Syllabi of High School Courses According to the Common Core Mathematics Standards

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The following are two sets of proposed syllabi for high school mathematics classes. Ever since the Common Core Mathematics Standards (CCMS) were released in June of 2010, many questions concerning the high school portion of these standards have surfaced. In response to these questions, the present syllabi are being offered as a reference for those who are involved in the implementation of CCMS.

This document had its origin in the challenge posed to me by Phil Daro and Marge Cappo at the end of March, 2011: could I write down a syllabus for grades 9-10, compatible with the CCMS, so that by the end of these two years, students who intend to pursue a college career in the STEM subjects will be done with all the topics in the traditional Algebra I and II and Geometry courses? My answer to that challenge is given in Part I below. I was informed that the Pearson Foundation was contemplating developing complete courses covering mathematics for grades K–10 and that their question was raised in that context. (Such an undertaking, done in partnership with the Bill & Melinda Gates Foundation, has since been officially announced on April 27, 2011.) In the course of the discussion with Phil Daro, he and I got the idea that I should also propose a syllabus for three years of high school mathematics for all the students who either will not pursue STEM subjects in college or have other interests in life. The goal of this three-year sequence is to make these courses maximally educational for these students. Such a syllabus is the content of Part II.

With minor exceptions, all the standards in the following pages are taken from CCMS—though sometimes not in the original formulation. This is the place to say a few words about the CCMS high school standards. Unlike the standards in grades K–8, the high school standards are grouped together under so-called *conceptual categories*, such as Number and Quantity, and Modeling. These are what the CCMS writers consider to be the main mathematical threads that tie together the high school mathematics curriculum; they are therefore not put forth in CCMS with the intention of making it easy for curriculum designers to do their work. Consequently, in the writing of a syllabus, the concern for natural development of the mathematical ideas must override the need to follow CCMS literally. For this reason, I did not hesitate to fill in some obvious curricular gaps that are not covered by the standards of CCMS, e.g., the explicit definition of *congruence* and *similarity transformation*, the explicit definition of an *inverse function*, an explicit explanation of why the graph of a linear equation of two variables is a line, etc. All the items in the following syllabi that have been added to CCMS out of necessity are given in **blue** color.

In my opinion, there is barely enough time as is to cover all the topics in the syllabus of the two-year sequence in Part I. Recall that this is a course-sequence for STEM students, and its primary obligation must be to serve their need of getting to know the *mathematics* inside out. The main weight of this syllabus is therefore the explication of the logical reasoning underlying the skills and the interrelationship between the concepts. I believe that if we try to do justice to these two aspects of mathematics, there will be little time left for statistics or learning the skill of making mathematical models out of real-world data. But I have every confidence that anyone who can master the mathematics of this syllabus will experience little difficulty in picking up statistics or modeling in science or engineering courses.

Here are a few guiding principles that were used in deciding what to put into the syllabus of Part II, which is a three-year sequence for non-STEM students; these principles are of course subjective. First of all, a few of the topics in Part I that are judged to be less likely to come up in a general, non-technical context are left out of Part II, e.g., Laws of Sines and Cosines, proof of the Binomial Theorem, rational expressions, proof that exponential functions are increasing or decreasing, complex quadratic polynomials, and a precise treatment of inverse functions. There are others too numerous to enumerate. On the other hand, I have added the following topics to Part II, and the reason for each topic varies:

A leisurely review of linear equations in grade 9 (essential for non-STEM students).

How to generate Pythagorean triples (fun topic)

Recurrence relations and Fibonacci sequence (fun topic)

Definition of a parabola (basic common sense)

Spherical geometry (the earth is almost spherical)

Historical context of the logarithms (basic common sense)

Statistics (essential for everyday life)

I should add that all I did about statistics was to copy the statistics standards in CCMS up to (but not including) probability. A truly knowledgeable statistician would undoubtedly be able to do this part better.

One thing I wish I could fit into this syllabus is a discussion of the Königsberg bridges problem and Euler's solution. It would be a marvelous introduction to mathematical abstraction and elementary graph theory. At the moment, there does not seem to be room for such a discussion as there is already too much to cover in three years, but there is always the possibility of letting it replace one of the last six topics.

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Part I

Part II

Part I

Grade 9

Use of Symbols

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.

   a. Interpret parts of an expression, such as terms, factors, and coefficients.

   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of* P *and a factor not depending on* P.

2. Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

Perform arithmetic operations on polynomials

3. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
5. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*
6. Factor quadratic polynomials in one variable with integer (or rational) coefficients.
7. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a

factor of $p(x)$.

Perform arithmetic operations on rational expressions

8. Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.
9. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

Rigid Motions and Congruences

Experiment with transformations in the plane

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Know the precise definitions of rectangle, parallelogram, trapezoid, and regular polygon, and describe the rotations and reflections that carry each onto itself.
4. Develop definitions of basic rigid motions (rotations, reflections, and translations) in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Know the meaning of composing rigid motions (rotations, reflections, and translations) and define congruence in terms of their composition.
6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to prove that two

triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Prove the criteria for triangle congruence (ASA, SAS, and SSS) from the definition of congruence in terms of rigid motions.

Prove geometric theorems

9. Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segments endpoints.
10. Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.
11. Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.

Dilation and Similarity

Understand similarity in terms of similarity transformations

1. Verify experimentally the properties of dilations given by a center and a scale factor:

   a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

2. Define a similarity transformation as the composition of a dilation followed by a congruence and prove that the meaning of similarity for triangles is the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

3. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar.
4. Use the properties of similarity transformations to establish the AA criterion and SAS criterion for two triangles to be similar.

Applications to coordinate geometry

5. Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx+b$ for a line intercepting the vertical axis at b.
6. Know that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). *For example, the graph of $x^2 + y^2 = r^2$ for a positive number r is the circle of radius r around the origin.*
7. Use similar triangles to explain why the graph of a linear equation of two variables is a line, and why every line is the graph of a linear equation.
8. Understand the relationship between the a linear equation of two variables and its graph: Given an equation, explain how to sketch its graph, and given certain data about a line, explain how to write down its equation.
9. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

Linear and Quadratic Equations

Solve linear equations and inequalities

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
2. Solve linear equations and inequalities in one variable, including equations involving absolute values and those with coefficients represented by letters.

3. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.
4. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
5. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
6. Know the definition of the graph of a linear inequality in two variables as the collection of all the point (x, y) which satisfy the inequality, and that the graph of a system of linear inequalities in two variables is the intersection of the graphs of the individual inequalities.

> *a*. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.
>
> *b*. Solve simple problems in linear programming.

Quadratic equations in one variable

7. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
8. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation.
9. Solve word problems that are modeled by quadratic equations.

Grade 10

Understanding Functions

Understand the concept of a function and use function notation

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is by definition the graph of the equation $y = f(x)$, i.e., the graph of f is the point (x, y) in the plane so that the second coordinate y is equal to $f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

> *a.* Graph linear and quadratic functions and show intercepts, maxima, and minima.
>
> *b.* Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
>
> *c.* Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

4. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.
5. Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.*

Understanding linear and quadratic functions

6. Understand that the graph of a linear function $f(x) = ax + b$ is the same as the graph of the linear equation in two variables $y = ax + b$.
7. Understand that locating the zeroes of a quadratic function $f(x) = ax^2 + bx + c$ is the same as solving the quadratic equation in one variable $ax^2 + bx + c = 0$.
8. Prove that every quadratic function $f(x) = ax^2 + bx + c$ can be written as $f(x) = a(x - p)^2 + q$ for some numbers p and q by the method of completing the square.

> a. Show that if the discriminant $b^2 - 4ac$ is negative f has no zero, if the discriminant is positive f has two distinct zeros, and if the discriminant is 0 f has a double zero.
>
> b. Use the quadratic formula to find the zeroes of f.
>
> c. Read off the extreme values of f and the maximum or minimum of f.
>
> d. Interpret items a to c above in terms of a context.
>
> e. Show the symmetry of the graph of f with respect to the vertical line $x = p$, and conclude in general that the graph of f is the translation of the graph of $y = ax^2$.

9. Prove that if r_1 and r_2 are the zeros of $f(x) = a(x - p)^2 + q$, then $f(x) = a(x - r_1)(x - r_2)$, and conclude that all quadratic polynomials which have a real zero can be factored by using the quadratic formula.

Exponential and logarithmic functions

10. Explain how the definition of the rational exponents of a positive number follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $5^{(1/3)3}$ must equal 5.
11. Rewrite expressions involving radicals and rational exponents using the properties of exponents.
12. Know and use the properties of exponents.

a. Know the laws of exponents, and know the proofs of simple cases, e.g., $a^{1/n}b^{1/n} = (ab)^{1/n}$.

b. Know how to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.

13. Prove the basic properties of exponential functions $f(x) = a^x$ for $a > 0$, including $f(s + t) = f(s)f(t)$ and that f is increasing if $a > 1$, decreasing if $a < 1$, and that f maps onto the real numbers in both cases.
14. Know the concept of an inverse function.

a. If a function f has an inverse function g, prove that the graphs of f and g plotted on the same x-y plane, are reflections of each other across the diagonal line $x - y = 0$.

b. Prove that for $a \neq 1$ and $a > 0$, a^x has an inverse function called the logarithmic function with base a, denoted by $\log_a x$.

15. Graph exponential and logarithmic functions, showing intercepts and end behavior.
16. Explain basic properties of the logarithmic function, and how they are related to the corresponding properties of its inverse exponential function.

Formal Algebra

Complex numbers

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
3. Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers. Represent complex numbers and their operations on the complex plane.

4. Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. For example, $(1+\sqrt{3}i)^3 = 8$ because $(1+\sqrt{3}i)$ has modulus 2 and argument 120°.
6. Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Use complex numbers in polynomial identities and equations

7. Understand that polynomials with complex coefficients form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials
8. Solve quadratic equations with real coefficients that have complex solutions.

a. Verify that the quadratic formula now holds in general.

b. Derive Viète's theorem relating the coefficients to the roots of the equation.

c. Prove that if r_1 and r_2 are the zeros of $f(x) = a(x-p)^2+q$, then $f(x) = a(x-r_1)(x-r_2)$. Thus the quadratic formula leads to a factorization of all quadratic polynomials with real coefficients.

9. Know the Fundamental Theorem of Algebra.

a. Make sense of the square roots of a complex number and prove the theorem for quadratic polynomials (with complex coefficients).

b, Write down the n roots of $x^n = 1$ (these are the so-called nth roots of unity).

The Binomial Theorem

10. Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascals Triangle.

   a. Know the technique of proof by induction.

   b. Prove the Binomial Theorem by induction.

Geometry

Define trigonometric ratios and solve problems involving right triangles

1. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
2. Explain and use the relationship between the sine and cosine of complementary angles.
3. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.
4. Prove the Laws of Sines and Cosines and use them to solve problems.

Circles

5. Prove that all circles are similar.
6. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.
7. Prove properties of angles for a quadrilateral inscribed in a circle.

Make geometric constructions

8. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric

software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.

9. Construct the inscribed and circumscribed circles of a triangle.

10. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

11. Construct a tangent line from a point outside a given circle to the circle.

Geometric measurement

Explain volume formulas and use them to solve problems

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.

2. Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

Part II

Grade 9

Linear Equations (With Review)

Solve linear equations in one variable

1. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.
2. Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).
3. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
4. Solve linear equations with coefficients represented by letters.

Linear equations in two variables

5. Know that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). *For example, the graph of* $x^2 + y^2 = r^2$ *for a positive number* r *is the circle of radius* r *around the origin.*
6. Solve simultaneous linear equations.

    a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

    b. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

c. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. For example, 3x + 2y = 5 and 3x + 2y = 6 have no solution because 3x + 2y cannot simultaneously be 5 and 6.

d. Solve systems of linear equations exactly and approximately (e.g., with graphs).

7. Solve the linear system $u + v = t/s$ and $u - v = s/t$ for positive integers s and t $(s < t)$ to get Pythagorean triples.

a. Explain the linear system by observing that if we have a Pythagorean triple $a^2 + b^2 = c^2$, then we also have $1 = (c/b)^2 - (a/b)^2 = ((c/b) + (a/b))((c/b) - (a/b))$, so that letting $u = c/b$ and $v = a/b$, we get $1 = (u + v)(u - v)$.

b. If the solutions are $u = c/b$ and $v = a/b$, where a, b, c are positive integers, then $a^2 + b^2 = c^2$.

c. Explain the significance of Pythagorean triples in the context of the Fermat equation $a^n + b^n = c^n$ for a positive integer $n \geq 2$.

Use of Symbols

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be*

factored as $(x^2 - y^2)(x^2 + y^2)$.

Perform arithmetic operations on polynomials

3. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
5. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*
6. Factor quadratic polynomials in one variable with integer (or rational) coefficients.
7. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

Rigid Motions and Congruences

Experiment with transformations in the plane

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Know the precise definitions of rectangle, parallelogram, trapezoid, and regular polygon, and describe the rotations and reflections that carry each onto itself.
4. Develop definitions of basic rigid motions (rotations, reflections, and translations) in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

5. Know the meaning of composing rigid motions (rotations, reflections, and translations) and define congruence in terms of their composition.
6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to prove that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Prove the criteria for triangle congruence (ASA, SAS, and SSS) from the definition of congruence in terms of rigid motions.

Prove geometric theorems

9. Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segments endpoints.
10. Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.
11. Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, and the diagonals of a parallelogram bisect each other.

Dilation and Similarity

Understand similarity in terms of similarity transformations

1. Verify experimentally the properties of dilations given by a center and a scale factor:

> *a*. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

2. Define a similarity transformation as the composition of a dilation followed by a congruence and prove that the meaning of similarity for triangles is the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar.
4. Use the properties of similarity transformations to establish the AA criterion and SAS criterion for two triangles to be similar.

Applications to coordinate geometry

5. Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx+b$ for a line intercepting the vertical axis at b.
6. Use similar triangles to explain why the graph of a linear equation of two variables is a line, and why every line is the graph of a linear equation.
7. Understand the relationship between the a linear equation of two variables and its graph: Given an equation, explain how to sketch its graph, and given certain data about a line, explain how to write down its equation.
8. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

Grade 10

Solve Linear Inequalities

1. Solve linear inequalities in one variable, including those involving absolute values.
2. Know the definition of the graph of a linear inequality in two variables as the collection of all the point (x, y) which satisfy the inequality, and that the graph of a system of linear inequalities in two variables is the intersection of the graphs of the individual inequalities.

a. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

b. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.

c. Solve simple problems in linear programming.

Solve Quadratic Equations in One Variable

1. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
2. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation.
3. Solve word problems that are modeled by quadratic equations.

Understanding Functions

Understand the concept of a function and use function notation

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is by definition the graph of the equation $y = f(x)$ i.e., the graph of f is all the point (x, y) in the plane so that the second coordinate y is equal to $f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

> *a*. Graph linear and quadratic functions and show intercepts, maxima, and minima.
>
> *b*. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
>
> *c*. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

4. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function h. .
5. Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.*

Understanding linear and quadratic functions

6. Understand that the graph of a linear function $f(x) = ax + b$ is the same as the graph of the linear equation in two variables $y = ax + b$.

7. Use technology to gain an intuitive understanding of how to approximate certain observable data, properly graphed, as the graph of a linear function. Discuss informally the method of least squares to arrive at a linear function that is a "best fit" for the data.
8. Understand that locating the zeroes of a quadratic function $f(x) = ax^2 + bx + c$ is the same as solving the quadratic equation in one variable $ax^2 + bx + c = 0$.
9. Prove that every quadratic function $f(x) = ax^2 + bx + c$ can be written as $f(x) = a(x - p)^2 + q$ for some numbers p and q by the method of completing the square.

(i) Show that if the discriminant $b^2 - 4ac$ is negative f has no zero, if the discriminant is positive f has two distinct zeros, and if the discriminant is 0 f has a double zero.

(ii) Use the quadratic formula to find the zeroes of f.

(iii) Read off the maximum or minimum of f.

(iv) Interpret items (i) to (iii) above in terms of a context.

(v) Prove that if r_1 and r_2 are the zeros of $f(x) = a(x - p)^2 + q$, then $f(x) = a(x - r_1)(x - r_2)$, and conclude that all quadratic polynomials which have a real zero can be factored by using the quadratic formula.

(vi) Discuss how the three expressions of f, $ax^2 + bx + c$, $a(x - p)^2 + q$, and $a(x - r_1)(x - r_2)$ reveal different properties of f.

(vii) Show the symmetry of the graph of f with respect to the vertical line $x = p$, and conclude in general that the graph of f is the translation of the graph of $y = ax^2$.

$(viii)$ Know the definition of a parabola as any curve in the plane similar to the graph of $y = x^2$. Prove that the graph of every quadratic function is a parabola, and exhibit a parabola that is not the graph of a quadratic function.

Sequences

1. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. For example, the Fibonacci sequence is defined

recursively by $F(0) = F(1) = 1, \ \ F(n+1) = F(n) + F(n-1)$ for $n \geq 1$.
2. Solve linear homogeneous recurrence relations of order 2 (with constant coefficients).
3. Prove the closed form expressions of the Fibonacci numbers, and use them to show intuitively the relation of Fibonacci numbers with the golden ratio.

Geometry

Define trigonometric ratios and solve problems involving right triangles

1. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
2. Explain and use the relationship between the sine and cosine of complementary angles.
3. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

Circles

1. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.
2. Prove properties of angles for a quadrilateral inscribed in a circle.

Make geometric constructions

3. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.

4. Construct the inscribed and circumscribed circles of a triangle.
5. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.
6. Construct a tangent line from a point outside a given circle to the circle.

Spherical geometry

1. Replace the plane by a fixed sphere S in Euclidean 3-Space around the origin, and let "lines" on S be the "great circles", which are the intersections of S with planes that go through the origin. Show that two points on S no longer determine a line, and that two distinct lines can intersect at more than one point.
2. Proves that there are no parallel lines on S.
3. Verify experimentally that the shortest curve between two points is part of a line (= great circle).
4. On a globe, verify that flights from the West coast to Europe necessarily get close to the arctic circle.
5. Let the angle between two lines at a point P be the angle of their tangent lines at P. Verify experimentally that the sum of the angles of a triangle on S is at least 180 degrees, and can exceed 180 degrees.
6. Verify experimentally that the Pythagorean Theorem does not hold on S; in general, $c^2 \neq a^2 + b^2$ for a right triangle on the sphere.

Grade 11

Exponential and Logarithmic Functions

Exponential functions

1. Explain how the definition of the rational exponents of a positive number follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $5^{(1/3)3}$ must equal 5.

2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

3. Know and use the properties of exponents.

   a. Know the laws of exponents, and know the proofs of simple cases, e.g., $a^{1/n}b^{1/n} = (ab)^{1/n}$.

   b. Know how to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.

4. Know that in terms of an exponential functions $f(x) = a^x$, a law of exponent states: $f(s+t) = f(s)f(t)$. Discuss intuitively the fact that f is increasing if $a > 1$, decreasing if $a < 1$, and that a can be any positive real number.

5. Distinguish between situations that can be modeled with linear functions and with exponential functions.

   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

d. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

6. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.
7. Discuss intuitively the concept of inverse functions, and that each function a^x for $a \neq 1$ and $a > 0$ has an inverse function called the logarithmic function with base a, denoted by $\log_a x$.

a. Show the graph of exponential and logarithmic functions, show intercepts and end behavior.

b. Discuss the use of logarithm in real life, such as the Richter scale to measure the energy of an earthquake.

8. Discuss basic properties of the logarithmic function, $\log xy = \log x + \log y$, and the historic role it played in the technology of the past before the advent of calculators.

Formal Algebra

Complex numbers

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
3. Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers. Represent complex numbers and their operations on the complex plane.
4. Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for

computation. For example, $(1 + \sqrt{3}i)^3 = 8$ because $(1 + \sqrt{3}i)$ has modulus 2 and argument 120°.
6. Calculate the distance between numbers in the complex plane as the modulus of the difference.

Use complex numbers in polynomial identities and equations

7. Solve quadratic equations with real coefficients that have complex solutions.

> *a*. Verify that the quadratic formula now holds in general.
>
> *b*. Prove that if r_1 and r_2 are the zeros of $f(x) = a(x-p)^2+q$, then $f(x) = a(x - r_1)(x - r_2)$. Thus the quadratic formula leads to a factorization of all quadratic polynomials with real coefficients.

8. Know the Fundamental Theorem of Algebra, and verify it in the case of quadratic polynomials with real coefficients.

The Binomial Theorem

9. Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascals Triangle.

Geometric measurement and dimension

Explain volume formulas and use them to solve problems

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.
2. Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

Basic Statistics

Summarize, represent, and interpret data on a single count or measurement variable

1. Represent data with plots on the real number line (dot plots, histograms, and box plots).
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).
4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

Summarize, represent, and interpret data on two categorical and quantitative variables

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.

    a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.

    b. Informally assess the fit of a function by plotting and analyzing residuals.

c. Fit a linear function for a scatter plot that suggests a linear association.

Interpret linear models

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.
8. Compute (using technology) and interpret the correlation coefficient of a linear fit.
9. Distinguish between correlation and causation.

Understand and evaluate random processes underlying statistical experiments

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population.
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?

Make inferences and justify conclusions from sample surveys, experiments, and observational studies

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.
6. Evaluate reports based on data.