The 1997 mathematics standards war in California

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Setting the stage

The controversy discussed in this article has its origin in the 1992 Mathematics Framework for California Public Schools ("the 1992 Framework")[1]. Published three years after the Curriculum and Evaluation Standards of the National Council of Teachers of Mathematics ("the NCTM Standards")[2], the 1992 Framework has come to symbolize the more extreme of the practices of the mathematics education reform initiated by the NCTM document. Its overemphasis on pedagogy at the expense of mathematical content knowledge spawned several new curricula that so enraged the parents of school children that grassroots revolt against the reform within California spread like wildfire in a hot summer day. Eventually, the Californian legislature bowed to the inevitable and called for an earlier than expected writing of a new Framework. It was also decided for the first time in the history of California that there should be a set of mathematics standards. By law, these standards have to be set at a level comparable with the best in the world. The California Academic Standards Commission ("the Commission") was created by the California legislature for the purpose of writing the standards.

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In October, 1997, the Standards Commission submitted to the State Board of Education ("the Board") a draft of Mathematics Content Standards which took the Commission more than a year to complete[3]. Under normal circumstances, the Board would approve such a document with no more than minor changes. However, a preliminary review of the Commission's Standards by several well-known research mathematicians was so devastating that the Board broke precedent by commissioning a group of Stanford University mathematics professors to revamp the Commission's draft. Within ten weeks, the Board issued the revised standards, first the portion on grades K–7 and then that on 8-12[4]. The reaction to the new version was swift and violent:

> *"[The Board's set of Standards] is 'dumbed down' and is unlikely to elicit higher order thinking from the state's 5.5 million public school students."*
>
> Delaine Eastin, as reported in the NY Times
>
> *"I will fight to see that California Math Standards are not implemented in the classrooms."*
>
> Judy Codding, as quoted at an NCEE conference
>
> *"The critics claimed the Board's 'back-to-basics' approach marked a return to 1950s-style methods. . . . Opponents characterized the [Board's] Standards as a 'return to the Dark Ages'".*
>
> as reported in the San Diego Union

The interest engendered by these two sets of Standards remained unabated in the ensuing months. For example, in the February 1998 issue of its *News Bulletin*, NCTM weighed in with unflattering comments about the Board's Standards[5]. Because education is a very political issue, it is expected that opinions would be delivered without relation to facts. However, a set of *mathematics* standards for schools also deserves a critical inspection from a mathematical as well as an educational perspectives, one that is based on facts and not on hyperbole. With this in mind, this article proposes to take a close look at both sets of standards from a scholar's perspective. Section 2 gives an overview of the comparison between the two versions. Section 3 details some of the mathematical flaws in the Commission's Standards, and

Section 4 contrasts these flaws with the clarity and the overall mathematical soundness of the Board's revision.

Section 5 discusses the problems that were in the Board's Standards and indicates whether and how these were addressed in the new California Mathematics Framework[6] that was approved in December 1998. Should the appearance of the new Framework in a discussion of the Standards come as a surprise, let it be noted that, in the state of California, the Standards were not designed to stand by themselves. Rather, the Standards and the Framework are required by law to function as a single unit in providing a blueprint for California's mathematics education.

The final section summarizes what we have learned from this battle.

It may be asked why a dispute within the state of California should be of interest to the rest of the nation. There are several reasons. One is that California, the most populous state of the nation, has been in the forefront of the current mathematics education reform. In the opinion of people outside California, "as California goes, so goes the nation". A second reason is that the inadequacy of mathematics education in this country has held a firm grip on the public's attention in the recent past, and the very visible act of groping for a solution in California adds enormously to cement this grip. Also the timing is just right: the Californian squabble came right in the midst of the revelation about the nonperformance of American students in the Third International Mathematics and Science Study (TIMSS). The Californian situation thus offers other states a glimpse of what to look forward to in their own battles to save mathematics education.

A third reason that makes the dispute between the two sets of Standards interesting to other states lies in the very nature of the dispute. The pervasive lack of precision in the Commission's Standards is symptomatic of a larger trend in contemporary mathematics education, which is to minimize the technical precision inherent in the discipline in order to make it more accessible to a wider audience. Would a state be fulfilling its basic mission in mathematics education if it promotes such a *modified version of mathematics* in order to reach out to a higher percentage of its students? California answered this question in the negative in 1998, when the Board's revision restored the precision and the technical materials. This very likely insures California's ability to continue to produce a robust corps of scientists and engineers. Whether or not it will also succeed in raising the mathematical achievements of the lower 50% of its students will be a matter of intense public interest. Other states would do well to look at the details of this dispute before they too fight the same battle in the near future.

The fourth and final reason is perhaps the most important because of its enormous practical consequences: the Board's Standards will generate a completely new set of mathematics textbooks by 2001. According to the latest information (Decemeber 1998) from the California Department of Education, there is more publisher interest, *by far*, in this new round of textbook adoption than in any other that has ever been conducted in this state. This means that by 2001, the nation will have a plethora of alternative textbooks that heed the call, not of the NCTM Standards, but of the California Standards and Mathematics Framework. The debate within California will become a national debate in a matter of two years.

An overview of the two sets of Standards

No critical inspection of the two sets of standards, the Commission's and the Board's, should engage in hairsplitting in order to search for perfection. Social documents generally do not fare too well when subjected to this kind of treatment. We expect flaws in both versions, and we shall find flaws aplenty. Yet, there is a fundamental difference between their flaws, and it is this difference that is our major concern.

The Commission's Standards is a thoughtful document. In both the Interim Report from the Commission Chair to the State Board and the Introduction to Mathematics Standards, one sees clearly the care that went into the enunciation of the goals, the work that had been done to achieve them, and the work that was envisioned in their implementation. Even if one disagrees with some of the details, one can applaud the overall soundness of purpose and the conscientious effort that went into the writing. The good intentions, however, are not abetted by flawless execution. Some parts of the document are controversial, such as the omission of the division algorithm in the lower grades[7], the omission of the Fundamental Theorem of Algebra in the upper grades, or the mixing of pedagogical statements with statements on content. There is also a pervasive ambiguity of language that makes the document unreadable in many places, e.g., the word "classify" has a precise meaning in a mathematical context which is not consistently respected. Or, what is a 7th grade teacher to make of "identify, describe, represent, extend and create linear and nonlinear number patterns"? But the most striking impression it makes on a mathematically knowledgeable reader would likely be the numerous mathematical errors that almost leap out of the pages.

The Board's Standards do not suffer from mathematical errors[8]. Mathematical accuracy thus assured, one can proceed to find fault from a higher perspective. It should be fairly obvious to the experienced eyes that the standards for each grade are not very "idiomatic": they are more like marching orders from an outsider than sure-handed utterances by a veteran of the classroom. There are occcasional (though very rare) linguistic ambiguities. There is an over-emphasis on pure mathematics in grades 8-12. The geometry curriculum in grades 8-12 is too much tilted towards synthetic Euclidean geometry. And so on. But perhaps the one quality of the document that stands out is its overall jaggedness; the various standards don't fit together too well. Is there an obvious explanation?

According to James Milgram—one of the Stanford mathematicians who helped revise the Commission's Standards—the revision was carried out under many constraints. The goals had been set for them: to rid the Commission's Standards of all the mathematical errors, to re-arrange the existing standards to make better sense of them and, above all, to clarify what was in there. There was also a strict order not to add anything new unless it was *absolutely* necessary, because the Board itself was under the same pressure. Milgram also added that, in fact, the Stanford mathematicians would not have minded if the Standards were a little less inclusive, but the choice of deleting the existing standards was not open to them or to the Board.

Retrofitting a set of standards is much more difficult than writing a new one, and it showed.

In spite of the controversy surrounding these two Standards, the verdict among mathematicians has been overwhelmingly in favor of the Board's version[9]. Could this be no more than a case of closing of ranks behind their own colleagues? At least one mathematician would venture the opinion that such is not the case, and that it is more a matter of triumph of substance over form, and clarity over vagueness.

The Board's Standards have the unmistakable virtue of being clear, precise, and mathematically sound overall. They describe clearly and precisely what is expected of students' mathematical achievement at each level, and the mathematical demands thus imposed conform to the conception of mathematics of most active working mathematicians. The qualities of clarity, precision, and correctness are the sine qua non of any mathematical standards worthy of the name, but the sad truth is that very few of the existing mathematics standards of other states can lay claim to any of them[10]. Even the influential NCTM Standards[2] is no exception[11]. These qualities will come to the fore in the comparison between the Commission's and the Board's Standards in the next two sections. At the end, there should be little mystery as to why, notwithstanding its flaws, the Board's Standards is the preferred version by far.

It may be puzzling to some as to why there is this great emphasis on the soundness of the mathematics in a set of mathematics standards. No doubt part of the puzzlement comes from a belief that the experts in mathematics education should be able to get the mathematics right. We know that such is not the case—as evidenced by the discussion of the Commission's Standards in the next section—and this discrepancy between perception and reality points to a serious problem in contemporary mathematics education: the divorce of mathematics from education. Too often mathematics educators and administrators lose touch with mathematics. Perhaps the publication of the Board's Standards and the publicity engendered by the accompanying fracas will inaugurate a new era of reconciliation between the two disciplines.

The large number of mathematical errors in the Commission's Standards also point to an intellectual problem far removed from the political fray. As the errors begin to pile up, they send out the unmistakable message that these standards were written by people whose mathematical understanding is inadequate for the task, and whose vision is therefore unreliable as a guide to lead students of California to a higher level of mathematical achievement. Such being the case, the so-called "conceptual understanding"[12] embedded in this document is thus of questionable value at best. The politicians and educators who rallied around the Commission's Standards and praised it for its emphasis on "conceptual understanding" were most likely unaware of this fact.

The Commission's Standards: its mathematical flaws

Local flaws

The mathematical flaws in the Commission's Standards are of two kinds: local and global. The *local* ones are those which contain obvious errors but which can be corrected without causing damage elsewhere. A colleague has estimated that there are over a hundred of these, and that is a conservative estimate. Since it is impossible to be exhaustive, only a few that are easily understood even when taken out of context will be discussed. Starting with the Glossary at the end, one finds, for example:

> Asymptote*: a straight line to which a curve gets closer and*

closer but never meets, as the distance from the origin increases

Since this definition of an asymptote does not specify that the distance between the curve and the straight line has to decrease to zero, it would make the line $y = -1$ an asymptote of $y = 1/x$ for $x > 0$.

> Axiomatic system*: system that includes self-evident truths: truths without proof and from which further statements, or theorem, can be derived*

By dictating that the "axioms" of an axiomatic system must be "self-evident truths", this definition excludes the axioms for non-Euclidean geometry from being an axiomatic system. After all, the statement that given a line and a point not on the line there are infinitely many lines from the point not intersecting the given line is certainly not a "self-evident truth".

> Recursive function*: in discrete mathematics, a series of numbers in which values are derived by applying a formula to the previous value*

This term has a precise technical meaning in symbolic logic, and its definition is nothing this simple. Perhaps the authors had in mind "recurrence relations" instead. Assuming this to be the case, then the correct definition would change "the previous value" to "previous values". Otherwise, even the Fibonacci numbers would not fit this description.

It has been argued by some people that it is inappropriate to criticize the Glossary with such mathematical precision because phrases such as "closer and closer" and "self-evident truths" are merely intended to be comprehensible to the layman in the same way a dictionary definition is. This kind of argument does not take into account the fact that an official state document on *mathematics* education which addresses not only the lay citizen but also the professional—the mathematics teacher—has the duty to aim higher than merely being informally correct. Moreover, for the case at hand, it is easy to be both informal *and* correct: just change "closer and closer" to "arbitrarily close" and "self-evident truth" to "statements to be taken on faith".

Next, we turn to the Commission's Standards proper and look there at some representative local flaws. It may be noted that the following examples do not include any that might have been the result of carelessness, such as that about the asymptotes of a polynomial (*Clarification and Examples*[13] for Standards 1.1 and 1.2 in Algebra and Functions of grades 11/12).

> **Grades K-8** *Problem Solving and Mathematical Reasoning*
> **2.1** *Predict outcomes and make reasonable estimates.*

It is not common to equate "predict outcomes" with mathematical reasoning. Could the authors have in mind "Make conjectures and provide justification"? One would gladly overlook this as an inadvertent error but for the fact that the same sentence appears nine times in grades K through 8

> **Grade 3** *Problem Solving and Mathematical Reasoning*
> Clarifications and Examples *Your friend in another classroom says that her classroom is "bigger" than yours. Find the answer, and prove that your solution is correct using mathematics you have learned this year. (Note: Students should be able to approach this task using concepts of perimeter and area.)*

This passage is supposed to *clarify* the content of the Standards, but it has achieved the opposite effect by obfuscating it. It would take many pages to write an analysis that does this passage justice, so here is a very abbreviated account[14]. First of all, mathematics deals with precise statements, and to the extent that we try to educate our children about mathematics, we would do well to teach them the necessity of eliminating the inherent vagueness in many everyday utterances before transcribing them into mathematical terms. "Her classroom is bigger" is clearly a case in point. Faced with such a statement, a set of mathematics standards has the responsibility to instruct children of grade 3 to make sense of the word "bigger" before proceeding any further. If they *interpret* "bigger" to mean "more area", then they should measure the respective areas. If they *interpret* "bigger" to mean "longer perimeter", then measure the perimeters. The basic message is therefore that each answer would be correct according to whichever interpretation is used. Furnishing such an explanation would seem to be the minimum requirement of a mathematics education for the young. Now look at the passage above: it tells teachers and students alike to accept an instruction that has no precise meaning ("bigger") and immediately proceed to "find the answer", and worse, "prove that your solution is correct using mathematics". If a teacher in an English class shows students a black box without telling them what is inside other than that it is an expensive piece of jewelry, and asks them to write an essay to describe the latter and *justify why their description fits the object*, there would be an uproar. Yet when the same thing happens in a set of mathematics standards, we have people leaping to its defense and calling it "world class". Why is that?

> **Grade 4** *Measurement and Geometry*
> **1.** *Students understand and use the relationship between the concepts of perimeter and area, and relate these to their respective formulas.*
> **Grade 5** *Measurement and Geometry*
> **1.** *Students understand the relationship between the concepts of volume and surface area and use this understanding to solve problems.*

The trouble with both standards is that there is no general relationship between perimeter and area, or between volume and surface area, except for the isoperimetric inequality. However, the latter would be quite inappropriate for students at this level. Moreover, how are students supposed to "use" this nonexistent relationship to help relate to the "respective formulas" of perimeter and area, etc.? What could the authors have in mind?

> **Grade 5** *Number Sense*
> Clarifications and Examples *What is the fractional value of each of the tangram pieces to the whole set of tangrams? Determine equivalences between one or more pieces and other pieces, based on the fractional values that you have determined.*

What does "fractional value" refer to in this case? Does each piece count as one unit, or is the area of each piece being sought in proportion to the whole area? What kind of "equivalence" between the pieces is intended here and why has it not been clearly defined?

> **Grade 6** *Number Sense*
> Clarifications and Examples *Emphasize how fractions and ratios as well as operations involving them are similar and how they can differ.*

Since a fraction is a ratio of integers, how can there be any difference between them with respect to their mathematical operations? Some educators, it is said, have begun to advocate that fractions are not ratios. If so, then we must redouble our efforts to produce better informed mathematics educators and not allow such ideas to creep into any mathematics standards.

> **Grade 6** *Measurement and Geometry*
> **1.2** *Determine estimates of* π (3.14*;* 22/7) *and use these values to estimate and calculate the circumference and the area of circles.*

There is no explanation of how a 6th grade student could "determine estimates of π" with this kind of accuracy, 3.14 or 22/7, especially the latter value. Is such a precise estimate even remotely conceivable?

> **Grade 7** *Algebra and Functions*
> Clarifications and Examples *Order of operations may be helpful when evaluating expressions such as* $3(2x+5)^2$*, recognizing the structure of the algebraic notation may be more helpful when evaluating* $3(2x+5)$*; both should be included as techniques.*

The order of operations to evaluate algebraic expressions is a matter of *definition*, and is not a technique. Moreover, to say in a mathematics standards document that knowing the simple definition of the notation is more helpful in the situation of $3(2x+5)$ than in $3(2x+5)^2$ is to undercut its own credibility.

> **Grade 8** *Algebra and Functions*
> Clarifications and Examples *Record and graph the relationship between time and the height of water in a cylindrical container when a drain on the bottom of the container is open and determine an equation which generalizes the situation.*

One would *guess* (although that is asking a lot of the general reader of the Standards) that the "relationship between the time and the height of water" is that the height is a well defined function of time. This function happens to be quadratic, but what could it mean to find an equation that would *generalize the situation*?

> **Grades 9/10** *Algebra and Functions*
> Clarifications and Examples *Students should understand the fact that equations in one variable (e.g.,* $(x-3)(x+1)(x-1)=0$*,* $3^{x-2}-81=0$*) have related two-variable counterparts (e.g.,* $(x-3)(x+1)(x-1)=y$*,* $3^{x-2}-81=y$*) and use this fact to solve or check the original equation and analyze the graph.*

If the intended message is that the zeros of a given polynomial can be approximated by examining the intersection of its graph and the x-axis, then this statement is very poorly phrased. If the intended message is something else, then obviously this statement needs to be completely re-written.

Global flaws

Next we examine the global flaws. Their corrections would involve changes in several related parts. The first such example occurs in grade 7:

> **Grade 7** *Measurement and Geometry*
> **3.2** *understand and use coordinate graphs to plot simple figures, determine lengths and areas related to them, and determine their image under simple transformations in the plane*

Now Grade 7 is not the usual place to find references to *simple transformations in the plane* and their *images*. What is meant by a "simple transformations"? Has it been defined? Has the image of a transformation been discussed? It turns out that "simple transformations" are defined nowhere in the Standards, but one could guess from related comments that the authors had in mind *reflections* and *translations*. It is difficult to decide whether the authors were unaware of the need to fulfill the minimum mathematical requirement of clarity or simply considered such matters unimportant. Could such negligence be nothing more than a momentary lapse? Not likely, because one also finds in grades 9/10 another reference to "transformations" in the plane with no explanation:

> **Grades 9/10** *Algebra and Functions*
> **1.** *Students classify and identify attributes of basic families of functions (linear, quadratic, power, exponential, absolute value, simple polynomial, rational, and radical).*
> **1.4** *demonstrate and explain the effect that transformations have on both the equation and graph of a function*

A pertinent related issue in connection with the above standard in grade 7 is how much coordinate geometry has been developed up to that point so that students may appreciate such a discussion. The answer appears to be "not enough". The first introduction of coordinates in the plane takes place in the 4th grade under, of all places, Algebra and Functions:

> **Grade 4** *Algebra and Functions*
> **1.** *Students use and interpret variables, mathematical symbols and properties to write and simplify expressions and sentences.*
> **1.4** *understand and use two-dimensional coordinate grids to find locations, and represent points and simple figures*

Special attention should be called to the fact that the important idea of "algebraicizing" the geometric plane occurs here almost as an after-thought in a discussion on variables and mathematical symbols. Since it is now fashionable to talk about "conceptual understanding", one can say unequivocally that such a set of mathematical standards displays a lack of conceptual understanding of mathematics. But, to continue with the present discussion, in the standards of grade 5, one finds "write the [linear] equation and graph the resulting ordered pairs of whole numbers on a grid" (*Algebra and Functions*, 2.2) and that, in grade 6, more graphing of linear functions and "single variable data" is called for in *Algebra and Functions* and *Statistics, Data Analysis and Probability*. This would seem to be the extent to which students have been exposed to coordinate geometry before they are asked to contemplate the image of a transformation in the plane.

Consider now a second example, which is the way the Commission's Standards approaches the Pythagorean theorems, a fundamental result in school mathematics. The first mention of this theorem is in grade 7:

> **Grade 7** *Measurement and Geometry*
> **3.3** *use the Pythagorean Theorem to find the length of the missing side of a right triangle and lengths of other line segments, and check the reasonableness of answers found in other ways.*
> Clarifications and Examples *Help students understand the relationship between the Pythagorean Theorem and direct measurement. Experience with both measurement tools and measurement on a coordinate grid should be included.*

(One's first reaction to the last sentence—"*Experience* with …"—is: "How very Californian!") This standard certainly makes it sound as though the Pythagorean theorem is a tool already familiar to the students. Yet, in fact, this is the first time the theorem is discussed. One could bend over backward to give a benign interpretation of this standard as: "State the Pythagorean theorem and verify it empirically by direct measurements". Few readers, however, would recognize that this is the intended message. Because this theorem is so surprising to a beginner, one would expect a demonstration of its truth early on. For example, the so-called "tangram proof" using four congruent right triangles nestled in a square is so elementary that it could be presented to 4th or 5th graders. One finds instead that when the theorem is mentioned again in grade 8 and for the first time in grades 9/10, no proof is mentioned:

Grade 8 *Measurement and Geometry*
1.3 *use the Pythagorean Theorem to determine distance and compare lengths of segments on a coordinate plane.*
Clarifications and Examples Include using the Pythagorean Theorem to confirm accuracy of scale drawings, and contexts involving coordinate graphing.

Grades 9/10 *Measurement and Geometry*
2.4 *use the Pythagorean Theorem, its converse, properties of special right triangles (e.g., sides in the ratio 3-4-5) and right triangle trigonometry to find missing information about triangles.*

It should be obvious that this standard in grade 8 merely repeats what is already in grade 7. What purpose does this serve? In addition, is it good education to ask students to believe in the converse of this theorem in grades 9/10, as indicated above, without first giving them a proof of the theorem itself? It remains to point out that only later in sub-Standard 4.4 of Measurement and Geometry, grades 9/10, do we find: "prove the Pythagorean Theorem using algebraic and geometric arguments".

It was mentioned earlier that the Commission's Standards omits the long division algorithm in the early grades except for the case of a single digit divisor (grade 4). With that in mind, let us look at what happens in grade 7.

Grade 7 *Number Sense*
1.3 *describe the equivalent relationship among representations of rational numbers (fractions, decimals and percents) and use these representations in estimation, computation and applications.*
Clarifications and Examples *Students should understand the relationship between terminating and repeating decimals and fractions.*

Yet the mere fact that a fraction yields a repeating decimal depends on the understanding of the sequence of remainders in the division algorithm. How are students going to *understand* that terminating and repeating decimals represent fractions without first knowing this algorithm by heart? Furthermore, in grades 11/12, we have:

Algebra and Functions
Clarifications and Examples *Graphing calculators, long and syn-*

> *thetic division may be used to factor polynomials and rational equations to verify attributes of the equation and graph.*

Perhaps not enough thought was given to the fact that, without learning the division algorithm for integers, it would be difficult to teach synthetic division for polynomials.

Incidentally, the preceding two examples from the Commission's Standards show an all-too-common sloppiness of language: "equivalent relationship among ...", "relationship between terminating and repeating decimals ...", and "attributes of the equation and graph" are too vague for a set of mathematics standards.

As a final example, let us look at how the Commission's Standards handles the concept of a function. The term "functional relationship" is used already in grades 4 and 5 ("the functional relationships within linear patterns" in grade 4, and "solve problems involving functional relationships" in grade 5). Now, there is nothing wrong with an informal discussion of a formal concept before a precise definition, but it is pedagogically untenable not to make it very clear that only an informal discussion is intended. (The Board's Standards simply deletes all such references.) Next, in grade 6 of Commission's Standards, one finds:

> **Grade 6** *Algebra and Functions*
> **2.** *Students analyze tables, graphs and rules to determine functional relationships and interpret, and solve problems involving rates.*
> **2.1** *identify and express functional relationships in verbal, numeric, graphical and symbolic form.*

Since it calls for a direct confrontation with the concept of a function itself, this standard is less likely to be ignored and the potential damage is consequently greater than before. Are students to learn about the definition of a function, or are they not? That is the question. The hazy conception of mathematics itself as exemplified in this instance (and elsewhere too, of course) is unnerving to the mathematically informed. If one cannot resolve this issue here, what about the next one in grade 8?

> **Grade 8** *Algebra and Functions*
> **1.1** *identify the input and output in a relationship between two variables and determine whether the relationship is a function.*

> *Clarifications and Examples Students should be able to identify key ideas when a relationship is expressed through a table, with symbols, or through a graph.*

Because this explicitly asks students to distinguish between a *relation* and a *function*, nothing short of a full-scale investigation of the functional concept would suffice. But should one do this in grade 8? And is this really what the authors had in mind? The answers seem to be supplied, however indirectly, by the following standard in grade 9:

> **Grades 9/10** *Algebra and Functions*
> **2.** *Students demonstrate understanding of the concept of a function, identify its attributes, and determine the results of operations performed on functions.*

It would appear that here is the first time that students learn what a function is. If this is to be believed, then what is one to make of all the rumblings on this topics in grades 6 through 8? But if not, *i.e.,* if a function is supposed to have been defined earlier, then what is such a standard doing in grades 9/10?

I hope the foregoing gives some idea of the magnitude of the problems besetting the Commission's Standards. At the same time, it should be pointed out that these problems are probably not detectable by someone who is not mathematically knowledgeable. The criticisms of the Board's Standards coming from educators and politicians are therefore understandable to a certain degree. By the same token, this gap in mathematical knowledge then imposes on professional mathematicians the obligation to serve as intermediaries between major decisons in mathematics education and the public. May the mathematics community as a whole take this responsibility seriously.

The Board's Standards

Now a brief look at the Board's Standards[4]. Some of the flaws of this document will be discussed in the next section. The main aim of this section is to contrast the mathematics here with the Commission's Standards. Let us first start with grades K–7. This portion is *very* close to the Commission's Standards, and the only difference between the two is that the Board's version eliminates the ambiguous and superfluous, corrects the erroneous, and deletes the *Clarifications and Examples* in the right column of the original. I will have more to say about this last concern presently, but let us sample some of the differences. It was mentioned above that in grade 4, the Commission's Standards incorrectly asks for "the relationship between the concepts of perimeter and area". By comparison, the Board's version now reads:

> **Grade 4** *Measurement and Geometry*
> **1.** *Students understand perimeter and area.*
> **1.1** *measure the area of rectangular shapes, using appropriate units (cm^2, m^2, km^2, yd^2, square mile)*
> **1.2** *recognize that rectangles having the same area can have different perimeters*
> **1.3** *understand that the same number can be the perimeter of different rectangles, each having a different area*
> **1.4** *understand and use formulas to solve problems involving perimeters and areas of rectangles and squares. Use these formulas to find areas of more complex figures by dividing them into parts with these basic shapes*

It is clear, and it is correct. More than that, 1.2 and 1.3 anticipate students' possible confusion, and 1.4 emphasizes the importance of applications and the general principle of progressing from the simple to the complex.

Another example is the Board's correction of the error committed in the Commission's version regarding the introduction of coordinates in the plane in grade 4. Now it is accorded a standard all its own and is placed correctly in the strand on Measurement and Geometry.

> **Grade 4** *Measurement and Geometry*
> **2.** *Students use two-dimensional coordinate grids to represent points and graph lines and simple figures*
> **2.1** *draw the points corresponding to linear relationships on graph paper (e.g., draw the first ten points*[15] *for the equation $y = 3x$ and connect them using a straight line)*
> **2.2** *understand that the length of a horizontal line segment equals the difference of the x-coordinates*
> **2.3** *understand that the length of a vertical line segment equals the difference of the y-coordinates*

Note that sub-standard 2.1 pays special attention to the *tactile* aspect of learning mathematics: use graph papers and draw ten points (by hand). We should be grateful that it does not say: enter these data in a graphing calculator and watch the graph emerge on the screen. Moreover, sub-standards 2.2

and 2.3 again anticipate students' confusion by singling out two key points for discussion. There is no question that this is an education document that truly tries to educate.

As a final example, let us look at how the Board's version discusses in one instance the issue of mathematical reasoning:

> **Grade 4** *Mathematical Reasoning*
> **3.** *Students move beyond a particular problem by generalizing to other situations.*
> **3.1** *evaluate the reasonableness of the solution in the context of the original situation.*
> **3.2** *note method of deriving the solution and demonstrate conceptual understanding of the derivation by solving similar problems.*
> **3.3** *develop generalization of the results obtained and extend them to other circumstances.*

In plain English—readable English—this standard lays out a step-by-step method of doing mathematics. Educational writing can be no better than this.

It is improvements of this nature that make the Board's Standards a superior document over the Commission's Standards in grades K–7. Yet, intense criticisms were already pouring in as soon as the K–7 portion of the Board's Standards appeared. Looking at the facts, how does one presume to claim that this set of standards is "basics only", or that it "almost cuts out almost everything that is not related to computation and the memorization of formulas" ? Obviously not on account of the standards themselves. But one explanation is that some people reacted strongly to the deletion of the *Clarifications and Examples* that are in the Commission's Standards.

It was pointed out earlier that whereas in other states the Mathematics Standards must stand alone as the sole guide-post for mathematics education, California has two documents: the Standards *and* the Framework[6]. In this arrangement, the curricular comments on the Standards, including examples, properly belong to the Framework, which at the time of the controversy was yet to be approved by the Board. It serves no purpose to criticize the absence of examples in the Board's Standards when they have merely been moved to a companion document.

Let us complete our brief survey of the Board's Standards by looking at grades 8–12. There is a basic change of format here, in that the grade-by-grade account in the Commission's version is replaced by a listing of topics in the traditional strands across the grades: Algebra I, Geometry, Algebra II, etc. The justification is that since at present an overwhelming majority of the schools teach mathematics in the traditional manner while others do so in an "integrated"[16] manner, listing only the content of each subject would provide maximum flexibility. Instead of prescribing one particular approach to the curriculum, it throws the door open to many approaches. Such a change is a defensible one, and is in any case not one to make a lot of fuss about. With this understood, one can immediately appreciate the clear and uncompromising demand that the Board's Standards places on students' all-around mathematical competence—not the formula-laden, rote-learning variety, but the genuine one. Students must be technically proficient, and they must also know what they are doing. For example, consider the discussion of the quadratic formula in Algebra I (which contains twenty-five standards):

> **Algebra I** *(Grades 8-12)*
> **14.** *Students solve a quadratic equation by factoring or completing the square.*
> **19.** *Students know the quadratic formula and are familiar with its proof by completing the square.*
> **20.** *Students use the quadratic formula to find the roots of a second degree polynomial and to solve quadratic equations.*

It does not say: derive the quadratic formula and use it to solve all quadratic equations. Instead, it makes students learn the important technique of completing the square first. Then it asks for a derivation of the formula. It is only after this that it mentions *using* the formula to solve equations. Does a document that handles the learning of a formula in this manner strike anyone as a "back-to-basics" document that emphasizes memorization and computation? Next, a similar example in a different subject:

> **Geometry** *(Grades 8-12)*
> **2.** *Students write geometric proofs, including proofs by contradiction.*
> **3.** *Students construct and judge the validity of a logical argument. This includes giving counterexamples to disprove a statement.*
> **4.** *Students prove basic theorems involving congruence and similarity.*

The unequivocal demand on students' ability to write down proofs and counterexamples is important in this day and age of diminished standards when proofs produce allergic reactions in many education circles. One can quibble with the precise meaning of standard 4—and more of this later—but that is not the same as insinuating that these Standards axe the development of mathematical understanding in the students. My personal opinion is that these are thoughtful standards, but their virtues are by no means apparent to the general public. Perhaps for this reason, the torrent of abuse heaped on these Standards took over the front pages of many newspapers for several weeks. Here are some reminders:

> *"I think [the Board's Standards are] half a loaf. We went from a world-class set of standards to one that cannot be characterized as world-class"*
> *"The reality is one set of standards had basics* and *problem-solving and conceptual understanding but what the Board adopted was the basics only."*

Delaine Eastin
Superintendent of Public Instruction

> *"When the State Board took a knife to the Commission's Standards, it cut out almost everything that was not related to computation and the memorization of formulas. What was gained? Nothing.... What the State Board deleted or weakened were Standards intended to make sure students understand the key concepts underlying mathematics."*

Judy Codding
Member, Academic Standards Commission

> *"While emphasizing important basics and memorization, [the Board's set of Standards] axes development of understanding, applications and critical thinking skills students will need to live in the 21th century.*
>
> *In one stroke, the Board discards the last three years' hard work and reasoned consensus among math professors and teachers, college professors who use math in their teaching (science and business) and public representatives."*

James Highsmith
Chair, Academic Senate; California State University
Letter to the editor, LA Times

"The Commission's Standards are the best set of mathematics standards in the U.S. ... The Board's Standards are most disappointing, [and are nothing more than] a 'back-to-basics' document that emphasizes memorization and computations."

William Schmidt
Executive Director for the U.S. Center for TIMSS

"The wistful or nostalgic 'back-to-basics' approach that characterizes the Board Standards overlooks the fact that the approach has chronically and dismally failed. It has excluded youngsters from engaging in genuine mathematical thinking and therefore true mathematical learning."

Luther Williams
NSF Director for Education and Human Resources
Letter to the California State Board

It may be noted that the NCTM editorial of February, 1998[5] endorsed the preceding statement by Luther Williams, and that none of the preceding writers is a mathematician.

One may ask, in light of all the flaws in the Commission's Standards and the obvious emphasis on mathematical understanding in the Board's version, how people could bring themselves to make indefensible statements about the high quality of the former and the unworthiness of the latter. There are probably political and psychological reasons that are beyond my power to probe. But as an educator, I would like to offer a speculation on how this has happened. I believe there is a fundamental misconception about mathematics education that has sprung up more or less in the past decade, which is that there are *conceptual understanding* and *problem solving ability* on the one hand and *basic skills* on the other. Furthermore, this misconception is based on the assumption that one can acquire the former without the latter. Thus when the Board's Standards explicitly call for fluency in basic skills, all kinds of red flags went up. Were these Standards not set up by elitists to thwart students' "mathematical empowerment"?

One *can* acquire some appreciation of mathematics without mastering technical skills, in much the same way that one can learn the main melodies of an opera by listening to recordings of "operas without the human voice"[17] and even enjoy them to some extent. But if we wish to educate students properly about the art of the opera, using such recordings "without the human voice" is not recommended. In the same way, a correctly written set of mathematics standards should not just talk about "the conceptual understanding in mathematics" without getting the mathematics straight. It must start and end with 100% correct mathematics, and will therefore be more like the Board's version rather than the Commission's. Mathematical understanding *goes through technique*, and technique is built on understanding. That is the way it is.

The new Framework

What were the problems with the Board's Standards? Without trying to be comprehensive, I will describe a few obvious ones and, at the end of the section, will look into how the new Framework[6] addressed them.

First, the terse statements of the Board's Standards need examples to clarify them. For example, standard 4 in Geometry (Grades 8–12)—"Students prove basic theorems involving congruence and similarity"—means many things to many people. Should one only assume SAS and prove SSS and ASA, or should all three be assumed for simplicity? Should the AA theorem for similar triangles be proved? Or take the case of the introduction of negative fractions and decimals in elementary school: exactly when should this take place? The preamble of the standards in Grade 5 states: "Students increase their facility with the four basic arithmetic operations applied to positive and negative numbers, fractions, and decimals". Is this to be taken literally so that "fractions" and "decimals" mean (as usual) *positive* fractions and *positive* decimals, or does it mean "positive and negative numbers, positive and negative fractions, and positive and negative decimals"? It does not help that this linguistic ambiguity persists in the subsequent enunciation of the detailed standards in both grades 5 and 6.

We must remember that these Standards are pioneering something new in California, and pioneers have to be transcendentally clear at each step or they run the risk of having no followers on their trail. I wish to drive home this point by comparing with what I consider a very admirable set of mathematics standards, the 1990 Mathematics Standards of Japan[18]. There the statement about similarity (in grade 8!) is equally terse:

> *To enable students to clarify the concepts of similarity of figures, and develop their abilities to find the properties of figures by using the conditions of congruence or similarity of triangles and confirm them.*
> **a.** *The meaning of similarity and the conditions for similarity of triangles.*
> **b.** *The properties of ratio of segments of parallel lines.*
> **c.** *The applications of similarity.*

There is a big difference, however. The Japanese change their standards every ten years and, because they already have a well established tradition, the changes are gradual and minor by comparison with the kind of sea change we have over here. Moreover, they have excellent textbooks[19] already in place, so there is no great need to spell out everything. By contrast, we are almost starting anew in California, especially in these turbulent times in education. There is therefore very great need for the Board's Standards to be absolutely clear.

The Board's Standards intentionally eschew any prescription on how to teach students in grades 8–12, whether in the traditional way or the "integrated" way[16]. The intention for greater flexibility was admirable, except that in the absence of a tradition, the added flexibility could be a curse. For example, the Standards specify that each discipline (Algebra I, Geometry, etc.) need not "be initiated and completed in a single grade". It would appear that this specification makes it possible to describe the desirable content of each discipline without undue regard to the time limitation of fitting everything into exactly one year. Perhaps for this reason, there are more topics in Algbra II than can be reasonably completed in a single year. How to teach this material in more than two semesters then becomes a challenge which few schools could meet. Also Algebra I asks that "Students [be] able to find the equation of a line perpendicular to a given line that passes through a given point." No matter how this is done, it would involve theorems about similar triangles. Does it then imply—contrary to the traditional curriculum—that Geometry may be taught simultaneously with Algebra I?

Finally, it appears that the forthcoming 10th or 11th grade statewide mathematics test would include some statistics. Was the Framework going to suggest ways of teaching statistics in the early part of secondary school if the traditional curriculum is followed?

Considerations of this nature bring out the fact that the traditional method of offering year long sequences on algebra and geometry is too rigid to be educationally optimal. While none of the current "integrated" models in this country seems to be entirely successful, the argument cannot be ignored that we should pursue the kind of integrated mathematics education that has been in use in Japan or Hong Kong for a long time[20]. The Framework might fulfill its basic function if it could nudge California in this direction in a forceful manner.

An idea that undoubtedly occurred to many people is how much the standards of grades 8–12 in the Board's Standards read like a "Manual for Pure Mathematics". One almost has the feeling that this document could not bring itself to face the relationship between school mathematics and practical problems. Thus the Framework needed to restore the balance between the pure and applied sides of school mathematics. While it is true that the reform exaggerates the role of "real-world" problems in mathematics, ignoring them altogether is for sure not a cure either. We would do well to remember that the overwhelming majority of school students will be *users* of mathematics, and that as future citizens they need to be shown the power of mathematics in the context of daily affairs. But all through grades 8–12, I seem to see only three explicit references to applications:

> **Algebra I**
> **15.** *Students apply algebraic techniques to rate problems, work problems, and percent mixture problems.*
> **23.** *Students apply quadratic equations to physical problems such as the motion of an object under the force of gravity.*
> **Trigonometry**
> **19.** *Students are adept at using trigonometry in a variety of applications and word problems.*

I hope I am not over-using the Japanese model if, again, I look at the corresponding situation in the 1990 Japanese Standards. The description of the *Content* of the Japanese Standards is every bit as abstract and "pure" as the Board's Standards, but *The Construction of Teaching Plans and Remarks Concerning Content* after each of grades K, 1–6, 7–9, and 10–12 pays careful attention to the bearing of "daily affairs" on the curriculum. For example, here is what is said after grades 7–9:

> *In the [8th and 9th] grades, problem situation learning should be included in a total teaching plan with an appropriate allot-*

> *ment and [implementation] for the purpose of stimulating students' spontaneous learning activities and of fostering their views and ways of thinking mathematically. Here, 'problem situation learning' means the learning to cope with a problem situation, appropriately provided by the teacher so that the content of each domain may be integrated or related to daily affairs.*

The tone makes it abundantly clear that this is no mere lip service to applications, but that the applied component is central to the whole curriculum.

A final comment is on the contentious subject of technology. From K to 12 in the Board's Standards, I could detect only the following two references to technology:

> **Grade 6** *Algebra and Functions*
> **1.4** *solve problems using correct order of operations manually and by using a scientific calculators*
> **Grade 7** *Statistics, Data Analysis and Probability*
> **1.** *Students collect, organize and represent data sets . . . both manually and by using an electronic spreadsheet program.*

This reticence is a de facto confession that we, as educators, do not know what the proper role of technology is in mathematics education. The reality is that computer and graphing calculators are here to stay, and the younger generation is besieged on all sides by them. It would not be an effective education policy to retreat and abdicate responsibility exactly when we were supposed to come forward to provide guidance. We do not want any kind of technological debauchery in the mathematics classroom, but neither do we want to make technological prudes out of our students. What we want are students who are technologically informed, especially about the role of technology in mathematics, but we won't get them if we continue to pretend that technology does not exist. I am being intentionally suggestive in my use of language in order to force the comparison with sex education. In both situations, it is better to keep our students informed than to let them pick up the wrong information in a state of prevailing ignorance.

Allow me to cite for the last time the 1990 Japanese Standards. Part of *The Construction of Teaching Plans and Remarks Concerning Content* also deals with the technological issue after each of grades K, 1–6, 7–9, and 10–12. Here is what is said after grades 1–6 and 10–12, respectively.

At the 5th Grade or later, the teacher should help children adequately use "soroban" or hand-held calculators, for the purpose of lightening their burden to compute and of improving the effectiveness of teaching in situations where many large numbers to be processed are involved for statistically considering or representing, or where they confirm whether the laws of computation still hold in multiplication and division of decimal fractions. At the same time, the teacher should pay attention to provide adequate situations in which the results of computation may be estimated and computation may be checked through rough estimation.

In teaching the content, the following points should be considered. The teacher should make active use of educational media such as computers, so as to improve the effectiveness of teaching.
In the teaching of computation, the teacher should have students use hand-held calculators and computers as the occasion demands, so as to improve the effectiveness of learning.

The Board had already wisely decided that no Standards-based state test in grades K–12 would use calculators. This general policy on technology, sensible as it is, needed to be supplemented by a more comprehensive one which gives guidance not only on when *not* to use it but also on *when* to use it. For example, encouraging teachers to use problems with more natural—and therefore more unwieldy—numerical data by enlisting the help of calculators is a beginning. In the presence of the no-calculator-in-tests rule, students would get a clear perspective on what they need to know regardless of technology, and on how they can use technology to their benefit when the need arises. Encouraging students in calculus to use calculator to estimate the limits of sequences while also holding them responsible for proofs of convergence is another example. Doubtlessly, thoughtful educators could formulate similar specific recommendations in other situations. As the preceding passages from the Japanese Standards indicate, we needed to make *active* use of calculators and computers to improve the effectiveness of teaching and learning.

With all this said, it is time to look at the new Framework[6] to see how it managed to address the foregoing problems in the Board's Standards. In this context, the foremost accomplishments of the new Framework would seem to be:

(1) It adopted a policy on the use of technology in the classroom that is as comprehensive as the available research allows. For example, it essentially recommends against the use of calculators in grades K–5, but encourages its judicious use starting with grade 6.

(2) It gives a detailed guide on how to teach the Standards in each grade of K–7, and for each discipline in grades 8–12 (see Chapter 3). In particular, the ambiguities regarding the introduction of negative fractions and negative decimals have been cleaned up.

A conscientious attempt was also made in the new Framework to emphasize applications in grades 8–12. Thus almost all the major concerns regarding the Board's Standards have been removed. *Almost*, except for two of them. It failed to directly address the issue of how to teach statistics in the traditional curriculum before grade 11. More seriously, it did not even take up the question of how to give Californian high school students a more integrated kind of mathematics education along the line of the Hong Kong or Japanese model.[20] These failures are blemishes in the new document to be sure, but considering how far it has outdistanced its 1992 predecessor[1] in terms of mathematical coherence and accuracy, one can afford to be philosophical about these blemishes. Social changes are rarely accomplished all at once. They take time.

What have we learned?

It is often forgotten in the war of words that mathematics education has a substantive component: mathematics. We have seen how a choice between the two versions of the Mathematics Content Standards in California came down to a mathematical assessment of the documents. The scant attention given to this component in the mathematics standards of an overwhelming majority of the states, as pointed out in the Fordham Foundation monograph[10], is nothing short of scandalous. One positive outcome of the current mathematics education reform may very well be the revival of the idea that mathematics is important in discussions of mathematics education. The battle over the Standards is a stunning illustration of this fact.

If there is anything that the Californian experience can teach policy makers in the other states, it is that without a solid mathematical input, it would be impossible to have a sound policy on mathematics education. California happened to benefit from such an input through entirely fortuitous circumstances. The accidental confluence of a group of enlightened State Board members and a group of knowledgeable mathematicians who are also educationally informed led to the writing of a set of quality standards and a Framework that is equally promising. So what can other policy makers do in order to bring about comparable happy results?

One can try to seek out mathematicians who are dedicated to the cause of education, but by itself this is not without risk. It suffices to recall that the New Math of the sixties was spearheaded by a small group of well-intentioned mathematicians.[21] A safer recommendation would be that policy makers cultivate standard channels of communication within the mathematical community as a whole, and seek consensus in that community at each major step of decision making. Back in the age of the New Math, much anguish and frustration would have been avoided had this guideline been followed.[22] The mathematical community, especially research mathematicians, should likewise do their share and make an effort to stay informed about mathematics education. Happily, recent events have proven that at least the latter seems to be taking place. Let us hope that in the near future, mathematicians would be alongside of educators in formulating major decisions in mathematics education.

It goes without saying that having a set of good standards and curriculum framework is only the first step towards improvement in education. The far more difficult issues of getting qualified teachers and administrative support for the implementation of the Standards lie ahead of California. However, these would be subjects of a different article.

Finally, let us return to the battle of the Standards for a moment. Few would disagree that this so-called math war is entirely senseless, but in the context of human affairs, it may be necessary. Destruction often has to precede progress. Needless to say, not everybody shares this view. When news of the U.S. 12th-grade performance on TIMSS was released on February 34, 1998, the then President of NCTM, Gail Burrill, made the following comment on the TIMSS result: "What's important is that we are working together toward a common goal of excellence in mathematics. The recent math wars have done nothing to improve mathematics education." These are sobering statements. On the one hand, Ms. Burrill's optimistic view that we are already working together toward a common goal in mathematics education could not have been based on the reckless public condemnations of the Board's Standards that had just transpired. NCTM's editorial[5] did not exactly contribute to promoting harmony either. On the other hand, the math war in California did manage to reverse the disastrous trend initiated by the 1992 Framework[1]. While much work remains to be done to achieve a balanced mathematics education in California, this achievement would give the lie to the assertion that the math wars have done nothing to improve mathematics education.

When all is said and done, educational reconstruction should be the common goal of all parties at this juncture, and the battle over the Standards is in this light nothing but a distraction. In his address before the Joint Annual Meeting of the American Mathematical Society and the Mathematical Association of America on January 8, 1998, Secretary Richard W. Riley had sounded the same theme of reconciliation: "This leads me back to the need to bring an end to the shortsighted, politicized, and harmful bickering over the teaching and learning of mathematics. I will tell you that if we continue down this road of infighting, we will only negate the gains we have already made—and the real losers will be the students of America." In all our education activities we should think of our children first. No, we *must*. If there is a main lesson to be learned from the battle of the Standards, it is that we should all learn to look at the facts and keep in mind the welfare of the students before we air our opinions.

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FOOTNOTES

1. *Mathematics Framework for California Public Schools*, California Department of Education, Sacramento, CA, 1992.

2. *Curriculum and Evaluation Standards for School Mathematics*, National Council of Teachers of Mathematics, Reston, 1989. Also available at http://www.enc.org/online/NCTM/280dtoc1.html

3. California Academic Standards Commission, *Mathematics Content Standards*, October 1, 1997. http://www.ca.gov/goldstandards/

4. *The California Mathematics Academic Content Standards* as adopted by the California State Board of Education, February 5, 1998. Available at *http://www.cde.ca.gov/board/board.html*

5. New California standards disappoint many, *NCTM News Bulletin*, Issue 7, 34 (1998), 1 and 5.

6. *Mathematics Framework for California Public Schools, Kindergarten Through Grade Twelve*, 1998. http://www.cde.ca.gov/cilbranch/eltdiv/

7. According to Commissioner Williamson Evers, "the omission of long division with two or more digit divisors was a conscious decision" by the Commission. See *California Mathematicians Respond*, at http://www.mathematicallycorrect.com

8. Unless one counts editorial howlers such as "1 square foot = 12 square inches" in Standard 2.4 of Grade 7, Measurement and Geometry.

9. There was some dissent, of course, but on February 2, 1998, an open letter to California State University Chancellor Charles Reed signed by over 100 mathematicians was released to the public; it expresses sentiments in support of the Board's Standards. See California Mathematicians Respond, http://www.mathematicallycorrect.com

10. See the survey of mathematical standards of 47 states in R. Raimi and L. S. Braden, *State Mathematics Standards,* Fordham Report Volume 2, No. 3, Thomas B. Fordham Foundation, Washington D.C., 1998.

11. H. Wu, Invited comments on the NCTM Standards, http://math.berkeley.edu/∼wu

12. Mathematicians are only concerned with whether students *understand* mathematics, *i.e.,* whether they know why something is true, why it is of interest, how to apply it, what its implications are, and whether something more general is still true. However, educators introduce the term "conceptual understanding" and make it one of the three pillars of a so-called balanced curriculum (the other items being "problem solving" and "basic skills"). To my knowledge, the meaning of "conceptual understanding" is as yet unclear.

13. The Commission's Standards is published in a two column format which displays the mathematics standards on the left and the "Clarifications and Examples' on the right.

14. But see H. Wu, The role of open-ended problems in mathematics education, *J. Math. Behavior* 13 (1994), 115-128.

15. There is an unfortunate linguistic slip here: "draw ten points" is undoubtedly what is meant. Subsequently this error was corrected.

16. The meaning of this word has to be carefully qualified because there are several "integrated" approaches to mathematics in secondary schools.

17. A popular undertaking by conductors such as Carmen Dragon and André Kostelanetz in the 50's and 60's.

18. *Mathematics Programs in Japan*, Japan Society of Mathematics Education, 1990.

19. Cf. K. Kodaira, ed., *Japan Grade 7 Mathematics*, *Japan Grade 8 Mathematics*, *Japan Grade 9 Mathematics*, The University of Chicago Mathematics Project, Chicago 1992; K. Kodaira, ed., *Mathematics 1*, *Mathematics 2*, (Japan Grade 11 Mathematics), American Mathematical Society, Providence 1997; and K. Kodaira, ed., *Algebra and Geometry*, *Basic Analysis*, (Japan Grade 11 Mathematics), American Mathematical Society, Providence 1997. The publication date of the Japanese original of these texts is 1984.

20. Cf. e.g., the texts cited in the preceding footnote.

21. However, the common perception that mathematicians were solely responsible for the New Math debacle is wrong. In fact, NCTM was also behind the New Math.

22. See, for example, On the mathematical curriculum of the high school, *Amer. Math. Monthly*, 69(1962), 189-193. This open letter was co-signed by 75 of the foremost mathematicians in this country.

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