What Should Be Taught in the Elementary Mathematics Curriculum*

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Some ten years ago, when the idea of creating a cadre of **math specialists**[1] in the upper elementary grades first made its way to the halls of the California legislature, a legislator pooh-poohed the need for any such legislation. What he said was something like: "All you have to do is add, subtract, multiply and divide numbers. How hard is that?"

On the other hand, Clive Goodall of the Royal Statistical Society in England recently made a comment about school mathematics, to the effect that even the most skilled teacher would struggle to convey the nuances, connections and usefulness of school mathematics unless they have a math background. Although Goodall did not specifically have the elementary mathematics curriculum in mind, his comment is particularly apt in this context.

Who is right? Today I will avoid making provocative pronouncements but will try instead to give you an idea of the content knowledge that is needed to teach elementary mathematics effectively. My charge is to discuss the mathematics curriculum of elementary school, but clearly 45 minutes is not long enough for that. Most likely, my host had in mind a discussion of the *essence* of the elementary curriculum. Such a task is easier. Insofar as the mathematics in elementary school is the foundation of all of K-12 mathematics and beyond, it should, in a grade-appropriate manner, respect the basic characteristics of mathematics. This sounds simple, but its implementation is anything but that, as the available evidence in the education literature shows. My personal conviction is that *the essence of the elementary school curriculum consists of coherence, precision, and reasoning.* In fact, the same

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[1] *Added May 31, 2007:* In this article, the term "math specialist" refers to a teacher who teaches students only mathematics in grades 4–6.

is true of the mathematics curriculum for all grades, not just the elementary grades. Perhaps I can give you some idea of these qualities.

First of all, the elementary curriculum is surprisingly **coherent**. If you dig beneath the surface, you will find that the main topics of this curriculum are not a collection of facts unrelated to each other like caged animals in a zoo; rather, they form a whole tapestry where each item exists as part of a larger design. For example, although whole numbers and fractions are related in such a way that fractions extend directly from whole numbers, and should be taught that way (cf. Wu, 2001), too often whole numbers and fractions are taught as if they are unrelated topics. If the smooth transition from whole numbers to fractions is properly made through classroom instruction, fraction-phobia could be greatly reduced.

Children should learn about this mathematics tapestry in a language that does not leave room for misunderstanding or guesswork. It should be a language sufficiently **precise**, so that they can reconstruct the tapestry step-by-step if necessary. Too often, such precision of language is not achieved. For example, if you tell a sixth grader that two objects are *similar* if they are the same shape but not necessarily the same size, it begs the question of what "same shape" means.

Above all, I would like to illustrate why it is important that elementary school mathematics, like all mathematics, be built on **reasoning**. Reasoning is the power that propels each move from one step to the next. When students are given this power, they learn that mathematics is something they can do, because it is done *according to some objective criteria*, in the same way that video games are played according to some objective criteria. When students are emboldened to make moves on their own in mathematics, they becomes sequential thinkers, and *sequential thinking is the engine that drives problem-solving*. If one realizes that almost the whole of mathematics is problem-solving, the centrality of reasoning in mathematics becomes all too obvious. By contrast, when reasoning is absent, mathematics becomes a black box, and fear and loathing set in. An example that immediately comes to mind is some children's failure to shift successive rows one digit to the left when multiplying whole numbers:

$$
\begin{array}{rrrrrr}
 & & & \mathbf{8} & \mathbf{2} & \mathbf{6} \\
\times & & & \mathbf{4} & \mathbf{7} & \mathbf{3} \\ \hline
 & & \mathbf{2} & \mathbf{4} & \mathbf{7} & \mathbf{8} \\
 & & \mathbf{5} & \mathbf{7} & \mathbf{8} & \mathbf{2} \\
+ & & \mathbf{3} & \mathbf{3} & \mathbf{0} & \mathbf{4} \\ \hline
 & \mathbf{1} & \mathbf{1} & \mathbf{5} & \mathbf{6} & \mathbf{4}
\end{array}
\qquad\qquad
\begin{array}{rrrrrrr}
 & & & & \mathbf{8} & \mathbf{2} & \mathbf{6} \\
\times & & & & \mathbf{4} & \mathbf{7} & \mathbf{3} \\ \hline
 & & & \mathbf{2} & \mathbf{4} & \mathbf{7} & \mathbf{8} \\
 & & \mathbf{5} & \mathbf{7} & \mathbf{8} & \mathbf{2} & \\
+ & \mathbf{3} & \mathbf{3} & \mathbf{0} & \mathbf{4} & & \\ \hline
 & \mathbf{3} & \mathbf{9} & \mathbf{0} & \mathbf{6} & \mathbf{9} & \mathbf{8}
\end{array}
$$

If no reason is ever given for the shift, it is natural that children would take matter into their own hands and make up new rules.

Learning cannot take place in the classroom if students are kept in the dark about why they must do what they are supposed to do.

If you are convinced as I am that the mathematics of elementary school is really this sophisticated, then you would see why children's faltering first steps in exploring the terrains of numbers and geometric figures need informed guidance. It is not fair that elementary teachers, in addition to all the impossible tasks we ask them to perform, are also asked to provide such guidance. This was the original motivation for the creation of math specialists in California. In the real world, however, math specialists in K-6 are no more than a pipe dream, and a more realistic compromise is to have math specialists only for the upper elementary grades.

For the next half hour or so, I will explain to you two simple pieces of mathematics: how to add whole numbers, and what it means to divide two numbers. At this point, you may feel as the California legislator did: "How hard is that?" So I will have to show you that addition and division are not as easy as you imagine. In so doing, I have put myself in a win-win situation. If you feel that at the end of the half-hour you understand everything, then clearly my presentation is a runaway success. If, however, you get all confused at the end of the half-hour, then I will have convinced you that elementary mathematics is not *that* elementary. Either way I will have made my point.

Consider then the seemingly mundane skill of adding two whole numbers, e.g.,

$$
\begin{array}{rrr}
 & \mathbf{4} & \mathbf{5} \\
+ & \mathbf{3} & \mathbf{1} \\ \hline
 & \mathbf{7} & \mathbf{6}
\end{array}
$$

Nothing could be simpler. But if you are the teacher, how would you convince your children that this is worth learning? Is it, as some have claimed, a rote-learning skill that stunts children's intellectual growth? Let us see. How else would you add these two numbers? In fact, *what does adding whole numbers mean?* This will be a dominant theme of my presentation: **before we do anything in mathematics, we must make clear what it is that we are doing.** Adding whole numbers means *iterated counting*. In this case, you count to 45, and then starting at 45, you count 31 times again and see where you end up. You end up at the number 76, and that is the sum. If you are the teacher, and if you know exactly what adding whole numbers means, you would begin by asking your children to act out the iterated counting: count to 45, and then count 31 more times to make sure they get to 76. This would keep them interested, but it will also make them feel frustrated. That is good, because you want them to know that

adding numbers is hard work.

Now you get to play the magician. You tell them it is not necessary to count so strenuously to get the answer to $\mathbf{45+31}$ (make them learn to write addition horizontally as well as vertically from the beginning!), because you are going to do two simple countings instead, one being $\mathbf{4+3}$ and the other $\mathbf{5+1}$, and these already give the *correct* answer. This should really perk them up!

You can demonstrate this effectively by bringing in two bags of marbles, one bag containing 45 and the other 31. You dump them on the mat, mix them up and ask them to count how many marbles there are together. They will have to count a long time, and of course the longer the better. Then you collect the 45 marbles, and put them into bags of 10; there will be 4 such bags with 5 stragglers. Do the same with the other 31 marbles. Now you dump these bags and stragglers on the mat again, and ask them how many marbles there are. It won't take long for them to figure out that

there are $\mathbf{4+3}$ bags of 10, and

$\mathbf{5+1}$ stragglers

They will figure out that 7 bags of 10 together with 6 stragglers give 76 again. Now ask them to compare the counting of the bags-and-stragglers with the magic you performed just a minute ago. If they don't see the connection (and some won't), you the teacher will patiently explain it to them. Of course this is the time to go over place value all over again and use it to explain the algorithm to them:

$$\begin{array}{rr} & \mathbf{4\ 5} \\ + & \mathbf{3\ 1} \\ \hline & \mathbf{?\ ?} \end{array} \iff \begin{array}{rr} & \mathbf{40} + \mathbf{5} \\ + & \mathbf{30} + \mathbf{1} \\ \hline & \mathbf{?} \quad\ \mathbf{?} \end{array} \iff$$

$$\begin{array}{rr} & \mathbf{40} + \mathbf{5} \\ + & \mathbf{30} + \mathbf{1} \\ \hline & \mathbf{70} + \mathbf{6} \end{array} \iff \begin{array}{rr} & \mathbf{4\ 5} \\ + & \mathbf{3\ 1} \\ \hline & \mathbf{7\ 6} \end{array}$$

They will listen more carefully this time to your incantations of place value because you have now given them more incentive to learn about this important topic.

So this is the essence of the addition algorithm: instead of doing the tedious, mind-numbing counting, you break up the task **digit-by-digit** and end up counting only two one-digit numbers in succession. Recall that the main goal of the elementary mathematics curriculum is to provide children with a good foundation for mathematics. In this context, the addition algorithm, when taught this way, serves as a splendid introduction. It teaches children an important skill in mathematics: **one always breaks up a complicated task into a sequence of simple easy ones if possible**. This is why we do not look at 45 or 31, but only 4 and 3 and also 5 and 1. We break up the numbers into single digits, add the single digits, and then reassemble the separate pieces of information to arrive at the final result. Of course, further down the road, you would encounter the phenomenon of "carrying", but that is just a sidelight, a little wrinkle on the fabric. "Carrying" is **not** the main idea of the addition algorithm as most textbooks would have you believe. The main idea is to break up any addition into the additions of one-digit numbers and then put these simple computations together to get the final answer. If you can make your children understand that, you would be doing fantastically well as a teacher, because you have taught them important mathematics.

You still think learning the standard algorithm stunts children's intellectual growth?

But you wouldn't be able to teach this way unless you have the necessary content knowledge. In the teaching of mathematics, it is usually the case that **content guides pedagogy**. (Cf. Wu, 2005.) In a real world, it would be far more realistic to expect a math specialist rather than an average elementary teacher to have this kind of content knowledge.

Division of numbers is next. We begin with the division of whole numbers. What does the division $\frac{\mathbf{54}}{\mathbf{6}} = \mathbf{9}$ mean? In the primary grades, it is certainly appropriate to talk exclusively about partitive or measurement divisions among whole numbers. As an example of measurement divisions, a third grader's understanding of $\frac{\mathbf{54}}{\mathbf{6}} = \mathbf{9}$ may be limited to thinking of a partitioning 54 into equal groups of 6 and then finding that there are 9 groups in all. In the upper elementary grades, however, this conception of division will not be enough. Why not? Because

(1) *Even in elementary school, we have to deal with the division of fractions, and the partitive or measurement meaning of division will not be able to handle a division such as*

$$\frac{\frac{\mathbf{1}}{\mathbf{3}}}{\frac{\mathbf{4}}{\mathbf{5}}}$$

(2) *The elementary curriculum does not exist in isolation; it must support the curriculum in K-12 and beyond. For the latter (e.g., the study of algebra), we must have a more symbolic definition of division.*

Sometimes, progress in mathematics is achieved, not by looking forward, but by looking back and understanding what we have done a little better. In the case of $\frac{\mathbf{54}}{\mathbf{6}} = \mathbf{9}$, its partitive meaning as a partition of 54 objects into *9* groups of 6's already points to a symbolic expression of the meaning of division:

$$\mathbf{54} = \underbrace{\mathbf{6} + \mathbf{6} + \cdots + \mathbf{6}}_{\mathbf{9}} = \mathbf{9} \times \mathbf{6}$$

The last equality uses the very definition of multiplication, of course. This is actually very clear from the way we teach division:

$$\begin{array}{r|l} & \mathbf{9} \\ \hline \mathbf{6} & \mathbf{5\ 4} \\ & \mathbf{5\ 4} \\ \hline & \quad\ \mathbf{0} \end{array}$$

What we tell children is that, to divide 54 by 6, we look for the number which, when multiplied by 6, gives 54. Therefore, in symbolic terms,

the meaning of the division $\frac{\mathbf{54}}{\mathbf{6}} = \mathbf{9}$ is that $\mathbf{54} = \mathbf{9} \times \mathbf{6}$.

In a similar fashion, the meaning of $\frac{\mathbf{48}}{\mathbf{6}} = \mathbf{8}$ is that $\mathbf{48} = \mathbf{8} \times \mathbf{6}$, the meaning of $\frac{\mathbf{36}}{\mathbf{12}} = \mathbf{3}$ is that $\mathbf{36} = \mathbf{3} \times \mathbf{12}$, etc. By using symbols (which is definitely appropriate for fifth graders), we can express this new understanding of the division of whole numbers as follows:

for whole numbers $\boldsymbol{m}$ and $\boldsymbol{n}$, where $\boldsymbol{m}$ is a multiple of $\boldsymbol{n}$ and $\boldsymbol{n}$ is nonzero, the meaning of the division $\frac{\boldsymbol{m}}{\boldsymbol{n}} = \boldsymbol{q}$ *is that* $\boldsymbol{m} = \boldsymbol{q} \times \boldsymbol{n}$

For fifth and sixth graders, we should ask them to reconceptualize division from this point of view. They should revisit division from the perspective of their new knowledge and reshape their thinking accordingly. Such is the normal progression of learning.

Note that this reconceptualization is *not* a rejection of the meaning of the division of whole numbers. On the contrary, it evolves from the latter, and makes it more precise.

The reason this reconceptualization is important is that the meaning of division, when reformulated this way, turns out to be *universal* in mathematics, in the following sense. If $\boldsymbol{m}$ and $\boldsymbol{n}$ are *any* two numbers (i.e., not just whole numbers) and $\boldsymbol{n}$ is nonzero, then the **definition** of "$\boldsymbol{m}$ divided by $\boldsymbol{n}$ equals $\boldsymbol{q}$" is that $\boldsymbol{m} = \boldsymbol{q} \times \boldsymbol{n}$. In other words,

$$\frac{\boldsymbol{m}}{\boldsymbol{n}} = \boldsymbol{q} \quad \textit{means} \quad \boldsymbol{m} = \boldsymbol{q} \times \boldsymbol{n}.$$

It is possible that some amplification of this sentence would enhance its clarity, so I will do that. What it says consists of two parts. The first part says no matter what the numbers $\boldsymbol{m}$ and $\boldsymbol{n}$ may be, the statement $\frac{\boldsymbol{m}}{\boldsymbol{n}} = \boldsymbol{q}$ for some number $\boldsymbol{q}$ is nothing but another way of expressing $\boldsymbol{m} = \boldsymbol{q} \times \boldsymbol{n}$. Furthermore, if we know $\boldsymbol{m} = \boldsymbol{q} \times \boldsymbol{n}$, then we may choose to rewrite it as

$\frac{\boldsymbol{m}}{\boldsymbol{n}} = \boldsymbol{q}$ if it suits our purpose. Thus in a very precise sense, *division is just multiplication in a different format.*

It would be *quite wrong* to consider this statement to be just another way of saying "division and multiplication are inverse operations." The latter statement signifies that, knowing what multiplication and division are, we merely make the observation about how they are related. Little or no thought is given to how this relationship might be the key to understanding division. By contrast, what we are now saying is that the *only* way we can get to know division is through multiplication, because the division statement $\frac{\boldsymbol{m}}{\boldsymbol{n}} = \boldsymbol{q}$ is just a different, but equivalent, way of writing the multiplication statement $\boldsymbol{m} = \boldsymbol{q} \times \boldsymbol{n}$, no less and no more. A teacher's understanding of this difference would be critical to how she could put this concept of division to effective use, as we proceed to show.

Why does

$$\frac{\frac{\mathbf{5}}{\mathbf{6}}}{\frac{\mathbf{9}}{\mathbf{4}}}$$

equal

$$\frac{\mathbf{5}}{\mathbf{6}} \times \frac{\mathbf{4}}{\mathbf{9}} \ ?$$

In other words, why invert and multiply? To give an explanation, we have to go back to the above definition of division; this is what a definition is for. It is easy to see that

$$\frac{\mathbf{5}}{\mathbf{6}} = \left\{ \frac{\mathbf{5}}{\mathbf{6}} \times \frac{\mathbf{4}}{\mathbf{9}} \right\} \times \frac{\mathbf{9}}{\mathbf{4}}$$

If we think of

$$\frac{\mathbf{5}}{\mathbf{6}} \quad \text{as} \quad \boldsymbol{m},$$

$$\frac{\mathbf{5}}{\mathbf{6}} \times \frac{\mathbf{4}}{\mathbf{9}} \quad \text{as} \quad \boldsymbol{q},$$

$$\frac{\mathbf{9}}{\mathbf{4}} \quad \text{as} \quad \boldsymbol{n},$$

then the preceding equality says $\boldsymbol{m} = \boldsymbol{q} \times \boldsymbol{n}$. By the definition of division, this is exactly the statement that $\frac{\boldsymbol{m}}{\boldsymbol{n}} = \boldsymbol{q}$, or,

$$\frac{\frac{\mathbf{5}}{\mathbf{6}}}{\frac{\mathbf{9}}{\mathbf{4}}} = \frac{\mathbf{5}}{\mathbf{6}} \times \frac{\mathbf{4}}{\mathbf{9}}$$

So you see that there is nothing to invert-and-multiply **once we know the meaning of division**. What is sobering is that the limerick, "Ours not to reason why/ Just invert and multiply," gets it all wrong. The key issue here is not the "why", because before we ask why invert and multiply, we have to know what division means. We are thus harking back to our earlier theme: before we do anything in mathematics, we must make clear what it is that we are doing. In other words, *we must have a precise definition of division before we can talk about its properties.*

In general, this reasoning explains why one inverts and multiplies in the division of fractions:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c}$$

Let us consider a slightly different problem. Take for instance, the division of 12 by -3. Students are told that $\frac{12}{-3} = -4$, but not why this is so. This creates a very dangerous situation, because in the absence of a valid reason, students' imagination runs wild. My own experience points to the fact that many of them are led to believe the reason $\frac{12}{-3} = -4$ is true is that, somehow, *the "fraction bar" is something you can ignore when it is not to your liking.* So since they don't know how to deal with $\frac{12}{-3}$, they simply push the minus sign in front beyond the fraction bar to get

$$\frac{12}{-3} = -\frac{12}{3} = -4$$

The truth is actually simpler: We want to know what the division $\frac{12}{-3}$ is equal to. We also remember that whatever it is equal to, let us say that $\frac{12}{-3} = q$, then this is actually the statement that $12 = q \times (-3)$. It remains to find out what q is. Now, *if we know how to multiply rational numbers*, we would recognize that $12 = (-4) \times (-3)$. Therefore $q = (-4)$ and the multiplication fact $12 = (-4) \times (-3)$, when written as a division, is exactly

$$\frac{12}{-3} = (-4)$$

The same reasoning explains in general why for any two numbers x and y, we always have:

$$\frac{x}{-y} = \frac{-x}{y} = -\frac{x}{y}$$

I am afraid I have tired you out with so much mathematics, so I will refrain from further exploiting the advantage of using a clear-cut definition of division to explain why the solutions of certain word problems require division (rather than some other arithmetic operation). An example of such a problem is the following:

> If 5 yards of ribbon are cut into pieces that are each $\mathbf{3/4}$ yard long to make bows, how many bows can be made?

Students usually recognize that this problem calls for a division of $\mathbf{5}$ by $\mathbf{3/4}$, but not the reason why "division" should be used. If we use the preceding definition of division, the reason would become all too apparent. (See Section 5 of Wu, 2005.)

Let me conclude with two observations. The first is that I began by asserting that the essence of the elementary mathematics curriculum consists of coherence, precision, and reasoning. Let us see if the preceding discussion reflects a little bit of these qualities:

> **Coherence:** One of the manifestations of the coherence of mathematics is the ubiquity of the general principle of reducing a complicated task to a collection of simple sub-tasks. This principle runs right through the addition algorithm. This algorithm is also based on the place value of the decimal system which weaves through all the discussions of whole numbers and later decimals and, in a reduced role, in polynomials. Similarly, we saw how one embracing definition of division clarifies the meaning of the division of whole numbers, fractions, and rational numbers. The same definition also underlies the validity of both invert-and-multiply and the equalities $\boldsymbol{\frac{x}{-y} = \frac{-x}{y} = -\frac{x}{y}}$.
>
> **Precision:** The bedrock of this discussion is the precise definitions we give to the addition of whole numbers and the division of arbitrary numbers. It is worth repeating that *before we do anything in mathematics, we must make clear what it is that we are doing.* Equally

noteworthy is the fact that the definitions given are precise and all inclusive: they are sufficient to provide precise mathematics, with no psychological overtones. Contrast this with a common definition of fraction as both part of a whole and a ratio. Beyond appealing to children's naive response to the mention of the word "ratio", what purpose does such a definition serve?

Reasoning: Every step of this discussion evolves from an earlier step by logical deduction. The addition algorithm follows from the place value of our numeral system, and invert and multiply is a logical consequence of the definition of division in terms of multiplication.

My belief is that the presence of these three qualities, coherence, precision, and reasoning, is a prerequisite to making school mathematics learnable. Too often, all three are absent from the elementary curriculum. More than that, too often they are absent from the elementary classroom. As I explained at the beginning, it is not the fault of the elementary teacher not to provide this kind of instruction. Indeed, I repeat that it is altogether unrealistic to expect our average elementary teacher to possess this kind of mathematical knowledge. One way or another, *we have to get math specialists into our school classroom.*

Finally, you may have noticed that in the whole discussion, I have placed a greater emphasis on the curriculum of the upper elementary grades while seemingly slighting the primary grades. I want to assure you that the same coherence, precision, and reasoning are of equal importance there. But if I were to also discuss in depth these qualities in the primary grades, then it would be necessary to more carefully qualify, for example, the kind of precision that would be optimal for the learning of young children, or how much formal reasoning is appropriate at each of the early grades. These are delicate issues that deserve a more detailed examination than I am capable of providing, or in any case, it would be unrealistic to do this in forty-five minutes. Nevertheless, there should be no letup in our insistence on these essential qualities of mathematics. Recall that the mathematics of elementary school serves as the foundation of the mathematics of K-12. So regardless of the difference in the preponderance of skills or the pedagogical needs in the primary grades, the principal concern is still with building a robust foundation for children's learning of mathematics. Even little kids need mathematics that is coherent, precise, and logical.

References

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