact on student achievement.*

11.6.2 Ratio, Percent, and Rate in TSM

Recall that the stated goal of the PD in the impact study was to raise *teachers'* content knowledge to the level of what "students should ideally have after completing the seventh grade," and its main focus was on improving the *teaching* of ratio, percent, rate, and proportion. Simply put, what the IES impact study aspired to do was *equip teachers with the best knowledge base that TSM had to offer students in the 7th grade curriculum,* with the hope that these teachers would then be able to boost student achievement by better pedagogy alone.

This strategy was bound to fail because percent, ratio, and rate are among the most feared topics by middle school teachers and students. *Why fear?* Because:

(1) Understanding ratio, percent, and rate requires a fluent knowledge of fractions in the first place. Since the knowledge of fractions for most students is precarious, they are handicapped before they begin.
(2) These three topics come *after* the division of fractions, and we have to remember: *"ours is not to reason why, just invert and multiply."*
(3) The presentations of *percent*, *ratio*, and *rate* in TSM are seriously flawed.

Let us elaborate on (3).

Ratio and Percent

We first consider *ratio* and *percent*. How are they defined for students in TSM? Percent and ratio are words in our daily conversation, and we all have a vague idea what "the ratio of Democrats to Republicans in this gathering is about 2 to 3" means: if there are about 200 Democrats in the gathering, there will be about 300 Republicans there. But what if there are about 2365 Republicans, can students still figure out roughly how many Democrats there are? Mathematics is a discipline of precision, so students need a precise meaning of "ratio." In TSM, ratio is "defined" in a variety of ways, but the following are typical:[29]

- A ratio is a comparison of two numbers, a and b, written as a fraction $\frac{a}{b}$. You can write a ratio in three ways.

  $$1 \text{ to } 45 \quad \text{or} \quad 1:45 \quad \text{or} \quad \frac{1}{45}$$

  You can write a ratio to compare two amounts—a part to a part, a part to the whole, or the whole to a part.
- A ratio is a comparison of two numbers. It may be written in three different ways. The ratio of the number of people who picked "hazardous waste material" (18) to the number of people who picked "greenhouse effect" (9) [in the preceding survey] can be written as:

  $$18 \text{ to } 9, \qquad 18:9, \qquad \frac{18}{9}$$

  ...If you think of a ratio as a fraction, then $\frac{18}{9} = \frac{2}{1}$. They are **equal ratios.**
- Ratios are encountered in everyday life. For example, there may be a 2-to-3 ratio of Democrats to Republicans on a certain legislative committee, a friend may be given a speeding ticket for driving 69 miles per hour, or eggs may cost 98 cents a dozen. Each of these illustrates a **ratio**. Ratios are written $\frac{a}{b}$ or $a : b$ and are usually used to compare quantities.

  A ratio of 1 : 3 for boys to girls in a class means that the number of boys is $\frac{1}{3}$ that of girls, that is, there is one boy for every three girls. Notice we could also say that the ratio of girls to boys is 3 : 1, or that there are three times as many girls as boys. The ratio of 1 : 3 for boys to girls in a class does not tell us how many boys and how many girls there are in the class. It only tells us the relative size of the groups.

Does any one of these abstruse "definitions" tell students clearly *what a ratio is?* In the first bullet, for instance, it suggests that a ratio is a comparison of two numbers that is written as a fraction. Since a fraction is a number, is ratio therefore also a

[29]These bulleted statements are taken directly from textbooks.

number? If so, why say it is a "comparison," which suggests that it is some kind of an "action"? But wait: a fraction $\frac{a}{b}$ implies that, by definition, a and b are both whole numbers. Does this mean we can only "compare" two whole numbers? Since people also use "ratio" to compare fractions—the ratio of flour to sugar in a recipe is $1\frac{3}{4}$ cups to $\frac{2}{3}$ cups—shouldn't the definition have been replaced by something like the following?

> A ratio is a comparison of two fractions, a and b, obtained by dividing the fractions: $\frac{a}{b}$.

In addition, the last sentence of the first bullet talks about comparing "two amounts" instead of two numbers. What is an "amount," and what is "a part to a part, a part to the whole, or the whole to a part" all about?

We can go further. For example, the statement that "the ratio of Democrats to Republicans in this gathering is about 2 to 3" has a well-known interpretation of "to every 2 Democrats there are 3 Republicans." Every student needs to know (1) what it means to say "to every 2 Democrats there are 3 Republicans" and, more to the point, (2) how does the definition of ratio lead to this interpretation? TSM does not even pretend to address these questions. (see Chapter 22 of Wu 2011a for some answers to the preceding questions.)

A *mathematical* definition must at least address these basic, mundane issues, but the first bullet does not. It is therefore easy to see why, given such a variety of "definitions" of a ratio, it is very difficult for learning to take place. This is why students fear ratios.

Without a definition of *ratio* that makes sense, there can be no reasoning to speak of (see the discussion of *learning about definitions* in Sect. 11.5.1). Similar comments can easily be made about the other two bullets.

We have thus seen that ratio is a confused concept both within TSM and in daily life. To introduce school students to ratio, we must find a definition that provides an entry point into such a vague concept that is *correct, simple, and therefore learnable.* If we introduce the concept of a *complex fraction* as the division of two fractions (see Chapter 19 of Wu 2011a), then a ratio of two fractions A and B can be simply defined to be the complex fraction $\frac{A}{B}$. All the standard ratio problems can then be easily solved *by the use of reasoning* (see Chapter 22 of Wu 2011a).

The sins of TSM on the subject of ratio run deeper. Implicit in the preceding "definitions" is the belief that *ratio* is an ineffable concept, so that even a verbose description will not "discharge its full meaning" (Lamon 1999, page 165). Yet, later in the school curriculum, TSM has no hesitation in defining a ratio simply as a division. For example, the *ratio* of the circumference to the diameter of a circle is the number π, the *ratio* of "rise over run" of a straight line in the coordinate plane is *one* number, the slope, and the *ratio* of the opposite side of an acute angle of t radians in a right triangle to the hypotenuse of the right triangle is also *one* number, $\sin t$. But TSM never addresses the disconnect between this and its earlier, inscrutable "definition" of a ratio,[30] and every middle school teacher is aware of that. This is why teachers, too, fear ratios.

Next, *percent*. Here are some typical TSM "definitions."[31]

- Percent is a special kind of ratio in which the second quantity is always 100.
- Remember that percent means "out of 100." You find a percent by first dividing to find a decimal.
- Percent: part of 100, or per hundred.

Do these "definitions" tell us what "percent" means? If students already have trouble understanding what a "ratio" is, how is the first bullet going to help them understand "percent"? Is "out of a hundred" or "of each hundred" a number, and if so, what number is it?

TSM is apparently indifferent to these concerns, but it will tell you how to solve "percent" problems by laying down some "rules." For example, the second bullet already has a built-in rule: divide to find a decimal. What is the *reasoning* that leads from "out of 100" to "divide to find a decimal"? None that we can see. It is an arbitrary rule dictated by TSM.

On the basis of the second bullet, the suggested way to solve "*what percent of* 80 *is* 45?" is this: $45 \div 80 = 0.5625$, "which when rounded to the nearest 100th" is 0.56. So the answer is: about 56%. Notice that we are using the fact that one can obtain the decimal representation of a fraction by long division to get the answer to this simple question. This fact about "dividing numerator by denominator to get a decimal" is in fact very difficult to prove at the level of school mathematics (for a preliminary explanation, see pp. 81-86 of Wu 2016a, and for the full proof, see Section 3.4 in the third volume of Wu 2020), and is in any case only taught by rote in TSM. Thus, to find an approximate answer to "*what percent of* 80 *is* 45?", students have to follow an arbitrary rule, *and* also use a difficult fact they could only memorize by rote. *No reasoning in any case.* What are we teaching and learning here except how to follow rules?

For another example, starting with the third bullet ("Percent: part of 100"), the answer to *what is* 45% *of* 80? can be found by one of two "rules," according to TSM. First rule: set up a proportion: $\frac{45}{100} = \frac{n}{80}$. By the CMA (cross-multiplication algorithm), $45 \times 80 = 100n$ and therefore $n = 36$. Second rule: multiply the number by the percent: $45\% \times 80 = 36$. TSM is silent about how to start with "part of 100" and arrive at either of these solution methods *by the use of reasoning*.[32] TSM is also silent on whether there is any connection between the two methods. As far as TSM is concerned, it is enough that both methods seem *intuitively* related to "part of 100," and that both methods yield the correct answer. This is why, in TSM, percent is not learnable as mathematics, and this is why teachers and students both fear it.

[30] A blatant example of incoherence (see the fundamental principles of mathematics in Sect. 11.3.1).

[31] These bulleted statements are taken directly from textbooks.

[32] See Nike's trademarked slogan: "Just do it."

One can consult Chapter 20 of Wu (2011a) for a simple definition of *percent* and see how—on the basis of this definition—reasoning leads to straightforward solutions of all such standard problems.

Rate

A great deal can be said about how the concept of *rate* is abused in TSM (see Section 7.2 of Wu 2016b), but we will limit ourselves here to discussing only *continuous rates*[33](those related to, e.g., motion, water flow, lawn-mowing). We already touched on the most troubling aspect of TSM's treatment of rate problems when we discussed Luis' walking problem in Sect. 11.5.1: its reliance on the fictitious concept of *proportional reasoning* to solve problems. But there are other issues.

TSM should make clear at the outset that *rate*, as it is understood intuitively, cannot be taught *as mathematics* in K–12 (it requires the use of the derivative and is therefore a calculus concept). But this message has never gotten out. As we noted in Sect. 11.5.1, what *can* be taught in K–12 are the concepts of *average rate* (over a fixed time interval) and *constant rate*, but TSM shows little or no inclination to define either precisely or teach either seriously. Instead, TSM serves up the following brew of "definitions" for *rate*:[34]

- A **rate** is a ratio that involves two different units. A rate is usually given as a quantity per unit such as miles per hour.
- A **rate** is a ratio that compares two quantities having different units of measure.
- A rate is the quotient of two quantities with different units. A quantity whose unit contains the word "per" or "for each" or some synonym.

Again, these "definitions" are not informative because, this time around, none of them even makes any pretense at trying to give students any *usable* mathematical information. In two out of three cases, these definitions are built on the concept of *ratio*. Since TSM never explains clearly what ratio is, how can students use what little they know about ratio to find out what "rate" is?

In Summary

The many mathematical flaws in the way percent, ratio and rate are taught in TSM make these topics *unlearnable*, and that is why they are feared by one and all.

When the 2011 IES impact study considered teachers' content knowledge to be adequate for teaching ratio, percent, rate, and proportion if it was equal to "the knowledge of topics in rational numbers that *students* should ideally have after completing the seventh grade" (page 21 of Garet et al. 2011), it *failed to recognize the damage TSM had done to teachers and to school mathematics education as a whole.* This is tantamount to saying that *TSM is good enough.* If the foregoing

[33]The concepts of what is "continuous" and what is "discrete" are well understood in mathematics, but for the case at hand, the *ad hoc* explanation of "continuous" given in the parentheses is sufficient.

[34]These bulleted statements are taken directly from textbooks.

analysis in this subsection means anything at all, it is that TSM is *not* good enough.[35] By not providing teachers with an improved content knowledge base, the IES impact study in effect forced them to present the standard unlearnable TSM to their students. *Was it the teachers' fault that they turned out to be ineffective in raising student achievement?*

In this light, the IES impact study—while its good intentions are undeniable—cannot serve as a reliable gauge of whether PD can improve teachers' effectiveness or student achievement. It would make sense to do a similar study with a PD program that teaches PBM to teachers in the first place.

11.6.3 Final Thoughts

We want to come to a real appreciation for why the 2011 IES impact study failed to raise student achievement.

Our first thought is that grade 7 is probably not the best grade to choose for the purpose of an impact study, for the following reason. We saw in the preceding subsection, Sect. 11.6.2, that ratio, rate, and percent are concepts that have been made more difficult by TSM than they actually are. Any attempt to improve teachers' mastery of these concepts must begin by providing teachers with precise definitions for these concepts and showing teachers—on the basis of these definitions—how to use reasoning to solve all the standard problems with ease. In other words, the PD must help teachers *relearn* these topics from the standpoint of PBM. There is a catch, however. Teachers cannot learn the new content knowledge about ratio, rate, and percent without first acquiring a new foundation for fractions such as that presented in Sect. 11.3.2. To help teachers of 7th grade to better teach ratio, rate, and percent, it is necessary to first revamp their knowledge of fractions. We are thus suggesting that the PD program of the 2011 impact study should have been more comprehensive in terms of content, and it should also be held over a longer time duration. A future impact study may try to avoid grade 7 and work with teachers in an earlier grade.[36]

If we accept the foregoing explanation that teachers' faulty knowledge of fractions and ratio, percent, and rate, based on TSM, cannot be used in a school classroom, then it immediately raises the question of whether the impact study's emphasis on SK (*specialized knowledge for teaching*) at the expense of CK (*common knowledge*) was a good decision (see Sect. 11.6.1). The impact study

[35]This conclusion is also consistent with the fact that, by the second year of the impact study, it became apparent to the researchers that the teachers were having trouble with the 7th-grade topics of ratio, percent, and rate.

[36]We may note that the 2016 impact study of Garet et al. (2016) did avoid this pitfall and chose to work with 4th grade teachers. However, the content component of the 2016 study has some issues of its own, including a lack of awareness of the damage TSM has done to teachers.

made the decision to essentially leave teachers' knowledge of TSM for 7th grade intact and empower them with some new pedagogical techniques, hoping that the same teachers would nevertheless raise student achievement. Having the benefit of hindsight, we would put the emphasis of the PD program on CK to help teachers eradicate their knowledge of TSM and replace it with PBM. Mindful of teachers' need for intensive content immersion, we would also increase the number of summer institute days[37] from 3 to 8, expand the length of the day from 6 to 8 h (including lunch), and give out daily homework assignments. (There is nothing like doing exercises to improve one's understanding of mathematics.) Of the eight summer institute days, a tentative suggestion would be to spend 5 days on the definition of fractions, the comparison of fractions and the arithmetic operations on fractions. Middle school teachers really need a firm foundation on fractions because almost everything they teach rests on this foundational knowledge. The remaining 3 days could then be devoted to a thorough discussion of *complex fractions* (see Chapter 19 of Wu 2011a), ratio, percent, and rate. We should add that complex fractions are absolutely essential for any discussion of ratio, percent, and rate. One of the reasons that TSM cannot make sense of these three concepts is precisely its neglect of the *concept* of complex fractions (see Wu 2016a, Section 1.7).

It goes without saying that the *content* of the eight summer institute days would be PBM rather than TSM (Sect. 11.3). Given the present lack of usable PBM materials (cf. the discussion in Sect. 11.4), allow me to suggest Chapter 1 of Wu (2016a), *without the long Section* 1.10 *on probability*, as a reference. (Wu 2016a was actually written explicitly for this kind of PD for middle school teachers.) A slightly different suggestion would be Part 2 of Wu (2011a), *without Chapters* 23 *and* 24.

The 10 days of coaching in the original PD design of the impact study is an excellent idea, but having to send so many coaches to different school districts poses a problem of getting qualified coaches. This also brings up a similar issue of how to get enough PD facilitators for the summer institutes to be given in the participating school districts. As we have indicated all along, most PD providers are themselves products of TSM. For them to be effective in helping teachers learn PBM, they themselves will have to undergo training to learn PBM first. Therefore, before teaching teachers, we will have to teach coaches and PD facilitators. There is no getting around this difficulty, the fact that the preparation for such an impact study—in addition to the usual logistical issues—would have to take place months before the study itself for the purpose of creating a corps of qualified coaches and facilitators. The content-intensive training for this purpose is similar to the content-intensive PD for the teachers, except that there will be less room for failure because the coaches and facilitators will be responsible for the PD after all. We will not go into the details of this training because much of it will take us far away from our main concern with PD for *teachers*.

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[37]If my own experience in PD is any guide, 8 days of content instruction do not really count as "*intensive* immersion." But we do have to compromise with reality.

In summary, this discussion may give a better idea of why one should not jump to conclusions about the non-effectiveness of inservice PD for improving school mathematics education on the basis of one or two impact studies. We hope there will soon be a follow-up impact study with a better design.

11.7 Miscellaneous Remarks

(1) As of 2019, elementary teachers are generalists, and it is impractical to ask all generalists to teach PBM. The reality is that learning and teaching PBM will take more dedication and time than a typical generalist can afford. Any real improvement in elementary mathematics education will inevitably require the creation of *mathematics teachers*—often called *math specialists*—to teach the mathematics of elementary school (see Wu 2009). These math specialists will certainly need the content-intensive training in PBM mentioned above.

(2) If we want more effective teachers, we cannot talk only about *inservice* PD because we must teach PBM to all preservice teachers so that the new teachers coming out of the pipeline will help solve the TSM-infestation problem rather than adding to it. Now preservice PD is understandably a different beast from inservice PD, and we will only *lightly* touch on a few of the major issues that complicate the preservice picture.

First of all, few colleges are willing to offer a mathematics course for teachers because teaching such a course on PBM will likely require extensive cooperation between the school of education and the department of mathematics. Given the often frosty relationship between these two units on many campuses, this obstacle can be overcome only by a leadership with intellectual vision and dedication to social justice. On top of that, there is the obvious problem with textbooks because an overwhelming majority of the available preservice PD textbooks are mired in TSM. The presence of these books is part of the reason that TSM is continually recycled in the world of education. There is also a less obvious personnel problem, as we now explain.

In year 2019, those with the requisite mathematical knowledge to teach PBM—regardless of grade level—are overwhelmingly found in mathematics departments. A typical mathematician is, however, ill-equipped to teach a course on correct school mathematics, for several reasons. Such a course is "elementary" in the sense of the usual mathematical hierarchy and will therefore be treated like calculus, and it is sad but true that calculus is usually taught as TSM. Moreover, teaching a course on PBM requires mathematical sophistication on a level with teaching an upper division course like introductory analysis (ϵ's and δ's) or abstract algebra (groups, rings, and fields). But teaching the former like an upper division course for math majors would be unfair to future teachers and ill-equip those teachers to teach their future students. Very often, the proofs (explanations) in a course for teachers—if done correctly—would be the most intuitive, not the shortest possible. Short proofs tend to be mathematically sophisticated and, therefore, *generally* not appropriate for school students. We would prefer that preservice teachers learn something close to what they will have to teach. For example, the explanation that $(-a)(-b) = ab$ for rational number a and b is something that bedevils most middle school students along with many of their teachers. In Wu (2016a), the explanation of this fact takes a full five pages (middle of page 166 to the middle of page 171). The usual three-line mathematical proof, expanded to half a page, is finally given in the lower half of the last page, page 171. A typical mathematician, approaching such a course for teachers as one in pure mathematics, would surely scoff at such verbosity as a waste of time. There are many such *mathematical* issues that can potentially reduce the relevance of such a course on PBM to future teachers. Ideally, one can smooth over such bumps on the road if there is a good working relationship between the School of Education and the Department of Mathematics, but such a spirit of cooperation is currently in short supply. Getting the right people to teach such a course will probably be a thorny issue for a long time to come.

(3) Finally, we have advocated for sustained PD to teach teachers correct school mathematics. While this seems not to be happening yet, several education centers around the country have been offering PD for mathematics teachers that focusses on solving hard problems or doing mathematical research. Any effort at raising the content knowledge of mathematics teachers is welcome, so there is no doubt that these centers are doing something right for a certain population in school mathematics education. Nevertheless, we must not lose sight of the fact that we have to raise the *general level* of content knowledge of the average teacher if better school mathematics education is our goal. If those centers that teach problem solving or helping teachers do research could convey the message that the problem solving and the mathematical research are means to an end, the end being the replacement of TSM by PBM, they would be making a major contribution to the cause of better school mathematics education for all.

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References

Aharoni, R. (2015). *Arithmetic for parents*. Singapore, Singapore: World Scientific Publishing.

Armstrong, B., & Bezuk, N. (1995). Multiplication and division of fractions: The search for meaning. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 85–120). Albany, NY, USA: State University of New York Press.

Askey, R. (2018). What content knowledge should we expect in mathematics education? In Y. Li, J. Lewis, & J. Madden (Eds.), *Mathematics matters in education. Essays in honor of Roger E. Howe* (pp. 219–231). Dordrecht, The Netherlands: Springer. https://tinyurl.com/y7bfhgqf

Baldridge, S. (2013). *Eradicating Textbook School Mathematics (TSM)*. Blog posted Retrieved February 6, 2013 from: https://tinyurl.com/yaegv9cn

Bezuk, N., & Kramer, K. (1989). Teaching about fractions: What, when and how? In P. R. Trafton & A. P. Schulte (Eds.), *New directions for elementary school mathematics*. Reston, VA, USA: National Council of Teachers of Mathematics.

Billstein, R., Liebeskind, S., & Lott, J. W. (2007). *A problem solving approach to mathematics for elementary school teachers* (9th ed.). Boston, MA, USA: Pearson-Addison Wesley.

Bingea, A. (undated). *Adding and Subtracting Rational Numbers on the Number Line*. Retrieved on October 30, 2019 from: https://teachers.yale.edu/curriculum/viewer/initiative_16.05.01_u

Common Core (2010). *Common Core State Standards for Mathematics*. Retrieved from: http://www.corestandards.org/Math/

Common Core (2012). *K-8 Publishers' Criteria for the Common Core State Standards for Mathematics*. Retrieved from: http://tinyurl.com/bpgx8ed

Common Core (2013). *High School Publishers' Criteria for the Common Core State Standards for Mathematics* (2013). Retrieved from: https://tinyurl.com/y8upgtcr

Cuoco, A., & McCallum, W. (2018). Curricular coherence in mathematics. In Y. Li, J. Lewis, & J. Madden (Eds.), *Mathematics matters in education. Essays in honor of Roger E. Howe* (pp. 245–256). Dordrecht, The Netherlands: Springer.

Darling-Hammond, L., Hyler, M. E., & Gardner, M. (2017). *Effective teacher professional development*. Palo Alto, CA, USA: Learning Policy Institute. https://tinyurl.com/y9qbdfmf

Douglas, A. (2015). *The Creation of a Mathematics Education Program at City Tech, CUNY*. Retrieved from: https://tinyurl.com/y8wn3rz4

Dynarski, M. (2018). *Why is accountability always about teachers? Brookings: Series–Evidence Speaks*. Retrieved February 25, 2018 from: https://tinyurl.com/y9d9zff4

Education Week (2014). *Researcher Isolates Common-Core Math Implementation Problems*. Retrieved November 14, 2014 from: https://tinyurl.com/y7hme29c

Garet, M. S., Cronen, S., Eaton, M., Kurki, A., Ludwig, M., Jones, W., … Sztejnberg, L. (2008). *The impact of two professional development interventions on early reading instruction and achievement* (NCEE 2008-4034). Washington, DC, USA: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, US Department of Education. https://ies.ed.gov/ncee/pdf/20084030.pdf

Garet, M., Wayne, A., Stancavage, F., Taylor, J., Eaton, M., Walters, K., … Dolittle, F. (2011). *Middle school professional development impact study: Findings after the second year of implementation* (NCEE 2011-4024). Washington, DC, USA: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, US Department of Education. https://ies.ed.gov/ncee/pubs/20114024/pdf/20114024.pdf

Garet, M. S., Heppen, J. B., Walters, K., Parkinson, J., Smith, T. M., Song, M., … Borman, G. D. (2016). *Focusing on mathematical knowledge: The impact of content-intensive teacher professional development* (NCEE 2016-4010). Washington, DC, USA: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, US Department of Education. https://ies.ed.gov/ncee/pubs/20164010/pdf/20164010.pdf

Hasiotis, D. (2015). What Does the Research Say on Professional Development, Anyway? *TNTP Blog*. Retrieved July 28, 2015 from: https://tinyurl.com/y8lbybkb

Jensen, G. (2003). *Arithmetic for teachers*. Providence, RI: American Mathematical Society.

Keeghan, A. (2012). *Afraid of Your Child's Math Textbook? You Should Be*. Blog posted on February 17, 2012. Retrieved from: https://tinyurl.com/yxo2a97h

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding*. Mahwah, NJ, USA: Lawrence Erlbaum.

Layton, L. (2015). Study: Billions of dollars in annual teacher training is largely a waste. *Washington Post*. Retrieved, August 4, 2015 from: https://tinyurl.com/y7zzr8t8

Little, J. W. (1989). District policy choices and teachers' professional development opportunities. *Educational Evaluation and Policy Analysis, 11*(2), 165–179.

Loewus, L. H. (2016). Gates Chief Acknowledges Common-Core Missteps. *Education Week*. Retrieved May 23, 2016 from: https://tinyurl.com/yd5juay5

Loewus, L. H. (2016). Study: Improving Teachers' Math Knowledge Doesn't Boost Student Scores. *Education Week*. Retrieved September 28, 2016 from: https://tinyurl.com/zh6hhez

Loewus, L. (2017). Majority of Teachers Say Reforms Have Been 'Too Much'. *Education Week*. Retrieved December 19, 2017 from: https://tinyurl.com/y7yw8lyd

Moynahan, J. (1996). Of-ing fractions. In D. Schifter (Ed.), *What's happening in math class?*. New York, NY, USA: Teachers College Press.

Namkung, J. M., & Fuchs, L. S. (2016). Cognitive predictors of calculations and number line estimation with whole numbers and fractions among at-risk students. *Journal of Educational Psychology, 108*(2), 214–228.

National Mathematics Advisory Panel (2008). *Foundations for success: The final report of the national mathematics advisory panel*. Washington, DC, USA: US Department of Education. Retrieved from http://tinyurl.com/hrzm3uu

National Research Council (2001). J. Kilpatrick, J. Swafford, and B. Findell (Eds.), *Adding it up.*. Washington, DC, USA: National Academy. https://tinyurl.com/y6mtcr5k

Otten, S., Herbel-Eisenman, B. A., & Males, L. M. (2010). Proof in algebra—Reasoning beyond examples. *Mathematics Teacher, 103*(7), 514–518.

Polya, G. (1957). *How to solve it; a new aspect of mathematical method.* Garden City, NY: Doubleday.

Poon, R. C. (2014). *Principle-Based Mathematics: An Exploratory Study*. Dissertation at University of California, Berkeley. Retrieved from: http://escholarship.org/uc/item/4vk017nt

Quinn, F. (2012). A revolution in mathematics? What really happened a century ago and why it matters today. *Notices of the American Mathematical Society, 59*, 31–37. https://www.ams.org/notices/201201/rtx120100031p.pdf

Romberg, T. A., & Carpenter, T. P. (1985). Research on teaching and learning mathematics: Two disciplines of scientific inquiry. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed.). New York, NY, USA: Macmillan.

Sawchuk, S. (2016). Are teachers getting the right kind of common-core PD? *Education Week*. Retrieved Febuary 1, 2016 from: https://tinyurl.com/y8238pzj

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., . . . Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide* (NCEE 2010-4039). Washington, DC, USA: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, US Department of Education. https://tinyurl.com/y66ffyec.

TIMSS 1995 Results (1995). *National Center for Education Statistics.* Retrieved from: https://nces.ed.gov/timss/results95.asp

TNTP (2015). *The mirage: Confronting the hard truth about our quest for teacher development*. New York, NY, USA: TNTP. https://tntp.org/assets/documents/TNTP-Mirage_2015.pdf

U. S. Department of Education (2009). *State and local implementation of the no child left behind act, volume VIII—Teacher quality under NCLB: Final report.* Washington, DC, USA: Office of Planning, Evaluation and Policy Development, Policy and Program Studies Service.

Wu, H. (1998). *Teaching fractions in elementary school: A manual for teachers*. Retrieved from: https://math.berkeley.edu/~wu/fractions1998.pdf.

Wu, H. (1999a). *Some remarks on the teaching of fractions in elementary school*. Retrieved from: https://math.berkeley.edu/~wu/fractions2.pdf

Wu, H. (1999b). Professional development of mathematics teachers. *Notices of the American Mathematical Society, 46*, 535–542. http://www.ams.org/notices/199905/fea-wu.pdf

Wu, H. (1999c). The joy of lecturing–With a critique of the romantic tradition of education writing. Appendix to *How to Teach Mathematics*, S. G. Krantz (pp. 261-271, 2nd ed.). Providence, RI, USA: American Mathematical Society.

Wu, H. (2002). *Chapter 2: Fractions (Draft)*. Retrieved September, 2002 from: https://math.berkeley.edu/~wu/EMI2a.pdf

Wu, H. (2006). How mathematicians can contribute to K-12 mathematics education, *International Congress of Mathematicians. Volume III.* (pp. 1676–1688). Zürich: European Mathematical Society. http://math.berkeley.edu/~wu/ICMtalk.pdf

Wu, H. (2009). What's sophisticated about elementary mathematics? *American Educator, 33*(3), 4–14. https://math.berkeley.edu/~wu/wu2009.pdf

Wu, H. (2011a). *Understanding numbers in elementary school mathematics.* Providence, RI, USA: American Mathematical Society.

Wu, H. (2011b). The mis-education of mathematics teachers. *Notices of the American Mathematical Society, 58*, 372-384. https://math.berkeley.edu/~wu/NoticesAMS2011.pdf

Wu, H. (2011c). Bringing the common core state mathematics standards to life. *American Educator, 35*(3), 3–13. http://www.aft.org/pdfs/americaneducator/fall2011/Wu.pdf

Wu, H. (2016a). *Teaching school mathematics: Pre-algebra*. Providence, RI, USA: American Mathematical Society. *Its **Index** is available at:* http://tinyurl.com/zjugvl4

Wu, H. (2016b). *Teaching school mathematics: Algebra*. Providence, RI, USA: American Mathematical Society. *Its **Index** is available at:* http://tinyurl.com/haho2v6

Wu, H. (2018). The content knowledge mathematics teachers need. In Y. Li, J. Lewis, & J. Madden (Eds.), *Mathematics matters in education. Essays in honor of Roger E. Howe* (pp. 43–91). Dordrecht, The Netherlands: Springer. https://math.berkeley.edu/~wu/Contentknowledge1A.pdf

Wu, H. (2020). *Rational numbers to linear equations, algebra and geometry, and pre-calculus, calculus, and beyond*. Providence, RI, USA: American Mathematical Society.