The Mathematics K-12 Teachers Need to Know

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Prologue

In 2001, the Conference Board for Mathematical Sciences published a volume to describe the mathematics that institutions of higher learning should be teaching prospective school teachers ([MET]). It recommends that the mathematical course work for elementary and middle school teachers should be at least 9 and 21 semester-hours, respectively, and for high school teachers it should be the equivalent of a math major plus a 6-hour capstone course connecting college mathematics with school mathematics. The major part of the volume is devoted to a fairly detailed description of the mathematics that elementary, middle, and high school teachers need to know.

Given the state of mathematics education in 2008, the recommendation on the course work for teachers by the [MET] volume is very sound, in my opinion. As to the description of the mathematics that teachers need to know, it is such a complicated subject that one would not expect what is in [MET] to be *the* definitive statement. At the very least, one would want an alternative view from the mathematical perspective. Certain essential features about mathematics tend to be slighted in general education discussions of school mathematics, but here is one occasion when these features need to be brought to the forefront. Mathematical integrity is important where *mathematics* is concerned, and this is especially true about school mathematics.

This paper begins with a general survey of the basic characteristics of mathematics (pp. 2–7). Some examples are then given to illustrate the general discussion (Part I). The bulk of the paper is devoted to a description of the mathematics that teachers of K–8 should know (Part II, pp. 22–69). The omission of what high school teachers should know is partly explained by the fact that a series of textbooks is being written about the mathematics of grades 8-12 for prospective teachers ([Wu2012]).

Mathematics for K–12 Teaching

This is the name we give to the body of mathematical knowledge a teacher needs for teaching in schools. At the very least, it includes a slightly more sophisticated version of **school mathematics**, i.e., all the standard topics in the school mathematics curriculum. In Part II of this article (pp. 22–68), there will be a brief but systematic discussion of what teachers of K–8 need to know about school mathematics. In other words, we will try to quantify as much as possible what this extra bit of "sophistication" is all about.

The need for teachers to know school mathematics at a slightly more advanced level than what is found in school textbooks is probably not controversial. After all, if they have to answer students' questions, some of which can be unexpectedly sophisticated, and make up exam problems, a minimal knowledge of school mathematics would not suffice to do either of these activities justice. Perhaps equally non-controversial is the fact that, even within mathematics proper, there is a little bit more beyond the standard skills and concepts in the school curriculum that teachers need to know in order to be successful in the classroom. Teachers have to tell a story when they approach a topic, and the story line, while it is about mathematics, is not part of the normal school mathematics curriculum. They have to motivate their students by explaining why the topic in question is worth learning, and such motivation also does not usually find its way to the school curriculum. To the extent that mathematics is not a collection of tricks to be memorized but a coherent body of knowledge, teachers have to know enough about the discipline to provide continuity from day to day and from lesson to lesson. These connecting currents within mathematics are likewise not part of the school curriculum. Teachers cannot put equal weight on each and every topics in the curriculum because not all topics are created equal; they need to differentiate between the truly basic and the relatively peripheral ones. Teachers cannot make that distinction without an in-depth knowledge of the structure of mathematics. And so on. All this is without a doubt part of the *mathematical* knowledge that should be part of every teacher's intellectual arsenal, but the various strands of this component of the mathematics for K–12 teaching have so far not been well articulated in the education literature. In the first part of this article, we will attempt such an articulation. To this end, we find it necessary to step back and examine the nature of mathematics education.

Beyond the crude realization that mathematics education is about both mathematics and education, we posit that mathematics education is **mathematical engineering**, in the sense that it is the customization of basic mathematical principles for the consumption of school students ([Wu] 2006). Here we understand "engineering" to be the art or science of customizing scientific theory to meet human needs. Thus chemical engineering is the science of customizing abstract principles in chemistry to help solve day-to-day problems, or electrical engineering is the science of customizing electromagnetic theory to design all the nice gadgets that we have come to consider indispensable. Accepting this proposal that mathematics education *is* mathematical engineering, we see that *school mathematics* is the product of the engineering process that converts abstract mathematics into usable lessons in the school classroom, and school mathematics teachers are therefore **mathematical engineering technicians** in charge of helping the consumers (i.e., the school students) to use this product efficiently and to do repairs when needed.

Just as technicians in any kind of engineering must have a "feel" for their profession in order to avert disasters in the myriad unexpected situations they are thrust into, mathematics teachers need to know something about the essence of mathematics in order to successfully carry out their duties in the classroom. To take a simple example, would a teacher be able to tell students that there is no point debating whether a square is a rectangle because it all depends on how one defines a rectangle, and that mathematicians choose to define rectangles to include squares because this inclusion makes more sense in various *mathematical* settings, such as the discussion of area and volume formulas? This would be a matter of understanding the role of *definitions* in mathematics. Or, if a teacher finds that the **slope** of a line L is defined in a textbook to be the ratio $\frac{y_2-y_1}{x_2-x_1}$ for two chosen points (x_1, y_1) and (x_2, y_2) on L, would she recognize the need to prove to her students that this ratio remains unchanged even when the points (x_1, y_1) and (x_2, y_2) are replaced by other points on L? In other words, if (x_3, y_3) and (x_4, y_4) are any other pair of points on L, then

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$$

In this case, the teacher has to be alert to the inherent *precision* of mathematics, so that a definition claiming to express the property of a line should not be formulated solely in terms of two pre-asssigned points on it. It is also about knowing the need to supply *reasoning* when an assertion is made about the equality of the two ratios.

At the moment, our teachers are not given the opportunity to learn about the basic topics of school mathematics, much less the the essence of mathematics ([Ball] 1990; [Wu1999a] and [Wu1999b]; [NRC], pp. 372-378; [MET] 2001, Chapters 1 and 2; [Wu2002a]). For example, the mathematics course requirements of pre-service elementary teachers in most education schools consist of one to two courses (e.g., [NCTQ], p. 25), which are far from adequate for a revisit of the elementary mathematics curriculum in greater depth. One can look at most mathematics textbooks written for elementary teachers, for example, to get an idea of how far we are from providing these teachers with the requisite mathematical knowledge (again, [NCTQ], pp. 35–37). Worse, anecdotal evidence suggests that some mathematics courses that are required may be about “college algebra” or other topics unrelated to the mathematics of elementary school. Along this line, some knowledge of calculus is usually considered a badge of honor among elementary teachers. While more knowledge is always preferable to less, it can be persuasively argued that so long as our elementary teachers don’t have a firm grasp of the mathematical topics they have to teach, any knowledge of calculus would be quite beside the point. As for secondary teachers, their required courses are usually taken in the mathematics departments. There is still a general lack of awareness in these departments, however, that the subject of school mathematics is about a body of knowledge distinct from what future mathematicians need for their research ([Wu1999a]), but that it nevertheless deserves their serious attention. At the moment, future secondary school teachers get roughly the same mathematics education as future mathematics graduate students, and the only distinction between these two kinds of education is usually in the form of some pedagogical supplement for the mathematics courses.

In general terms, such a glaring lacuna in the professional development of future mathematics teachers is partly due to the failure to recognize that *school* mathematics is an engineering product and is therefore distinct from the mathematics we teach in standard college mathematics courses. Teaching school mathematics to our prospective teachers requires extra work, and "business as usual" will not get it done. There is another reason. Education itself is beset with many concerns, e.g., equity, pedagogical strategies, cognitive developments, etc. In this mix, schools of education may not give the acquisition of mathematical content knowledge the attention that is its due. And indeed mathematics often gets lost in the shuffle.

To further the discussion, more specificity would be necessary. We therefore propose that the following five **basic characteristics** capture the essence of mathematics that is important for K–12 mathematics teaching:

Precision: Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.

Definitions: They are the bedrock of the mathematical structure. They are the platform that supports reasoning. No definitions, no mathematics.

Reasoning: The lifeblood of mathematics. The engine that drives problem solving. Its absence is the root cause of teaching- and learning-by-rote.

Coherence: Mathematics is a tapestry in which all the concepts and skills are interwoven. it is all of a piece.

Purposefulness: Mathematics is goal-oriented, and every concept or skill is there for a purpose. Mathematics is not just fun and games.

These characteristics are not independent of each other. For example, without definitions, there would be no reasoning, and without reasoning there would be no coherence to speak of. If they are listed separately, it is only because they provide easy references in any discussion.

It may not be out of place to amplify a bit on the characteristic of purposefulness. One reason some students do not feel inspired to learn mathematics is that their lessons are presented to them as something *they are supposed to learn*, willy-nilly. The fact is that mathematics is a collection of interconnecting chains in which each concept or skill appears as a link in a chain, so that each concept or skill serves the purpose of supporting another one down the line. Students should get to see for themselves that the mathematics curriculum does move forward with a purpose.

We can give a first justification of why these five characteristics are important for teaching mathematics in schools. For students who want to be scientists, engineers, or mathematicians, the kind of mathematics they need is the mathematics that respects these basic characteristics. Although this claim is no more than professional judgment at this point, research can clearly be brought to bear on its validity. Accepting this claim for the moment, we see that students are unlikely to learn this kind of mathematics if their teachers don't know it. Apart from the narrow concern for the nation's technological and scientific well-being, we also see from a broader perspective that *every* student needs to know this kind of mathematics. This is because, if school mathematics education is to live up to its educational potential of providing the best *discipline of the mind* in the school curriculum, then we would want to expose all students to precise, logical and coherent thinking. This then gives another reason why teachers must know the kind of mathematics that respects these basic characteristics.

However, the ultimate justification of why mathematics teachers must know these five characteristics must lie in a demonstration that those who do are better teachers, in the sense that they can make themselves better understood by their students and therefore have a better chance of winning their students' trust. These are testable hypotheses for education research, even if such research is missing at the moment. In the meantime, life goes on. Instead of waiting for research data and doing nothing, we proceed to make a simple argument for the case that teachers should know these basic characteristics, and also give several examples for illustration.

The simple argument is that many students are turned off by mathematics because they see it as one giant black box to which even their teachers do not hold the key. Therefore teachers who can make transparent what they are talking about (cf. *definitions* and *precision*), can explain what they ask students to learn (cf. *reasoning* and *coherence*), and can explain why students should learn it (cf. *purposefulness*) have a much better chance of opening up a dialogue with their students and inspiring them to participate in the doing of mathematics.

We divide the remaining discussion into two parts. In Part I, we use examples from several standard topics in school mathematics to show how teachers who know the basic characteristics of mathematics are more likely to be able to teach these topics in a meaningful way. Part II highlights the main points in the school curriculum that are often misrepresented in school mathematics. These, therefore, should be the focus of professional development.

Part I: Some Examples

EXAMPLE 1. **Place value.**

Consider a number such as

$$\underline{\underline{7}} \quad 8 \quad \underline{\underline{7}} \quad 5 \quad \underline{\underline{7}}$$

We *tell* students that the three 7's represent different values as a matter of convention, and yet we expect them to have *conceptual understanding* of place value. When all is said and done, it is difficult to acquire conceptual understanding of a set of rules. This incongruity between our pedagogical input and the expected outcome causes learning difficulties.

Teachers who know the way mathematics is developed through ***reasoning*** would look for ways to explain the reason for such a rule. When they do, they will discover that, indeed, the rules of place value are logical consequences of the way we *choose to count*. This is the decision that we count using only ten symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Many older numeral systems, for example the Egyptian numeral system, pretty much made up symbols for large numbers as they went along: one symbol for a hundred, another one for a thousand, and yet another one for ten thousand, etc. But by limiting ourselves to the use of ten symbols and no more, we are forced to use more than one position (place) in order to be able to count to large numbers.

To illustrate the underlying reasoning and at the same time minimize the enumeration of numbers, we will use *three* symbols instead of ten: 0, 1, 2. Of course, counting now stops after three steps. To continue, one way is to repeat the three symbols indefinitely:

0	1	2	
0	1	2	
0	1	2	etc.

This allows us to continue counting all the way to infinity, but the price we pay is that we lose track of where we are in the endless repetitions. In oder to keep track of the repetitions, we label each repetition by a symbol to the *left*:

00	**0**1	**0**2
10	**1**1	**1**2
20	**2**1	**2**2

Adding one symbol to the left of each group of

0 1 2

allows us to count, without ambiguity, up to nine numbers (we only have three symbols to add to the new position on the left!). Then we are stuck again. To keep going, we again try to repeat these nine groups of symbols indefinitely:

00	01	02	10	11	12	20	21	22	
00	01	02	10	11	12	20	21	22	
00	01	02	10	11	12	20	21	22	etc.

This has the same drawback of ambiguity as before unless we again label each repetition by a symbol to the *left*:

000	**0**01	**0**02	**0**10	**0**11	**0**12	**0**20	**0**21	**0**22
100	**1**01	**1**02	**1**10	**1**11	**1**12	**1**20	**1**21	**1**22
200	**2**01	**2**02	**2**10	**2**11	**2**12	**2**20	**2**21	**2**22

By adopting the **convention** of omitting the 0's on the left, we obtain the first 27 numbers ($27 = 9 + 9 + 9$) in our counting scheme:

0	1	2	10	11	12	20	21	22
100	101	102	110	111	112	120	121	122
200	201	202	210	211	212	220	221	222

The next step is to repeat these 27 numbers indefinitely, and then label each of them by labeling each group of 27 numbers with a 0 and 1 and 2 to the left, thereby obtaining the first 81 (= 27 + 27 + 27) unambiguous numbers in our counting scheme. And so on.

This way, students get to see the *origin of place value*: we use three places only after we have exhausted what we can do with two places. Thus the *2* in *2*01 stands not for 2, but the *third* round of repeating the *9* two-digit numbers, i.e., the *2* in *2*01 signifies that this is a number that comes after the 18th number 200 (in daily life we start counting from 1), and the second and third "digits" 01 signify more precisely that it is in fact the 19th number $(9 + 9 + 1)$.

In the same way, if we use ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and adapt the preceding reasoning, we see that for a number such as 374,

> the *3* in *3*74 signifies the *fourth* round of repeating the 100 two-digit numbers and therefore 374 is a number that comes after the 300th number, 300, while the second and third "digits" 74 signify more precisely that 374 is the 300th-and-70th-and-4th number.

When teachers know the underlying reasoning of place value, they will find a whole host of pedagogical options opened up for them. Instead of simply laying down a set of rules that each place stands for a different value, they can now lead their young charges step-by-step through the counting process and make them see for themselves why each *place* of a number has a different meaning. Moreover, they can also invite their students to experience the counting process in a different context by using any number of symbols (just as we used three above). In mathematics, it is the case that content knowledge heavily influences pedagogy ([Wu] 2005).

EXAMPLE 2. **Standard algorithms.**

The teaching of whole number standard algorithms was a flash point of the Math Wars. As key theorems in the study of whole numbers, there is absolutely no doubt that these algorithms and their explanations should be taught. Nevertheless, their great merit may not always be obvious to elementary students, and the need for teachers to plead their case has thus become a necessity. A teacher must be aware of the two characteristics of ***definitions*** and ***coher-***

ence in mathematics in this situation. A minimum requirements for success in teaching these algorithms is to always make explicit the definitions of the four arithmetic operations. For example, **whole number addition** is by definition *continued counting*, in the sense that the meaning of $1373+2615$ is counting 2615 times beyond the number 1373. A teacher who appreciates the importance of definitions would emphasize this fact by making children add manageable numbers such as $13+9$ or $39+57$ by brute force continued counting. When children see what kind of hard work is involved in adding numbers, the addition algorithm comes as a relief because *this algorithm allows them to replace the onerous task of continued counting with numbers that may be very large by the continued counting with only single digit numbers.* Armed with this realization, they will be more motivated to memorize the addition table for all single digit numbers as well as to learn the addition algorithm. It will also give them incentive to learn the reasoning behind such a marvelous labor-saving device.

Multiplication being repeated addition, a simple multiplication such as 48×27 would require, by definition, the addition of $27 + 27 + \cdots + 27$ a total of 48 times. In this case, even the addition algorithm would not be a help. Again, a teacher who wants to stress the importance of definitions would, for example, make children get the answer to 7×34 by actually performing the repeated addition. Then teaching them the multiplication algorithm and its explanation becomes meaningful. Because this algorithm depends on knowing single-digit multiplications, children get to see why they should memorize the multiplication table. (***Purposefulness***.) Similar remarks can be made about subtraction and division.

The sharp contrast between getting an answer by applying the clumsy definition of each arithmetic operation and by using the relatively simple algorithm serves the purpose of highlighting the virtues of the latter. Therefore a teacher who emphasizes definitions in teaching mathematics would at least have a chance of making a compelling case for the learning of these algorithms, and these algorithms deserve nothing less. More is true. It was mentioned in passing that the addition and multiplication algorithms depend on single-digit computations. It is in fact a unifying theme that *the essence of all four standard algorithms is the reduction of any whole number computation to the computation of single-digit numbers.* This is a forceful illustration of the ***coherence*** of mathematics, and a teacher who is alert to this basic characteristic would stress this commonality among the algorithms in her teaching. If a teacher can provide such a conceptual framework for these seemingly disparate algorithmic procedures, she would increase her chances of improving student learning (cf. similar discussions in [Pesek-Kirschner], 2000; [Rittle-Johnson-Alibali], 1999).

Incidentally, for an exposition of the division algorithm that brings out the fact that this algorithm is an iteration of single digit computations, see the discussion below in item (A) of **Whole numbers**, Part II.

EXAMPLE 3. **Estimation.**

In recent years, the topic of estimation has become a staple in elementary mathematics instruction. In the form of **rounding**, it enters most curricula in the second grade. Textbooks routinely ask students to round whole numbers to the nearest one, nearest ten, nearest hundred, etc., without telling them when they should round off or why. A teacher who knows about the ***purposefulness*** of mathematics knows that if a skill is worth learning, then it cannot be presented as a meaningless rote exercise. She would introduce in her lessons examples in daily life that naturally call for estimation. For example, would it makes sense to say in a hilly town that the temperature of the day is 73 degrees? (No, because the temperature would depend on the time of the day, the altitude, and the geographic location. Better to round to the nearest 5 or nearest 10. "Approximately 70 degrees" would make more sense.) If Garth lives two blocks from school, would it make sense to say his home is 957 feet away from school? (No, because how to measure the distance? From door to door or from the front of Garth's garden to the front of the school yard? Measured along the edge of the side walk or along the middle of the side walk? Or is the distance measured as the crow flies? Etc.) Better to round to the nearest 50, or at least to the nearest 10. Many other examples such as a city's population or the length of a students' desk (in millimeters) can also be given. When a teacher can present a context and a need for estimation, the rounding of whole numbers becomes a meaningful, and therefore learnable, *mathematical* skill.

Textbooks also present estimation as a tool for checking whether the answer of a computation is reasonable. Here is a typical example. Is $127 + 284 = 411$ likely to be correct? One textbook presentation would have students believe that, since rounding to the nearest 100 changes $127 + 284$ to $100 + 300$, which is 400 and 400 is close to 411, therefore 411 is a reasonable answer. A teacher who is aware of the need for ***precision*** in mathematics would be immediately skeptical about such a presentation. She would ask what is meant by "400 is close to 411". If we change the problem to $147 + 149 = 296$, how would this approach to estimation check the reasonableness of the answer? Rounding to the nearest 100 now changes $147 + 149$ to $100 + 100$, which is 200, should we consider 200 to be "close" to 296? The teacher therefore realizes that even in estimation, there is a need to be precise. She would therefore forsake such a cavalier approach to estimation and teach her students instead about the inevitable **errors** that come with each estimation. Each rounding to the nearest 100 could bring either an over-estimate or an under-estimate up to 50, and she would teach the notation of $\pm\mathbf{50}$ to express this fact. When one adds two such estimations of rounding to the nearest 100, the error could therefore be as high as ± 100. And this was exactly what happened with the estimation of $147+149$: the error of 200 compared with the exact value 296 is 96, which is almost 100. If we use rounding to the nearest 100 to check whether an addition of two 3-digit numbers is reasonable, we *must expect an error possibly as high as 100.* In this way, she shows her students that the declaration of "closeness" for this way of checking an addition is completely meaningless. She would tell her students that if they really want a *good* estimate, in the sense of the estimation being within 10 of the true value, they should round to the nearest 10, in which case the previous reasoning yields an error of ± 10. By rounding to the nearest 10, the addition $147 + 149$ becomes $150 + 150 = 300$, and since 296 is within 10 of 300, one may feel somewhat confident of the answer of 296.

In this case, teaching the concept of the error of an estimation helps to steer the teacher away from teaching something that is mathematically incorrect. Because the mathematics in school textbooks is often unsatisfactory (compare [Borisovich], or [NMPb], Appendix B of Chapter 3, pp. 3-63 to 3-65), a knowledge about the basic characteristics of mathematics in fact becomes indispensable to the teaching of mathematics. Incidentally, if the consideration of estimation is part of a discussion in a sixth or seventh grade class, then the concepts of "absolute error" and "relative error" should also be taught.

EXAMPLE 4. **Translations, rotations, reflections.**

We will refer to these basic concepts in middle school geometry as **basic**

rigid motions. The way these concepts are usually taught, they are treated as an end in itself. Students do exercises to learn about the effects of these basic rigid motions on simple geometric figures. They are also asked to recognize the translation, rotation, and reflection symmetries embedded in patterns and pretty tessellations. This is more or less the extent to which the basic rigid motions are taught in the middle grades. One gets the impression that the basic rigid motions are offered as an aid to art appreciation.

A teacher cognizant of the ***purposefulness*** of mathematics would try to direct the teaching of these concepts to a mathematical purpose. She would teach the basic rigid motions as the basic building blocks of the fundamental concept of **congruence**: two geometric figures are by definition congruent if a finite composition of a translation, a rotation, and/or a reflection brings one figure on top of the other. The basic rigid motions are *tactile* concepts; many hands-on activities using transparencies can be devised to help students get to know them. In school mathematics, congruence is usually defined as "same size and same shape". To the extent that there is no discernible systematic effort in professional development to correct such a lapse of ***precision***, this hazy notion of congruence is what our teachers have be forced to live with.[1] By contrast, the definition of congruence in terms of the basic rigid motions is mathematically accurate, is (as noted) tactile, and therefore by comparison learnable. A teacher who knows the basic characteristics of mathematics would have a much better chance of making students understand what congruence is all about.

A teacher aware of the ***coherence*** of mathematics would also make an effort to direct students' attention to the role played by congruence in other areas of mathematics. She would underscore, for instance, the fact that a basic requirement in the definition of **geometric measurements** (i.e., length, area, or volume) of geometric figures is that congruent figures have the same geometric measurements. By emphasizing this property of geometric measurement vis-a-vis congruence, a teacher can greatly clarify many of the usual area or volume computations. (See the discussion in item (D) below on **Geometry**.) Such considerations enable students to see why they should learn about congruence. (***Purposefulness*** again.)

EXAMPLE 5. **Similarity.**

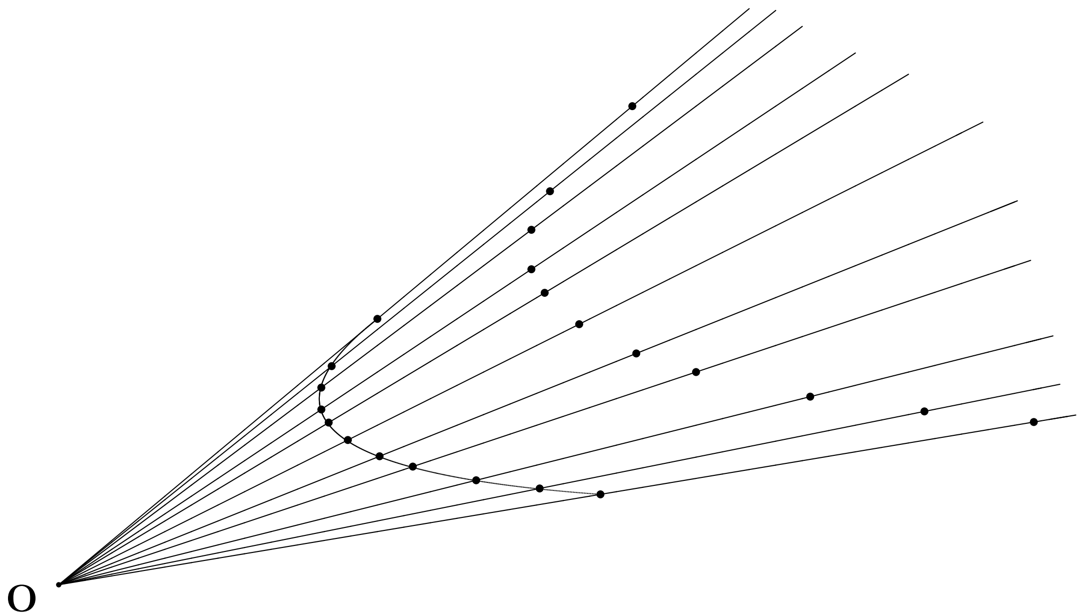
[1]This is another striking example of the absence of mathematical engineering.

The concept of *similarity* is usually defined to be "same shape but not necessarily the same size". This phrase carries as much (or if one prefers, as little) information as "same size and same shape". A teacher who values the ***precision*** of mathematics would know that this is not a mathematically valid definition of similarity that she can offer to her students. For example, how would "same shape but not necessarily the same size" help decide whether the following two curves are similar, and if so, how?

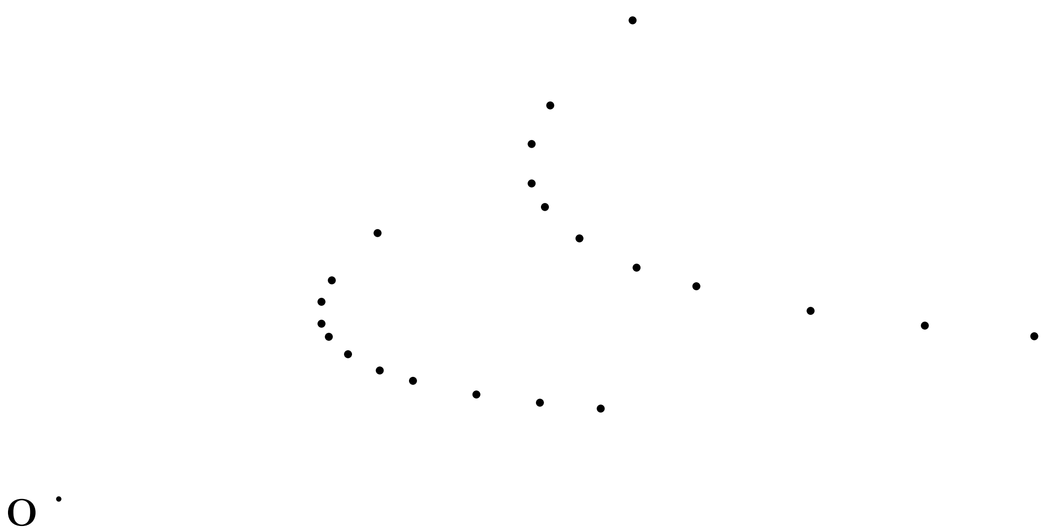
A correct definition of similarity, one that can be taught to middle school students, turns out to be both elementary and teachable. For convenience, restrict ourselves to the plane. A **dilation** with **center** O and **scale factor** r ($r > 0$) is the transformation of the plane that leaves O unchanged, but moves any point P distinct from O to the point P' so that O, P, P' are collinear, P and P' are on the same half-line relative to O, and the length of OP' is equal to r times the length of OP. Here is an example.

Consider the dilation with center O and scale factor 1.8. What does it do to the curve as shown?

We take a few points on the curve (in this case, eleven, as shown) and proceed to dilate every one of these points according to the definition.



What we get is a “profile” of the dilated curve consisting of eleven points.



If we use more points on the original curve and dilate each point the same way, we would get a better approximation to the dilated curve itself. Here is an example with 150 points chosen on the original curve.

In the middle school classroom, a teacher can capture students' imagination by telling them that, through the use of dilation, they can *expand* (scale factor > 1) or *contract* (scale factor < 1) any geometric figure. In fact, this is the basic principle behind digital photography for expanding or contracting a picture. Both facts are likely to be new to the students.

Now the definition of similarity: two geometric figures are **similar** if a dilation followed by a congruence bring one figure on top of another. Again, such a precise definition makes similarity a tactile and teachable concept. To go back to the two curves at the beginning of this discussion, it turns out that they are similar because, after dilating the right curve by a scale factor of 1.5, we can make it coincide with the curve on the left using the composition of a 90° counter-clockwise rotation, a translation, and a reflection across a vertical line.

EXAMPLE 6. **Fractions, decimals, and percent**.

Here we focus on the teaching of these topics in *grade 5 and up.* We do so because this is where the informal knowledge of fractions in the primary grades begins to give way to a formal presentation, and where students' drive to achieve algebra begins to take a serious turn. This is where abstraction becomes absolutely necessary for the first time and, not coincidentally, this is where non-learning of mathematics begins to take place on a large scale (cf. [Hiebert-Wearne] 1986, [Carpenter-Corbitt et. al] 1981).

In the following discussion, we use the term **fraction** to stand for numbers of the form $\frac{a}{b}$, where a and b ($b \neq 0$) are whole numbers, and the term **decimals** to stand for *finite decimals*.

In broad terms, standard instructional materials ask students to believe that

> a **fraction** is a piece of pizza, part of a whole, a division, and a ratio;
>
> a **decimal** is a number one writes down by using the concept of *extended place value*: 43.76 is 4 tens, 3 ones, 7 tenths, and 6 hundredths.
>
> a **percent** is part of a hundred.

Teachers who have taught decimals this way are well aware of the elusiveness (to a student) of the concept of extended place value, to the point that students cannot form a concrete image of what a decimal is. One should be able to get research data on this observation directly. And of course, students are also urged to "reason mathematically" using these concepts to solve problems when all they are given is this amorphous mess of information.

A teacher who knows the ***coherence*** characteristics of mathematics would know that, insofar as fractions, decimals and percents are *numbers*, the concept of a number should not be presented to students in such a fragmentary manner as suggested by the above sequence of "definitions". For example, the suggested "definition" of a fraction has too many components, and some of them don't even make sense. What is a "ratio"? If students already know what a "ratio" is, would they need a definition of a "fraction"? If a fraction is just a piece of pizza, then how to multiply two pieces of pizza? And these are only two of the most naive concerns. Moreover, since decimals and percent are numbers, it is important that students feel at ease about computing with them. Therefore, if we try to relate this notion of 43.76 to something familiar to students, could it be $40+3+\frac{7}{10}+\frac{6}{100}$? If so, isn't a decimal a fraction, and therefore why not say explicitly that a decimal is a fraction obtained by adding the above fractions? By the same token, is "part of a hundred" supposed to mean a fraction whose denominator is 100? If so, why not use this as the definition instead of the imprecise phrase "part of a hundred"?

The teacher would recognize the need for a ***definition*** of a fraction that is at once precise and correct. One such definition is to say that a fraction is **a point on the number line** constructed in a precise, prescribed manner, e.g., $\frac{2}{3}$ is the 2nd division point to the right of 0 when the segment from 0 to 1 is divided into 3 segments of equal length (see the discussion in item (B) in **Fractions**, Part II, below; for an extended treatment, see [Wu2002b], and in a slightly different form, [Jensen] 2003). Building on this foundation, she can define decimal and percent as special kinds of fractions in the manner described above. After these definitions have been put firmly in place, she would then using *reasoning* to explore other implications and representations of these concepts. Here are the relevant definitions:

> a **decimal** is any fraction with denominator equal to a power of 10, and the decimal point notation is just an abbreviation for the power, e.g. 3.52 and 0.0067 are, by definition, $\frac{352}{10^2}$ and $\frac{67}{10^4}$, respectively, and a **percent** is a fraction of the form $\frac{N}{100}$, where N is a fraction. (Notation: $\frac{N}{100}$ is written as $\boldsymbol{N\%}$.)

Note that the $\frac{N}{100}$ above is the fraction obtained by dividing the fraction N by the fraction $\frac{100}{1}$. The division $\frac{A}{B}$ of two fractions A and B ($B \neq 0$) is called in school mathematics a **complex fraction**, and B will continue to be called the **denominator** of the complex fraction $\frac{A}{B}$. Therefore a percent is strictly speaking a *complex fraction whose denominator is 100.*[2]

What this discussion suggests is that a teacher well aware of the importance of ***coherence*** in mathematics would be more likely to find a way to present the concepts of fraction, decimal, and percent from a unified perspective, thereby lighten the cognitive load of students and make these traditionally difficult concepts more transparent and more learnable. Since such a suggestion does not as yet have support by data, it should be a profitable topic for education research.

What are the advantages of having a coherent concept of fractions, decimal, and percent? Consider what is supposed to be a difficult problem for sixth and seventh graders:

> What percent of 76 is 88?

How might a teacher who is at ease with the basic characteristics of mathematics handle this? Knowing that percent is a fraction, she would first suggest to her students to look at an easier cognate problem: *what fraction of 76 is 88?* This

[2]Complex fractions are extremely important for the discussion of ratio and rates in school mathematics and in preparing students for algebra. Their neglect in the school curriculum is inexplicable and inexcusable. See §9 of [Wu2002b].

problem is, by comparison, more straightforward , namely, if $\frac{k}{\ell}$ is the fraction so that $\frac{k}{\ell}$ of 76 is 88, then a direct translation of "$\frac{k}{\ell}$ of 76 is 88" is[3]

$$\frac{k}{\ell} \times 76 = 88$$

From this, she gets $\frac{k}{\ell} = \frac{22}{19}$. Now she returns to the original problem: suppose $N\%$ of 76 is 88. Since a percent is a fraction, the same reasoning should be used to get

$$N\% \times 76 = 88$$

Therefore $N = \frac{8800}{76} = 115\frac{15}{19}$. Thus the answer is $115\frac{15}{19}\%$.

We give two more examples. The same teacher would use the definition of a decimal to give a simple explanation of the multiplication algorithm for decimals. For example, the algorithm says that to multiply 2.6×0.105,

(α) multiply the corresponding whole numbers 26×105, and

(β) put the decimal point 4 (= 1+3) places to the left of the last digit of 26×105 because the decimal point in 2.6 (resp., 0.105) is 1 place (resp., 3 places) to the left of the last digit of 26 (resp., 105).

She would use this opportunity to illustrate the use of precise definition in mathematical ***reasoning***. She would calculate 2.6×0.105 this way:

$$\begin{aligned} 2.6 \times 0.105 &= \frac{26}{10} \times \frac{105}{10^3} && \text{(by definition)} \\ &= \frac{26 \times 105}{10 \times 10^3} && \text{(this is } (\alpha)) \\ &= \frac{2730}{10^{1+3}} \\ &= 0.2730 && \text{(this is } (\beta)) \end{aligned}$$

For the second example, recall that we brought up the concept of *extended place value* in the usual definition of a decimal. In the education literature, this *ad hoc* idea has to be accepted on faith for the understanding of a decimal. A teacher

[3]This requires a thorough understanding of fraction multiplication. See the discussion in item (B) below in **Fractions**.

who believe in the ***coherence*** of mathematics might ask whether the original concept of place value is not already sufficient for the same purpose. She would find out that indeed it is. For example, why is 3.712 equal to $3 + \frac{7}{10} + \frac{1}{100} + \frac{2}{1000}$? This is because if we know the *ordinary* concept of place value, then $3712 = 3000 + 700 + 10 + 2$. Therefore:

$$\begin{aligned} 3.712 &= \frac{3712}{1000} \\ &= \frac{3000 + 700 + 10 + 2}{1000} \\ &= \frac{3000}{1000} + \frac{700}{1000} + \frac{10}{1000} + \frac{2}{1000} \\ &= 3 + \frac{7}{10} + \frac{1}{100} + \frac{2}{1000} \end{aligned}$$

EXAMPLE 7. **The equal sign**.

Education research in algebra sees students' defective understanding of the equal sign as a major reason for their failure to achieve algebra. It is said that students consider the equal sign

> an announcement of the result of an arithmetic operation

rather than as

> expressing a relation.

The conclusion is that the notion of "equal" is complex and difficult for students to comprehend.

A teacher who values ***precision*** in mathematics would immediately recognize such abuse of the equal sign as a likely result of too much sloppiness in the classroom. She knows how tempting such sloppiness can be. For example, it is so convenient to write "27 divided by 4 has quotient 6 and remainder 3" as

$$27 \div 4 = 6 \ \text{ remainder } \ 3$$

Here then is a prime example of using the equal sign as "an announcement of the result of an arithmetic operation". But the teacher also recognizes the sad truth that this way of writing division-with-remainder is given in all the standard textbooks as well as in too many professional developmental materials for comfort.[4] She knows all too well if teachers are taught the wrong thing, then they will in turn teach their students the wrong thing. If classroom practices and textbooks encourage such sloppiness, students brought up in such an environment would naturally inherit the sloppiness. The so-called misconception of the equal sign is thus likely the inevitable consequence of flawed mathematics instruction that our teachers received from their own teachers and textbooks, which they impart, in turn, on their own students. Again, this is something education research could confirm or refute.

From a mathematical perspective, the notion of "equal" is unambiguous and is *not* difficult to comprehend. The concept of **equality** is a matter of precise definitions. If teachers can emphasize the importance of ***definitions***, and always define the equal sign in different contexts with ***precision*** and care, the chances of students abusing the equal sign would be much smaller. The principal concern for any misunderstanding of the equal sign is therefore something professional development should address.

To drive home the point that the concept of equality is a matter of definition, here is the list of the most common *definitions* of $\boldsymbol{A = B}$ that arise in school mathematics:

- A and B are expressions in **whole numbers**: A and B are verified to be the same number by the process of counting (e.g., $A = 2+5$, $B = 4+3$). If whole numbers are already placed on the number line, then $A = B$ means A and B are the same point on the number line.

- A and B are expressions in **fractions**: same point on number line (e.g., $\frac{1}{2} + \frac{1}{3} = 2 - 1\frac{1}{6}$).

- A and B are expressions in **rational numbers**: same point on number line (e.g., $\frac{1}{3} - \frac{1}{2} = 2 - 2\frac{1}{6}$).

- A and B are two **sets**: $A \subset B$ and $B \subset A$.

- A and B are two **functions**: A and B have the same domain of definition, and $A(x) = B(x)$ for all elements x in their common domain.

[4] The correct way of expressing the division-with-remainder is of course $27 = (6 \times 4) + 3$.

- $A = (a, a')$, $B = (b, b')$ are **ordered pairs** of numbers: $a = b$ and $b = b'$. (Same for ordered triples of numbers.)
- A and B are two **abstract polynomials**: pairwise equality of the coefficients of the same power of the indeterminate.

Part II: The Mathematics for Teachers of K–8

Let us next examine in some detail what mathematics teachers need to know about school mathematics. Perhaps as a consequence of the hierarchical nature of mathematics, the core content of school mathematics seems to be essentially the same in all the developed countries as far as we know. See [NMPb], Chapter 3, pp. 3-31 to 3-32. True, there are observable minor variations, but these variations all seem to be related to the grade level assigned to each topic or the sequencing of a few of the topics. For example, most Asian countries require calculus in the last year of school, whereas such is rarely the case in the U.S. Or, the U.S. curriculum generally dictate a year of algebra followed by a year of geometry, and then another year of algebra, but such an artificial separation is almost never followed in foreign countries (cf. e.g., [Kodaira], 1996 and 1997). We will have more to say about the later presently. Such differences are, however, insignificant compared with the overall agreement among nations on the core topics in school mathematics *before calculus*. In terms of grade progression, these core topics are essentially the following:

> whole number arithmetic, fraction arithmetic, negative numbers, basic geometric concepts, basic geometric mensuration formulas, coordinate system in the plane, linear equations and quadratic equations and their graphs, basic theorems in plane geometry, functions and their graphs, exponential and logarithmic functions, trigonometric functions and their graphs, mathematical induction, binomial theorem.

Once we accept that these core topics are what our teachers must teach, the hierarchical nature of mathematics mentioned above dictates, in the main, what teachers must learn and in what order they should learn it. In this sense, teacher's content knowledge is not circumscribed by education research but must also be informed by mathematical judgment.

Before we get down to the specifics of the mathematics we want teachers to learn, it behooves us to reflect on the nature of this body of knowledge as it has a bearing on many ongoing discussions about teachers. If we agree that teachers should know a more sophisticated version of school mathematics, the fact that school mathematics is an engineering product (see the discussion of mathematical engineering on page 3) means that what teachers should know must satisfy two seemingly incompatible requirements, namely,

(i) it is mathematics that respects the five basic characteristics, and

(ii) it is sufficiently close to the established school curriculum so that teachers can make direct use of it in the school classroom without strenuous effort.

Let us address specifically the school mathematics of K-8. There is probably no better illustration of the dichotomous nature of school mathematics than the case of fractions, a topic that has been discussed in Example 6 of Part I. From the standpoint of advanced mathematics, the concept of a fraction, and more generally the concept of a rational number, is simplicity itself. Junior level algebra courses in college deal with rational numbers and their arithmetic in less than a week. Given the notorious non-learning of fractions and rational numbers in grades 5–7, it is natural to ask why we don't just use what works in college to teach school students. The simple reason is that the college treatment of fractions requires that we define a fraction as an equivalence class of ordered pairs of integers. It is not just that our average fifth graders, in terms of mathematical maturity, are in no position to work on such an abstract level, but that more pertinently, fifth graders, through their experience, conceive of fractions as parts of a whole, and this conception is worlds apart from ordered pairs of integers or equivalence relations. To facilitate student learning, a theory of fractions for fifth graders would have to take into account such cognitive developments.

A course for teachers on fractions, if it is to be useful to them in the school classroom, therefore cannot adopt the abstract approach and ask each prospective teacher to do research of their own in order to bring such abstract knowledge down to the elementary and middle school classsrooms. This kind of research, nontrivial as it is, is best left to professional mathematicians. By the same token, a mathematics course for elementary teachers also must not teach fractions in the usual chaotic and incomprehensible manner (cf. Example 6) and then expect our prospective teacher to miraculously transform such chaos into meaningful lessons in the classroom. Least of all should we expect to achieve improvement in school mathematics education simply by exhorting our teachers to teach for conceptual understanding while continuing to feed these same teachers such chaotic information. ***We must teach our teachers materials that have gone through the process of mathematical engineering.***

There have been some attempts to bridge the chasm between the abstract approach and what is useful in the fifth grade classroom. For example, one textbook for professional development defines a fraction $\frac{a}{b}$ to be the solution of the equation $bx = a$, but then *it goes on to discuss fractions without once making use of this definition for logical reasoning.* In this case, because the "definition" is detached from the logical development, it ceases to be a definition in the mathematical sense. Such a development ill serves both mathematics and education.

What was said about the subject of fractions is of course true for most other topics of school mathematics: the concepts of negative numbers, straight line, congruence, similarity, length, area, volume, together with all their associated logical developments. For example, one cannot teach in the school classroom, at any level, the area of a region in the plane as its so-called Lebesgue measure or even the value of an integral. Nor for that matter can one define a line as the graph of a linear equation in two variables (as is done in advance mathematics). What pre-service professional development needs are, as we said, courses in mathematical engineering. In other words, these should be courses which are devoted to mathematics but which have gone through a careful engineering process. Such courses unfortunately have been in short supply in university campuses up to this point, and one can only hope that the situation will change for the better in the near future.

Currently, there has been a lot of interest in mathematics teachers' content knowledge, especially in its effect on student achievement. One measure of this content knowledge is the number of mathematics courses teachers have taken. For example, in [Kennedy-Ahn-Choi], it is assumed that "courses in mathematics represent content knowledge". One purpose of the present discussion is to call attention to the fact that, until mathematics departments and schools of education take mathematical engineering seriously, mathematics courses are not likely to be very relevant to teachers' ability to teach better in the school classroom, and the number of mathematics courses teachers have taken will continue to be a defective measure of their content knowledge for teaching.

But to go back to our task at hand, we now describe, from the standpoint of mathematical engineering, what needs to be taught to teachers of grades K-8, more or less in accordance with the list of core topics enunciated above. It will be clear to one and all that the description itself meticulously observes requirement (*ii*) above of what teachers should know, namely, it is always close to the average school curriculum.

(A) **Whole numbers** The basis of all mathematics is the whole numbers. In particular, a complete understanding of the whole numbers and its arithmetic operations is the core of the knowledge teachers need in K-3. What is often not recognized is the fact that an adequate understanding of place value, the central concept in discussions of whole numbers, only comes with an understanding of how to *count* in the Hindu-Arabic numeral system. If teachers have difficulty convincing children that, for example, the 3 in 237 stands not for 3 but for 30, it may be because children only know it as one among hordes of other rules that they have to memorize. Suppose now we explain to teachers the fundamental idea of the Hindu-Arabic numeral system as the use of *exactly* ten symbols {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} to count indefinitely. Then after the first round of using these symbols to count from 0 up to 9, we would be stuck unless we are allowed to also these ten symbols in a place to its left, as follows. We "recycle" these ten symbols ten times, and each time we systematically place one of the ten symbols {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} to its left so as to keep track of the continued counting. Thus we begin with

$$00 \quad 01 \quad 02 \quad 03 \quad 04 \quad 05 \quad 06 \quad 07 \quad 08 \quad 09$$

Then we continue with the same row of numbers but with a 1 placed to the left in place of 0:

$$10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18 \quad 19$$

We continue with the same row of numbers once more but with a 2 instead of 1 placed to the left:

$$20 \quad 21 \quad 22 \quad 23 \quad 24 \quad 25 \quad 26 \quad 27 \quad 28 \quad 29$$

We continue this way until we get to

$$90 \quad 91 \quad 92 \quad 93 \quad 94 \quad 95 \quad 96 \quad 97 \quad 98 \quad 99$$

At this point we can count no more unless we agree to allow the placement of the same ten symbols in another place to the left. We do the same by "recycling" the 100 numbers {00, 01, 02, . . . , 09, 10, 11, ... , 98, 99} and by pacing each of the ten symbols {1, 2, 3, 4, 5, 6, 7, 8, 9} in succession in the place to its left so as to keep track of the continued counting. Thus, after

$$000 \quad 001 \quad 002 \quad \cdots \quad 009 \quad 010 \quad \cdots \quad 097 \quad 098 \quad 099$$

we continue with the same row of 100 numbers but with a 1 replacing the 0 on the left:

$$100 \quad 101 \quad 102 \quad \cdots \quad 110 \quad 111 \quad \cdots \quad 197 \quad 198 \quad 199$$

Then we replace the 1 on the left with a 2:

$$200 \quad 201 \quad 202 \quad \cdots \quad 210 \quad 211 \quad \cdots \quad 297 \quad 298 \quad 299$$

And so on. We remark that in normal usage, we omit the writing of 0's on the left, so that 001 is just 1, 091 is just 91, etc.

As a teacher, the advantage of learning how to count in this fashion is that she now sees the appearance of each digit in each place to the left *as a necessity*, so that there is no more doubt as to why the 3 in 237 represents 30 and not 3. This is because in the above counting process, we don't get to 37 until we have recycled the *10* symbols {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} a *3*rd time, and then count 7 more. With the counting process clearly understood, the teacher would have a far better chance of clearly explaining what place valuer means. More than that, she can also explain, for two whole number m and n, what it means for one to be bigger than the other. Precisely, **m is smaller than n,** or in symbols**, $m < n$**, if m comes before n in the counting process. It then becomes possible to explain why a two digit number is smaller than a three-digit number: a two-digit number is of course a three digit number with a 0 to the left, so in the way counting as done above, those three digit numbers with a leading 0 comes before those with a leading 1, and the latter come before those with a leading 2, etc. In the same way, we see why a number with 12 digits is smaller than one with 14 digits, etc.

Without knowing how to count, the fact that a 12 digits number is smaller than a 14 digit number would of course be strictly a matter of faith.

A second overriding theme about whole numbers that should be part of the basic knowledge of elementary teachers is the fact that *all the standard algorithms* ($+$, $-$, $\times$, $\div$) *reduce whole number computations to single-digit computations*. This fact was already discussed in Example 2 of Part I. A recurrent theme of mathematics is in fact to always try to break down complex concepts or skills to simpler ones. The fact that such a simplification is possible in the Hindu-Arabic numeral system is a fantastic achievement. This overriding theme should be emphasized in teaching the standard algorithms because it gives a conceptual framework for the learning of these algorithms.

A common failing in the teaching of the standard algorithms is the lack of emphasis on the *definitions* of the arithmetic operations. (Again, see Example 2 of Part I.) For example, the teaching of the addition algorithm usually goes straight to the column-by-column mechanism and the technique of carrying without a word said about what it means to add two whole numbers. For sure, such teaching necessarily demeans the algorithm to a trick, and nothing more. If we begin by explicitly defining addition as *continued counting*, so that $1373 + 2615$ is counting 2615 times beyond 1373, then students would more likely recognize that such an addition problem is no easy task. When this is understood, then the extremely simple procedure of obtaining the answer by doing four single-digit additions, $3 + 5$, $7 + 1$, $3 + 6$, and $1 + 2$, becomes truly impressive. All teachers should be able to convey this sense of wonderment to their students, and this would not happen if teachers are not taught this knowledge.

Once the teachers are secure in this knowledge, then they would recognize the extra bit of work to cope with the phenomenon of *carrying* in the addition algorithm is just that, an extra bit of work. The same remark applies to the subtraction algorithm and trading.

Because the long division algorithm is the most challenging of the four standard algorithms, a few comments may be in order. First of all, the name of the "long division algorithm" is misleading: it is not about *division* per se but about division-with-remainder; the latter is not a "division" in the usual mathematical sense. In the context of school mathematics as of 2008, division-with-remainder needs careful explanation for the reason that it is so carelessly taught in general as to not even give a proper definition of the concept of the *remainder*. But as is well-known, division-with-remainder is an important topic because it underlies the Euclidean algorithm (see below) and the division algorithm of polynomials.

But to return to the long division algorithm, it is an **iteration** of divisions-with-remainder. Here is an example to clarify this comment.

Consider the division of 586 by 3. At the outset, it should be made clear that what the long division algorithm is about: it is an algorithm to compute, *digit by digit*, the quotient and the remainder of the division-with-remainder of 586 by 3 which, when expressed correctly, states:

$$586 = (195 \times 3) + 1$$

School mathematics has a long tradition of expressing this fact as $583 \div 3 = 195\ R1$. *This is an outright abuse of the equal sign*, which contributes to much confusion in education research and should be studiously avoided (see Example 7 of Part I). Now, the usual way to express the long division algorithm is the following:

$$\begin{array}{rcccc} & & 1 & 9 & 5 \\ \cline{2-5} 3 &) & 5 & 8 & 6 \\ & & 3 & & \\ \cline{3-5} & & 2 & 8 & \\ & & 2 & 7 & \\ \cline{3-5} & & & 1 & 6 \\ & & & 1 & 5 \\ \cline{3-5} & & & & 1 \end{array}$$

It is not difficult to see that the algorithm, in this special case, is *completely* captured by the following three simpler divisions-with-remainder, which will be referred to as the **procedural description** of the algorithm. Observe that the quotient can be read off, *digit-by-digit*, vertically from the first entries on the right sides and that the remainder is the last number in the last line:

$$\begin{aligned} 5 &= (\boxed{1} \times 3) + 2 \\ 28 &= (\boxed{9} \times 3) + 1 \\ 16 &= (\boxed{5} \times 3) + \boxed{1} \end{aligned}$$

The mechanism for going from one division-with-remainder in this array to the next is the following: the first division-with-remainder takes the *left digit* of the dividend (586) as its own dividend, and in general, the dividend of each succeeding division-with-remainder is obtained by taking the remainder of the preceding one, multiply it by 10, and add to it the next digit in the original dividend (586 in this case). It is now a simple exercise to make use of the procedural description together with the expanded form of 586 as $586 = (5 \times 10^2) + (8 \times 10) + 6$ to *derive* the desired conclusion that, in fact, $586 = (195 \times 3) + 1$. Insofar as this explanation does not depend on the specific numbers 586 and 3, it gives a general understanding of why the long division algorithm always yields the correct division-with-remainder for any two whole numbers. Teachers should get to understand this beautiful algorithm in such depth before good teaching can take place in the school classroom.

The basic technique of division-with-remainder has multiple implications in the school classroom. The first are the various **divisibility rules**, such as a whole number n is divisible by 3 if and only if the whole number obtained by adding the digits of n is divisible by 3. Emphasize that *all such divisibility rules are nothing more than a consequence of the behavior of a power of 10 when it is divisible by a single digit number* (such as 3). (Incidentally, the divisibility rule for 7 is so complicated that it does not deserve to be taught in the professional development of elementary teachers.) Because these divisibility rules are usually taught as unexplained tricks in the elementary classroom, their explanations are overdue. A more substantial application of division-with-remainder is the **Euclidean algorithm**, which is an algorithm for finding the **GCD** (greatest common divisor) of two whole numbers. (It is not necessary, but it is easier if the concept of an *integer* is available at this juncture.) In the process of developing this algorithm, the following basic fact will be uncovered: the GCD k of two whole numbers a and b can be expressed as:

$$k = ma + nb$$

for some integers m and n. From this, it follows that if a prime number p divides a product of whole numbers ab, and p does not divide a, then p must divide b. The Fundamental Theorem of Arithmetic is now a simple consequence. Students in grades 5-7 may not fully grasp the significance of the *uniqueness* statement, but elementary teachers must make an effort to come to an understanding of this fact. The concept of uniqueness, while subtle (Euclid did not have it, for example), is of fundamental importance in modern mathematics, but it is rarely taught in K–12. If elementary teachers can begin to informally teach this idea, and middle and high school teachers can continue to keep this idea alive in classroom discussions, all students would benefit from such instruction.

In recent years, the subject of estimation has been emphasized in K–6, and rightly so, but what has found its way into textbooks on the subject tends to misrepresent the reason for this emphasis. (See Example 3 of Part I.) Let us begin with an enumeration of some of the troubling issues. The first one is that it is difficult to get a correct description of the rounding of numbers to the nearest 10, nearest 100, nearest 1000, etc. In standard textbooks, students are usually taught to *round a whole number n to the nearest 10* by the following algorithm: if the ones digit of n is ≤ 4, change it to 0 and leave the other digits unchanged, but if the ones digit is $b \geq 5$, then change it to 0 but also increase the tens digit by 1 and leave other digits unchanged. This is correct in most cases, but collapses completely in the case of a number such as 12996. A correct formulation of rounding a whole number n to the nearest 10 is the following:

> Write n as $N + \overline{n}$, where $\overline{n}$ is the single-digit number equal to the ones digit of n (and hence N is the whole number obtained from n by replacing its ones digit with 0). Then rounding n to the nearest ten yields the number which is equal to N if $\overline{n} < 5$, and equal to $N + 10$ if $n \geq 5$.

One can give an analogous formulation for rounding to other powers of 10. A more fundamental issue is that estimation is taught as a rote activity with no thought given to convincing students that this is something worth learning. Students are not told why sometimes they should estimate (e.g., uncertainties in a measurement), under what circumstance an approximate answer is all that makes sense (e.g., built-in imprecision in the concept, such as distance from house to school or temperature of the day), or under what circumstance an estimation becomes an aid to achieving precision (e.g., the process of carrying out the long division algorithm). An even more serious concern is that students are not alerted to the need of always finding out about the **error** that comes with each estimation. In grades up to five (approximately), it is understood that only the concept of *absolute error* can be discussed, but starting with roughly grade six, students should be taught to routinely estimate the *percentage error*.

Professional development would do well to take these potential pitfalls into account and provide teachers with the kind of instruction that would enable them to avoid such pitfalls.

(B) **Fractions** Fractions are positive rational numbers for this discussion. Compare [Wu 2008], especially with regard to the research literature. We will address fractions in this section, and negative rational numbers in the next.

Before we begin the detailed discussion of teachers' knowledge of fractions, it should be stated up front that the kind of knowledge discussed below is essentially one that can be used in the classroom of grades 5–7 without drastic changes. Where then does this discussion leave the primary teachers? We believe that all elementary teachers, including primary teachers, should acquire such knowledge about fractions. The most obvious reason is that no elementary teacher can guarantee that they will teach only the primary grades the rest of their lives. The real reason is, however, the fact that what a teacher teaches in the primary grades may be simple, but it should still be a simplified version of *correct* mathematics. For example, a teacher familiar with a logical development of fractions would recognize the futility of relying exclusively on cutting pizzas in order to teach fractions. Such a teacher would be more likely to introduce the number line as early as possible. If a teacher knows how fractions can be developed in a way that is consonant with the basic characteristics of mathematics, then she would be immeasurably better equipped to provide primary students with the mathematical foundation they need in the later grades.

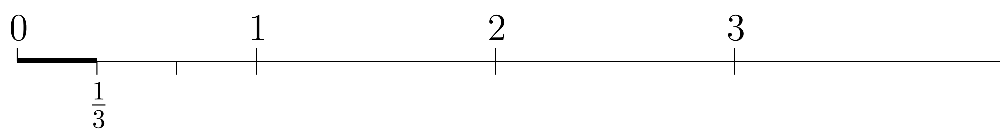
Traditionally, students' failure to learn fractions is explained by the disconnection in their understanding of the conceptual complexity of fractions (e.g., [Behr-Lesh-Post-Silver] 1983, [Bezuk-Bieck] 1993). Our teachers are therefore exhorted to develop a strong number sense about fractions and to develop an ability to think about fractions "in other ways" beyond part-whole. A recurrent theme in the education literature is that, to achieve flexibility in working with rational numbers, one must acquire a solid understanding of the different representations for fractions, decimals, and percents. Another theme, equally prized, is that a deep understanding of rational numbers should be developed through experiences with a variety of models, such as fraction strips, number lines, 10×10 grids, and area models.

There are two things fundamentally wrong with such a view of fractions. The first is that, if students are not told what a fraction is, any talk about their "different representations" of a fraction would be akin to talking about the spots on the skin of a unicorn. It is appealing, but it is educationally unsound. (See Example 6 of Part I.) The second thing wrong is that, up to this point, there has been hardly any *mathematically correct* presentation of the subject of fractions in the school classroom or pre-service professional development. Against this background, to talk about developing a "deep understanding" of fractions through hands-on experiences is therefore to concede that an incoherent presentation of fractions is a law of nature, so that we must settle for *ultra*-mathematical methods for the learning of fractions. Although such a view happens to be consistent with one commonly held in mathematics education research (cf. papers from the Rational Number Project, e.g., [Behr-Harel-Post-Lesh] 1992; also [Kieren] 1976 and [Vergnaud] 1983), it is not a mathematically acceptable view of a topic in *mathematics*.

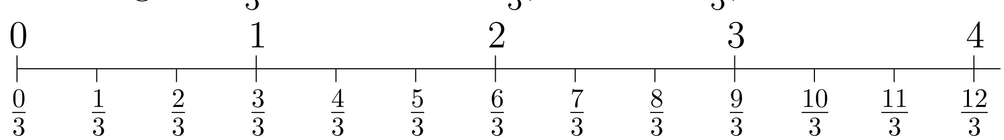
Teachers' understanding of fractions *as a mathematical concept* will not improve until we can provide them with a mathematical framework in which all these "representations" emerge as logical consequences of a clearly enunciated conception (definition) of a fraction. (Compare the discussion of *coherence* in Examples 2, 4, and 6 of Part I.) We want teachers to see that the subject of fractions is one that is infused with the aforementioned five characteristic properties of mathematics. Because the tradition of teaching by rote in fractions is so entrenched, unless we can make our teachers buy into the idea that the subject of fractions is logical rather than whimsical, there would be little hope that their students would perceive fractions as a learnable subject.

We need to approach fractions from a viewpoint that is consonant with requirements (*i*) and (*ii*) above. In other words, we need an approach that is rooted in mathematical engineering. Such an approach has been available for some time now ([Wu 2002]; for a similar but slightly different one, see [Jensen 2003]). Regardless of how soon school textbooks will teach fractions as part of mathematics (rather than as part experimental science and part royal decree), there is an urgent need for our teachers to learn a mathematically valid presentation of fractions.

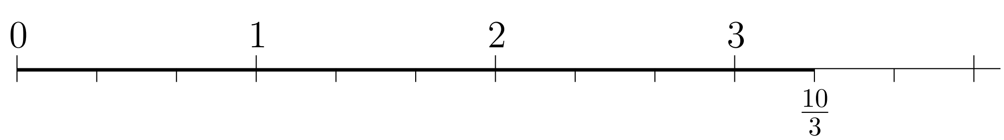
Such a presentation begins with a definition of fractions as points on the number line constructed in a specific manner and, on this basis, provides ***all*** the properties we expect of fractions with mathematical explanations. This is not the place to give a detailed treatment or even a complete summary of such a presentation. We can, however, try to give a flavor of what this presentation tries to accomplish. For example, let us see what is a reasonable definition of the fractions with denominator equal to 3. We first fix some terminology. If a and b are two points on the number line, with a to the left of b, we denote the segment from a to b by $[\boldsymbol{a}, \boldsymbol{b}]$. The points a and b are called the **endpoints** of $[a, b]$. The special case of the segment $[0, 1]$ occupies a distinguished position in the study of fractions; it is called the **unit segment** and its *length* is, intuitively, our "whole". The point 1 is called the **unit**. Since the fraction $\frac{1}{3}$ is one-third of the whole, we see from the picture of the number line below that the length of any of the three smaller segments of equal length between 0 and 1 qualifies as $\frac{1}{3}$. However, the right endpoint of the thickened segment is sufficient, in an intuitive sense, to indicate the length of this thickened segment, so this right end-point will be chosen as the representative of $\frac{1}{3}$.



If we divide, not just $[0, 1]$, but every segment between two consecutive whole numbers — $[0, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, etc. — into three equal parts, then these division points together with the whole numbers form an infinite sequence of equi-spaced points, to be called the **sequence of thirds**. The point in this sequence to the right of $\frac{1}{3}$ will be called $\frac{2}{3}$, the third $\frac{3}{3}$, etc.



Observes that each point in this sequence gives the length of the segment from 0 to the point itself. For example, $\frac{10}{3}$ is the length of the segment $[0, \frac{10}{3}]$, as the latter is 10 times the length of $[0, \frac{1}{3}]$ (just count!).



We have now used intuitive reasoning to locate the fractions with denominator equal to 3 on the number line.

In a formal mathematical introduction to fractions, we would therefore first create the sequence of thirds in exactly the same way, and then *define* the fractions with denominator equal to 3 to be exactly these points. In other words, as far as mathematics is concerned, the fraction $\frac{10}{3}$ (for example) is just the tenth point in this sequence to the right of 0, no more and no less. If we want to say anything about this fraction, *we must start with the fact that it is the tenth point in the sequence of thirds.*

In like manner, the fractions with denominator equal to n are by definition the points in the following sequence: we divide each of $[0, 1]$, $[1, 2]$, $[2, 3]$, $\ldots$, into n equal parts, then these division points together with the whole numbers form the sought-for infinite sequence of equi-spaced points. This sequence is called **the sequence of $\boldsymbol{n}$ths**. The fraction $\frac{m}{n}$ is then the m-th point to the right of 0 in this infinite sequence.

Here we can point to an immediate advantage of having such a precise definition of a fraction. Given two fractions A and B, we define $\boldsymbol{A < B}$, and say $\boldsymbol{A}$ **is less than** (or **smaller than**) $\boldsymbol{B}$ if A is to the left of B on the number line. Why this is significant is that, in the traditional presentation of fractions, *there is no definition of what it means for one fraction to be smaller than another.* Rather, students are told to do something first (e.g., get a common denominator for both fractions) and then *decide* after the fact that one is smaller than the other.

In the following, the whole number n in a fraction symbol $\frac{m}{n}$ will be automatically assumed to be nonzero.

A special class of fractions are those whose denominators are all positive powers of 10, e.g.,

$$\frac{1489}{10^2}, \quad \frac{24}{10^5}, \quad \frac{58900}{10^4}.$$

These are called **decimal fractions**, but they are usually abbreviated to

$$14.89, \quad 0.00024, \quad 5.8900$$

respectively. The rationale of the notation is clear: the number of digits to the right of the so-called **decimal point** keeps track of the power of 10 in the respective denominators, 2 in 14.89, 5 in 0.00024, and 4 in 5.8900. In this notation, these numbers are called **finite** or **terminating decimals**. In context, we usually omit any mention of “finite” or “terminating” and just say “decimals” if there is no danger of confusion. One would like to think that 5.8900 is the same as 5.89, as every school student is taught about this at the outset, but we have already agree *by definition* that 5.8900 is $\frac{58900}{10^4}$ whereas 5.89 is (again by definition) $\frac{89}{10^2}$. How do we know that they are equal?

$$\frac{58900}{10^4} = \frac{589}{10^2}$$

We must give a proof! It turns out to be more enlightening to first prove a general fact which is fundamental to the whole subject of fractions. This is the theorem known in school mathematics as “equivalent fractions”. First, we define two fractions to be **equal** if they are the same point on the number line.

The **Theorem on Equivalent fractions** is the statement that *given two fractions* $\frac{m}{n}$ *and* $\frac{k}{\ell}$, *if there is a whole number* j *so that* $k = jm$ *and* $\ell = jn$, *then* $\frac{m}{n} = \frac{k}{\ell}$.

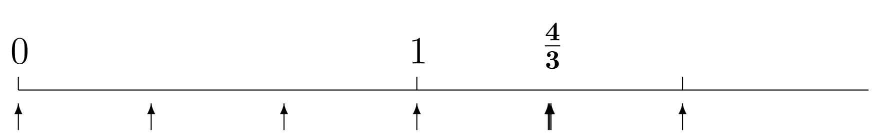
It is common to state this theorem in the following form: *for all whole numbers* j, m, *and* n (*so that* $n \neq 0$ *and* $j \neq 0$),

$$\frac{m}{n} = \frac{jm}{jn}$$

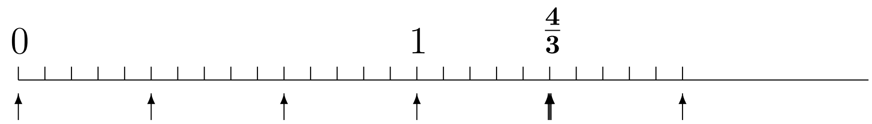
This fact can be simply proved with the definition of a fraction available. We give the reasoning for the special case

$$\frac{4}{3} = \frac{5 \times 4}{5 \times 3}$$

but this reasoning will be seen to hold in general. First locate $\frac{4}{3}$ on the number line:



We divide each of the segments between consecutive points in the sequence of thirds into 5 equal parts. Then each of the segments $[0, 1]$, $[1, 2]$, $[2, 3]$, ... is now divided into 15 equal parts and, in an obvious way, we have obtained the **sequence of fifteenths** on the number line:



The point $\frac{4}{3}$, being the 4-th point in the sequence of thirds, is now the 20-th point in the sequence of fifteenths. The latter is by definition the fraction $\frac{20}{15}$, i.e., $\frac{5\times 4}{5\times 3}$. Thus $\frac{4}{3} = \frac{5\times 4}{5\times 3}$.

Observe that without a precise definition of a fraction, it would be difficult to make sense of the statement of equivalent fractions for *arbitrary* j, m and n.

The first application of the theorem on equivalent fractions is to bring closure to the discussion about the decimal 5.8900. Recall that we had, by definition,

$$\frac{58900}{10^4} = 5.8900$$

We now show that $5.8900 = 5.89$ and, more generally, *one can add or delete zeros to the right end of the decimal point without changing the decimal.* Indeed,

$$5.8900 = \frac{58900}{10^4} = \frac{589 \times 10^2}{10^2 \times 10^2} = \frac{589}{10^2} = 5.89,$$

where the middle equality makes use of equivalent fractions. The reasoning is of course valid in general, e.g.,

$$12.7 = \frac{127}{10} = \frac{127 \times 10^4}{10 \times 10^4} = \frac{1270000}{10^5} = 12.70000$$

It is commonly stated that a conceptual understanding of fractions should include the fact that a fraction is both parts-of-a-whole and a division. We hasten to point out that *this statement is meaningless as a statement in mathematics.*

Indeed, one is supposed to understand, for example, that the fraction $\frac{5}{7}$ is not only 5 parts when the whole is divided into 7 equal parts, but also "5 divided by 7". The first questionable aspect of this statement is that if $\frac{5}{7}$ does mean "5 divided by 7", then one must be able to give a reason. In other words, there is a theorem to prove, although none is ever offered. A second questionable aspect is that, among whole numbers, there is no division such as "$5 \div 7$", only divisions of the form $12 \div 4$, $25 \div 5$, $48 \div 6$, or in general, $a \div b$ *when a is a multiple of b*. Therefore just to make sense of $\frac{m}{n}$ as "$m \div n$", we must first precisely define for *any* two whole numbers m, n ($n \neq 0$), that

$\boldsymbol{m \div n}$ is the length of one part when a segment
of length $\boldsymbol{m}$ is partitioned[5] into $\boldsymbol{n}$ equal parts.

Now we are at least in a position to make an assertion that is mathematically meaningful, namely, we assert *the equality of two numbers*, $\frac{m}{n}$ and $m \div n$:[6]

$$\frac{m}{n} = m \div n$$

This is the correct meaning of the so-called **division interpretation of a fraction**. And, of course, we still need to give a proof! The advantage of having done all this work to clarify this statement is that we see more clearly how to prove it. To divide $[0, m]$ into n equal parts, we express $m = \frac{m}{1}$ as

$$\frac{nm}{n}$$

using equivalent fractons. That is, nm copies of $\frac{1}{n}$, which is equivalent to n copies of $\frac{m}{n}$. So one part out of these n equal parts is just $\frac{m}{n}$.

We note once again that such a precise explanation could be given only because we have a precise definition of a fraction.

No professional development for elementary teachers can afford to avoid a discussion of whether a teacher can insist on **always reducing a fraction to lowest terms**. Implicit in this stance is the statement that

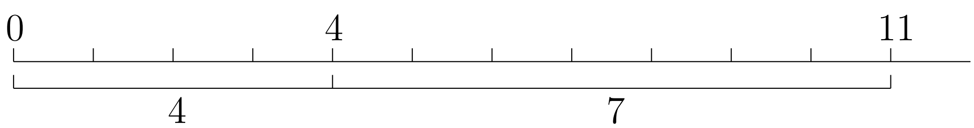
[5] To avoid the possibly confusing appearance of the word "divide" at this juncture, we have intentionally used "partition" instead.

[6] Notice how careful we are in using the equal sign! Compare Example 7 in Part I.

every fraction is equal to a unique fraction (*one and only one fraction*) *in lowest terms.*

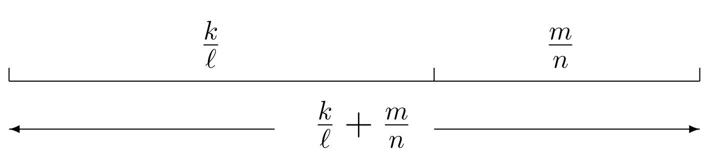
First of all, we must recognize the fact that it is quite nontrivial to prove this statement. This is where the Euclidean algorithm comes in. While a proof should be given, it cannot be given in grade 5 or even in grade 6 in most schools because of the mathematical sophistication involved. In addition, teachers should also know that, a fraction such as $\frac{12}{9}$ is every bit as good as $\frac{4}{3}$, so that the insistence that $\frac{4}{3}$ be use rather than $\frac{12}{9}$ must be recognized as a preference but not a mathematical necessity. Finally, it is sometimes not immediately obvious whether a fraction is in lowest terms or not, e.g., $\frac{68}{51}$. (It is not.) For all these reasons, a more flexible attitude towards unreduced fractions would consequently make for a better mathematics education for students.

One would go on to define the addition and subtraction of fractions, the multiplication of fractions, and the division of fractions. Here, *we want teachers to appreciate the coherence of mathematics by exhibiting the fundamental similarity between the arithmetic operations on fractions and those on whole numbers.* See [Wu 2001]. Teachers need to appreciate the fact that fractions are not "another kind of numbers". To *define* the addition of two fractions, we first consider how we add whole numbers when whole numbers are considered as points on the number line. Take for example, the addition of 4 to 7. In terms of the number line, this is just the total length of the two segments joined together end-to-end, one of length 4 and the other of length 7, which is of course 11, as shown.



We call this process the **concatenation** of the two segments. Imitating this process, we define, given fractions $\frac{k}{\ell}$ and $\frac{m}{n}$, their **sum** $\boldsymbol{\frac{k}{\ell} + \frac{m}{n}}$ by

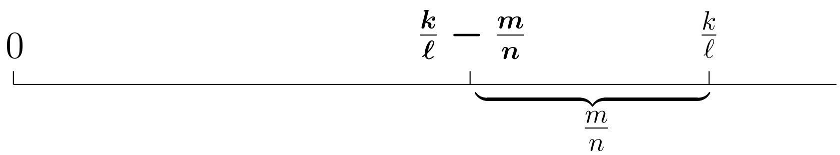
$$\frac{k}{\ell} + \frac{m}{n} = \textit{the length of two concatenated segments, one of length } \tfrac{k}{\ell}\textit{, followed by one of length } \tfrac{m}{n}$$



Then one proves the **addition formula** for any two fractions $\frac{k}{\ell}$ and $\frac{m}{n}$:

$$\frac{k}{\ell} + \frac{m}{n} \;=\; \frac{kn + \ell m}{\ell n}$$

For the subtraction $\boldsymbol{\frac{k}{\ell} - \frac{m}{n}}$ to make sense, we first have to make sure that $\frac{k}{\ell} > \frac{m}{n}$. Once done, the subtraction is then defined to be the length of the remaining segment when a segment of length $\frac{m}{n}$ is taken away from one end of a segment of length $\frac{k}{\ell}$.



Next, given two fractions A and B, we will define $A \times B$ and $\frac{A}{B}$. We first define $\boldsymbol{\frac{k}{\ell}}$ **of a number $\boldsymbol{x}$** to be the number which is the length of the concatenation of k parts when the segment $[0, x]$ of length x is partitioned into ℓ parts of equal length. Then, by definition, the product $\boldsymbol{\frac{k}{\ell} \times \frac{m}{n}}$ is $\frac{k}{\ell}$ of $\frac{m}{n}$. On the basis of this definition, one proves the well-known **product formula** for any two fractions $\frac{k}{\ell}$ and $\frac{m}{n}$:

$$\frac{k}{\ell} \times \frac{m}{n} = \frac{km}{\ell n}$$

Observe that when $\ell = 1$, $\frac{k}{\ell} = k$, so that (if m is a whole number) "k of m" is according to the above definition exactly the length of k copies of $[0, m]$, i.e., $k \times m$ is km, which is the definition of multiplication among whole numbers. It should mentioned that the product formula leads to the second, and equally important meaning of fraction multiplication:

The area of a rectangle with sides of lengths $\frac{k}{\ell}$ and $\frac{m}{n}$ is equal to

$$\frac{k}{\ell} \times \frac{m}{n}$$

One proves this by first proving it for unit fractions, i.e., the special case where $k = m = 1$, and then using this special case to prove the general case.

Finally, given fractions A and B ($B \neq 0$), the **division** or **quotient** $\boldsymbol{\frac{A}{B}}$ is by definition the fraction C so that $A = CB$. If this doesn't sound familiar, consider the division of whole numbers such as $\frac{36}{9}$. We tell children that $\frac{36}{9} = 4$ because $36 = 4 \times 9$. Now if replace 36 by A, 9 by B, and 4 by C, then we would get exactly the definition of $\frac{A}{B}$. (We should add that, in the preceding definition of $\frac{A}{B}$, the existence of a unique fraction C that satisfies $A = CB$ must be proved.) The classical rule of invert-and-multiply now becomes a theorem.

Once the concept of division is available, we can introduce the important concept of a **complex fraction**, i.e., the division $\frac{A}{B}$, where A and B are fractions ($B \neq 0$). Now a complex fraction is just a fraction, so why single it out for discussion? To see this, consider the sum of the two complex fractions such as

$$\frac{2.8}{\frac{4}{5}} + \frac{\frac{12}{7}}{2.5}$$

We know how to do the addition: express each complex fraction as a fraction by the invert-and-multiply rule,

$$\frac{2.8}{\frac{4}{5}} = \frac{\frac{28}{10}}{\frac{4}{5}} = \frac{28 \times 5}{10 \times 4} = \frac{140}{40}$$

and similarly,

$$\frac{\frac{12}{7}}{2.5} = \frac{120}{175}.$$

Then the above addition becomes a routine problem:

$$\frac{2.8}{\frac{4}{5}} + \frac{\frac{12}{7}}{2.5} = \frac{140}{40} + \frac{120}{175} = \frac{7}{2} + \frac{24}{35} = \frac{293}{70}$$

However, suppose we *make believe* that the complex fractions are just ordinary fractions and we add them as we would ordinary fractions. Then the addition formula for fractions yields the same answer:

$$\frac{2.8}{\frac{4}{5}} + \frac{\frac{12}{7}}{2.5} = \frac{(2.8 \times 2.5) + (\frac{4}{5} \times \frac{12}{7})}{\frac{4}{5} \times 2.5} = \frac{7 + \frac{48}{35}}{2} = \frac{7}{2} + \frac{24}{35} = \frac{293}{70}$$

Two thoughts immediately come to mind. One is that although the second strategy is blatantly illegal at this point (the addition formula has been proved only for ordinary fractions), it nevertheless gives the correct answer. Is it just luck? We will show that it is not. A second thought is that, since the first strategy always works, why bother with the second one? The superficial reason is that because the second strategy uses the same mechanical procedure for both ordinary fractions and complex fractions, it has the advantage of saving some wear-and-tear of the brain. But the real reason is that when we come to the manipulation of rational expressions in algebra, we will be *forced* to use the second strategy and will no longer have a choice.

This leads us to the arithmetic of complex fractions: can we add, subtract, multiply, and divide them as if they were ordinary fractions (see above)? The answer is yes. Textbooks and the education literature take this fact for granted and make use of it without a word of explanation. They do not consider it necessary to point out that such an extension of the arithmetic of fractions to complex fractions has taken place and, even more importantly, that it is correct. What we wish to affirm is that, indeed, every single one of the computational formulas involving fractions can be *proved* to be valid for complex fractions (though the proofs are mechanical and not interesting). Our main point is, however, that it is bad policy for school mathematics to be so cavalier about this generalization — from fractions to complex fractions — and we must at least get our teachers to understand why this is a bad policy.

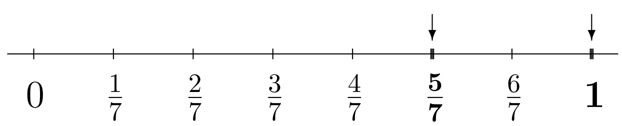
Students should learn *not* to overstep the bounds of what they know. If they want to claim more than they know, they should be immediately aware of the need to prove it. In this instance, it is a matter of luck that the extrapolation from fractions to *complex* fractions turns out not to cause any problems. One cannot, however, expect this kind of luck to persist. For example, among fractions, it is true that for any fraction A, the fact that $B \geq C$ implies that $AB \geq AC$. Students who have formed the habit of claiming more than they know would assume, when they come to rational numbers (i.e., positive and negative fractions, see item (C), **Rational numbers**, below), that for any rational numbers A, B, C, $B \geq C$ also implies $AB \geq AC$. This would be a mistake, because while $3 \geq 2$, it is false that $(-4)3 \geq (-4)2$ because $(-4)3 = -12$, which is less than $(-4)2 = -8$. If we do not want students to fall into this bad habit, then we are obligated to make sure that our teachers do not form such bad habits in the first place.

Complex fractions are critical to the study of fractions and should be singled out and systematically taught to teachers. To demonstrate their importance, let us introduce the concepts of percent, ratio, and rate in general.

A **percent** is a complex fraction whose denominator is 100. By tradition, a percent $\frac{N}{100}$, where N is a fraction, is often written as $N\%$. By regarding $\frac{N}{100}$ as an ordinary fraction, we see that the usual statement **$\boldsymbol{N}$% of a number $\boldsymbol{\frac{m}{n}}$** is (by the definition of fraction multiplication) exactly $N\% \times \frac{m}{n}$. With this precise definition of percent available, all questions about percent can be routinely computed. See Example 6 in Part I.

Next, given two fractions A and B ($B \neq 0$), both referring to the same unit (i.e., they are points on the same number line), the **ratio of** A **to** B, sometimes denoted by $\boldsymbol{A : B}$, is by definition the complex fraction $\boldsymbol{\frac{A}{B}}$. In connection with ratio, there is a common expression that needs to be made explicit. To say that **the ratio of boys to girls in a classroom is 3 to 2** is to say, *by convention*, that if B (resp., G) is *the number of* boys (resp., girls) in the classroom, then the ratio of B to G is $\frac{3}{2}$.

When A and B are whole numbers, we want to show that this definition of the ratio $5 : 7$ has the same intuitive meaning as "5 parts to 7 parts". Indeed, $5 : 7$ is by definition the fraction $\frac{5}{7}$ which, by the definition of a fraction, is the 5th division point when the unit segment $[0, 1]$ is divided into 7 equal parts:



We therefore see that $5 : 7$ (i.e., $\frac{5}{7}$) is 5 parts (a part being $\frac{1}{7}$) compared with 7 parts.

The significance of this definition of the ratio of a fraction A to a fraction B is that, by first establishing the meaning of a fraction as a point on the number line, we show that when A and B are both whole numbers, the meaning of $A : B$ is exactly the fraction $\frac{A}{B}$. As is well-known, one of the traditional definitions of a fraction $\frac{A}{B}$ (A, B are whole numbers, $B \neq 0$) is that it is the ratio of A to B. What we have done is therefore to clarify the relationship between these two concepts by turning the table: we define fractions first and then define ratio in terms of a fraction.

In school mathematics, the most substantial application of the concept of division is to problems related to *rate*, or more precisely, *constant rate.* The precise definition of the general concept of "rate" requires more advanced mathematics, and in any case, it is irrelevant in school mathematics whether we know what a *rate* is or not. What is important is to know the precise meaning of "constant rate" in specific situations, and some of the most common ones will now be described.

The most intuitive among the various kinds of rate is *speed.* A motion is of **constant speed** v (v being a fixed number) if the distance traveled, d, from time 0 to *any* time t is $d = vt$. Equivalently, in terms of the concept of division, a motion is of constant speed if there is a fixed number v, so that for *any* positive number t, the distance d (feet, miles, etc.) traveled in *any* time interval of length t (seconds, minutes, etc.) satisfies

$$\frac{d}{t} = v$$

Notice that $\frac{d}{t}$ is a complex fraction, from which, one can infer that most of the computations in speed problems involve the arithmetic of complex fractions.

What is noteworthy about the preceding equation is the fact that we are dividing two numbers, d and t, ostensibly from different number lines. In greater detail, d is a number on the number line where 1 is the chosen unit of length (foot, mile, etc.) while t is on the number line whose unit 1 is the chosen unit of time (second, minute, etc.). What we have done, at least implicitly, is to identify the two units of length and time, so that d and t are now points on the same number line. If this sounds strange, it could only be because it is rarely explicitly pointed out, although it is done all the time. For example, suppose a rectangle has area 48 ft^2 and one side is 8 ft. The length of the other side is then $\frac{48}{8} = 6$ ft. Here the division makes sense only because we have identified the unit ft^2 with the unit of length, one foot. In any case, it is the need of identifying two number lines that distinguishes *rate* from *ratio.*

In the language of school mathematics, speed is the "rate" at which the work of going from one place to another is done. Other standard "rate" problems which deserve to be mentioned are the following. One of them is painting (the exterior of) a house. The rate there would be the number of square feet painted per day or per hour. A second one is mowing a lawn. The rate in question would be the number of square feet mowed per hour or per minute. A third is the work done by water flowing out of a faucet, and the rate is the number of gallons of water coming out per minute or per second. In each case, the concept of **constant rate** can be defined in a manner that is identical to the case of constant speed. For example, a **constant rate of lawn-mowing** would mean: there is a constant r (with unit equal to square-feet-per-hour) so that if A is the total area mowed after T hours, then $A = rT$ *no matter what* T *is.* Equivalently, the lawn-mowing is of constant rate if there is a fixed number r so that the number of square feet A mowed in T hours satisfies

$$\frac{A}{T} = r$$

no matter what T *is.*

Without knowing the precise meanings of division and multiplication among fractions, it would be impossible to detect the fact that all these constant rate problems are identical problems. For example, assuming constant rate in each situation, the problem of "if I walk 287 meters in 9 minutes, how many meters do I walk in 7 minutes?" is identical to "if I mow 287 square meters of lawn in 9 minutes, how many square meters do I mow in 7 minutes?". This is one argument for emphasizing the importance of definitions.

Finally, we take up the topic of converting a fraction to a finite or infinite decimal. We take this up last because of its deceptive subtlety. Consider first the case of those fractions which are equal to a finite decimal. (Teachers should learn how to prove the theorem that a reduced fraction is equal to a finite decimal if and only if its denominator only has 2 or 5 as its prime factors. This proof needs the Fundamental Theorem of Arithmetic.) Let us prove, for example, that

$$\frac{3}{8} = 0.375$$

By itself, this equality is unremarkable. Indeed, the definition of 0.375 is $\frac{375}{1000}$, so that, by equivalent fractions, we get

$$0.375 = \frac{375}{1000} = \frac{3 \times 125}{8 \times 125} = \frac{3}{8}$$

However, the algorithm that converts a fraction to decimals asserts that *one obtains the decimal* 0.375 *from the fraction* $\frac{3}{8}$ *by doing the long division of* 3×10^5 (*or* 3×10^n *for any large* n) *by 8 and then placing the decimal point in the quotient in some prescribed way*. Thus what is at issue here is not so much that the two numbers $\frac{3}{8}$ and 0.375 are equal,[7] but that *the method of long division of 300000 by 8 would yield the correct answer*. This can be done simply as follows:

$$\frac{3}{8} = \frac{1}{10^5} \times \frac{3 \times 10^5}{8}$$

By the long division algorithm, $\frac{3 \times 10^5}{8} = 37500$. Therefore, using the definition of a decimal, we have:

$$\frac{3}{8} = \frac{1}{10^5} \times 37500 = \frac{37500}{10^5} = 0.37500 = 0.375$$

Clearly, we would obtain the same answer if 10^5 is replaced by 10^3, or any power of 10 greater than 3. In general, we just try to multiply the numerator and denominator of the fraction under consideration by a large power of 10, where "large" means "large enough to see the decimal terminate in 0's". The same reasoning is applicable to all other cases.

In general, a fraction is equal to an infinite repeating decimal. For example,

$$\frac{3}{7} = 0.428571428571428571428571428571\ldots$$

The task of proving this is not so simple. It involves (i) making sense of an infinite decimal as a point on the number line, (ii) showing that *through the process of long division* the fraction $\frac{3}{7}$ is equal to the above infinite decimal, and (iii) proving that the infinite decimal is necessarily repeating. In school textbooks, basically none of these three steps is proved (though there may be some half-hearted attempt at explaining (iii)), which is understandable considering the advance nature of the mathematics involved. Professional development materials usually concentrate on explaining (iii) but completely ignore (ii) and (i). It would be a good idea to tread lightly on (i), at least for elementary teacher, but (ii) should be taken up seriously in professional development. One shows

[7] But note the importance in this context of having precise definitions of both $\frac{3}{8}$, a fraction, and 0.375, a decimal, as *numbers*. Anything less (such as only knowing a fraction as a piece of pizza) and this equality wouldn't even make sense.

directly using the mechanism of the long division algorithm (see part (A) on **Whole numbers**) that

$$\begin{aligned}\frac{3}{7} \;=\; & \frac{4}{10}+\frac{2}{10^2}+\frac{8}{10^3}+\frac{5}{10^4}+\frac{7}{10^5}+\frac{1}{10^6}+ \\ & \frac{1}{10^6}\left(\frac{4}{10}+\frac{2}{10^2}+\frac{8}{10^3}+\frac{5}{10^4}+\frac{7}{10^5}+\frac{1}{10^6}\right)+\cdots\end{aligned}$$

This then is the meaning of $\frac{3}{7} = 0.428571428571\ldots$ on one level. On a deeper level, we need to prove the convergence of the infinite series.

(C) **Rational numbers** Like the teaching of fractions, the teaching of rational numbers (positive and negative fractions) is usually nothing more than the presentation of a collection of rules to be memorized, with an occasional pseudo-explanation thrown in (such as the many analogies purporting to show why *negative* $\times$ *negative* = *positive*). Rational numbers present a higher level of abstraction than fractions, and can be understood only if the abstract laws of operations, especially the distributive law for both positive and negative fractions, are taken seriously. Few of our teachers get this message in their college courses. This is a subject littered with plausible statements promoted unceremoniously as truths without explanations, e.g., the statement that $\frac{-x}{y} = \frac{x}{-y} = -\frac{x}{y}$ for all *rational numbers* (not just whole numbers) x and y. Nowhere is it more important that we carefully attend to the clarity of the definition of each concept and the proof of every assertion.

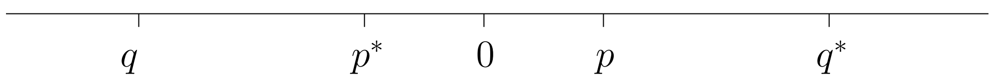
With a **number** understood to be a point on the number line, we now look at all the numbers as a whole. Take any point p on the number line which is not equal to 0; such a p could be on either side of 0 and, in particular, it does not have to be a fraction. Denote its mirror reflection on the opposite side of 0 by p^*, i.e., p and p^* are equidistant from 0 and are on opposite sides of 0. We simply call p^* the **mirror reflection** of p. If $p = 0$, we define

$$0^* = 0$$

Then for any points p, it is clear that

$$p^{**} = p$$

This is nothing but a succinct way of expressing the fact that reflecting a nonzero point across 0 twice in succession brings it back to itself (if $p = 0$, of course $0^{**} = 0$). Here are two examples of mirror reflections:



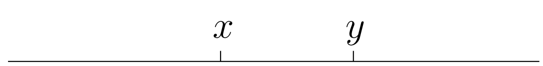
Because the fractions are to the right of 0, the numbers such as 1^*, 2^*, or $(\frac{9}{5})^*$ are to the left of 0. The set of all the fractions and their mirror reflections with respect to 0, i.e., the numbers $\frac{m}{n}$ and $(\frac{k}{l})^*$ for all whole numbers k, l, m, n ($l \neq 0$, $n \neq 0$), is called the **rational numbers**. Recall that the whole numbers are a sub-set of the fractions. The set of whole numbers and their mirror reflections,

$$\ldots 3^*,\ 2^*,\ 1^*,\ 0,\ 1,\ 2,\ 3, \ldots$$

is called the **integers**. We therefore have:

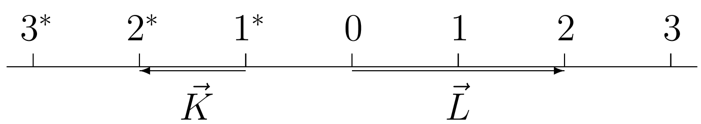
whole numbers $\subset$ integers $\subset$ rational numbers

We now extend the **order** among numbers from fractions to all numbers: for any x, y on the number line, $x < y$ means that x is to the left of y. An equivalent notation is $y > x$.



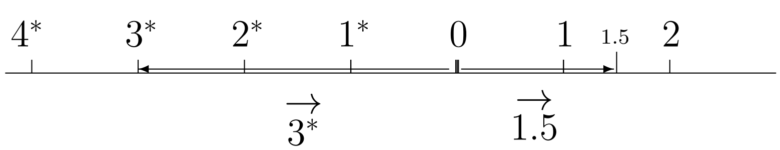
Numbers which are to the right of 0 (thus those x satisfying $x > 0$) are called **positive**, and those which are to the left of 0 (thus those that satisfy $x < 0$) are **negative**. So 2^* and $(\frac{1}{3})^*$ are negative, while all nonzero fractions are positive, but if y is a negative number to begin with, y^* would be positive. *The number 0 is, by definition, neither positive nor negative.* As is well-known, a number such as 2^* is normally written as -2 and $(\frac{1}{3})^*$ as $-\frac{1}{3}$, and that the "$-$" sign in front of -2 is called the **negative sign**. However, it is better to avoid mentioning the negative sign until we get to subtraction, because we should develop one concept at a time.

For teachers' need in the classroom, it would be a good idea to begin the discussion of the arithmetic of rational numbers with a concrete approach to the addition of rational numbers. To this end, define a **vector** to be a segment on the number line together with a designation of one of its two endpoints as a **starting point** and the other as an **endpoint**. We will continue to refer to the length of the segment as the **length** of the vector, and call the vector **left-pointing** if the endpoint is to the left of the starting point, **right-pointing** if the endpoint is to the right of the starting point. The **direction** of a vector refers to whether it is left-pointing or right-pointing. We denote vectors by placing an arrow above the letter, e.g., $\vec{A}$, $\vec{x}$, etc., and in pictures we put an arrowhead at the endpoint of a vector to indicate its direction. For example, the vector $\vec{K}$ below is left-pointing and has length 1, with a starting point at 1^* and an endpoint at 2^*, while the vector $\vec{L}$ is right-pointing and has length 2, with a starting point at 0 and an endpoint at 2.



By definition, **two vectors being equal** means exactly that they have the same starting point, the same length, and the same direction.

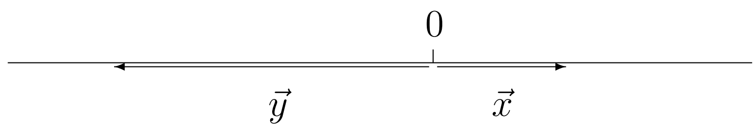
For the purpose of discussing the addition of rational numbers, we can further simplify matters by restricting attention to a special class of vectors. Let x be a rational number, then we define the vector $\vec{x}$ to be one with starting point at 0 and endpoint at x. It follows from the definition that, *if x is a nonzero fraction, then the segment of the vector $\vec{x}$ is exactly* $[0, x]$. Here are two examples of vectors arising from rational numbers:



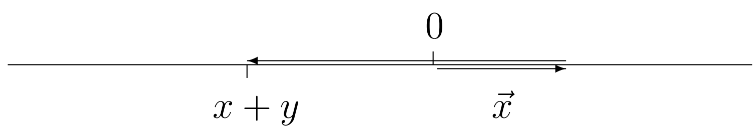
With this notation understood, we now describe how to add such vectors. Given $\vec{x}$ and $\vec{y}$, where x and y are two rational numbers, the **sum** vector $\vec{x} + \vec{y}$ is, by definition, the vector whose starting point is 0, and whose endpoint is obtained as follows:

slide $\vec{y}$ along the number line until its starting point (which is 0) is at x, then the endpoint of $\vec{y}$ in this new position is by definition the endpoint of $\vec{x}+\vec{y}$.

For example, if x and y are rational numbers, as shown:



Then, by definition, $x+y$ is the point as indicated,



We are now in a position to define the addition of rational numbers. The **sum** $\boldsymbol{x+y}$ for any two rational numbers x and y is by definition the endpoint of the vector $\vec{x}+\vec{y}$. In other words,

$$x+y = \text{ the endpoint of } \vec{x}+\vec{y}.$$

Put another way, $x+y$ is defined to be the point on the number line so that its corresponding vector $\overrightarrow{(x+y)}$ satisfies:

$$\boldsymbol{\overrightarrow{(x+y)} = \vec{x}+\vec{y}.}$$

We proceed to prove that the addition of rational numbers is commutative, i.e., $x+y = y+x$ for all rational numbers. Of course this is equivalent to checking $\vec{x}+\vec{y} = \vec{y}+\vec{x}$. Remembering that two vectors are equal if and only if they have the same length and the same direction, we simply check that $\vec{x}+\vec{y}$ and $\vec{y}+\vec{x}$ do have the same length and same direction. The checking is straightforward.

One can also prove that the addition of rational numbers is associative, i.e., $(x+y)+z = x+(y+z)$ for all rational numbers x, y, z. However the reasoning this time is much more tedious and not so instructive.

With this definition of the addition of rational numbers, one can prove in a hands-on manner the following basic facts for all *positive* fractions s and t:

$$\begin{cases} s+t &= \text{ the old addition of fractions.} \\ s^*+t^* &= (s+t)^* \qquad\qquad\qquad\qquad (\text{e.g., } 2^*+5^*=(2+5)^*) \\ s+t^* &= (s-t) \qquad \text{if } \ s\geq t \qquad (\text{e.g., } 6+(\tfrac{1}{2})^*=(6-\tfrac{1}{2})) \\ s+t^* &= (t-s)^* \qquad \text{if } \ s<t \qquad (\text{e.g., } 2\tfrac{1}{2}+7^*=(7-2\tfrac{1}{2})^*) \end{cases}$$

Because $s^*+t=t+s^*$, by the commutative law of addition, the above four cases exhaust all the possibilities of the addition of any two rational numbers. We have just explicitly determined how to add any two rational numbers in terms of the addition and subtraction of fractions.

We now must confront the fact that rational numbers are on a higher level of abstraction than fractions. A fact not mentioned in the brief discussion of fractions is that the addition and multiplication of fractions satisfy the associative, commutative, and distributive laws, but now things are going to change. We have just brought out the commutativity and associativity of addition among rational numbers. At this point, these laws must come to the forefront, because while the addition of rational numbers can be directly defined using the concept of vectors, there will be no such analog for multiplication. For the latter, we have to approach it from a different vantage point. Therefore, to prepare for multiplication, we forget the preceding definition of addition in terms of vectors and start from the beginning.

We now take the attitude that although we do not know what the negative numbers are, the collection of rational numbers simply "must" satisfy the associative, commutative, and distributive laws with respect to addition and multiplication. Historically, this was what happened, and of course our intellectual inertia welcomes the status quo! Such being the case, one reasonable way to develop the addition of rational numbers is to make three **fundamental assumptions** about the addition of rational numbers at the outset. The first two fundamental assumptions are entirely noncontroversial:

(A1) *Given any two rational numbers x and y, there is a way to add these to get another rational number $x+y$ so that, if x and y are* fractions, *$x+y$ is the same as the usual sum of fractions. Furthermore, this addition of rational numbers satisfies the associative and commutative laws.*

(A2) $x + 0 = x$ *for any rational number x.*

The last assumption explicitly prescribes the role for all negative fractions:

(A3) *If x is any rational number, $x + x^* = 0$.*

On the basis of (A1)–(A3), one can proceed to compute the sum of two rational numbers in terms of the addition and subtraction of fractions as before. Let s and t be any two *positive* fractions. By (A1),

$$s + t = \text{ the old addition of fractions.}$$

Then one can prove, with some effort, that (A1)–(A3) imply that

$$\begin{cases} s^* + t^* & = & (s+t)^* & \\ s + t^* & = & (s-t) & \text{if } \ s \geq t \\ s + t^* & = & (t-s)^* & \text{if } \ s < t \end{cases}$$

We now pause to amplify on the second equality above by rewriting it as

$$s - t = s + t^* \quad \text{when} \quad s \geq t.$$

The ordinary *fraction* subtraction $s - t$ now becomes the addition of s and t^*. This fact prompts us to **define**, in general, the subtraction between any two *rational numbers* x and y to mean:

$$\boldsymbol{x - y \stackrel{\text{def}}{=} x + y^*}$$

Note the obvious fact that the meaning of the subtraction of (say) the two *rational numbers* $\frac{6}{5} - \frac{3}{4}$ is, according to this definition,

$$\frac{6}{5} + \left(\frac{3}{4}\right)^*$$

which, on account of "$s + t^* = (s - t)$ if $s \geq t$", is just the *fraction subtraction* $\frac{6}{5} - \frac{3}{4}$. More generally, when x, y are fractions and $x \geq y$, the meaning of $x - y$ *as a subtraction of rational numbers* coincides, according to this definition, with the old meaning of subtracting fractions. In other words, we have not created a *new* concept of subtraction, merely made it more comprehensive. To repeat, $\frac{6}{5} - \frac{3}{4}$ has exactly the same meaning whether we look at it as a subtraction between fractions *or* between rational numbers; this is reassuring. On the other hand, we are now free to do a subtraction between any two fractions such as $\frac{3}{4} - \frac{6}{5}$, whereas before (i.e., in item (B), **Fractions**) we could not carry out the subtraction because the first fraction is smaller than the second. We now see for the first time the advantage of having rational numbers available: we can as freely subtract any two fractions as we add them. But this goes further, because we can even subtract any two *rational numbers*.

The main message to come out of this definition is, however, the fact that *subtraction is just a different way of writing addition* among rational numbers.

As a consequence of the definition of $x - y$, we have

$$0 - y = y^*$$

because $0 + y^* = y^*$. Common sense dictates that we should **abbreviate $\boldsymbol{0 - y}$ to $\boldsymbol{-y}$**. So we have

$$-y = y^*$$

It is only at this point that we can abandon the notation of y^* and replace it by $-y$. Many of the preceding equalities will now assume a more familiar appearance, e.g., from $x^{**} = x$ for any rational number x, we get

$$-(-x) = x,$$

and from $x^* + y^* = (x + y)^*$, we get

$$-(x + y) = -x - y$$

We now come to the multiplication of rational numbers, and we see the payoff from the more abstract approach to fraction addition. For multiplication, we make the following similar **fundamental assumptions** that

> (M1) *Given any two rational numbers x and y, there is a way to multiply them to get another rational number xy so that, if x and y are fractions, xy is the usual product of fractions. Furthermore, this multiplication of rational numbers satisfies the associative, commutative, and distributive laws.*

(M2) *If x is any rational number, then* $1 \cdot x = x$.

We note that (M2) must be an assumption because we would not know what $1 \times 5^*$ means without (M2). The equally "obvious" fact, which is the multiplicative counterpart of (A2), to the effect that

(M3) $0 \cdot x = 0$ *for any* $x \in \mathbf{Q}$.

turns out to be provable once (M1) and (M2) are assumed to be true.

We want to know explicitly how to multiply rational numbers. Thus let x, y be rational numbers. What is xy? If $x = 0$ or $y = 0$, we have just seen from (M3) that $xy = 0$. We may therefore assume both x and y to be nonzero, so that each is either a fraction, or the negative of a fraction. Letting s, t be *positive* fractions, one can prove on the basis of (M1)–(M3):

$$\begin{array}{lcll} (-s)t & = & -(st) & (\text{e.g.,}\ \ (-3)(\frac{1}{2}) = -\frac{3}{2}) \\ s(-t) & = & -(st) & (\text{e.g.,}\ \ 3\,(-\frac{1}{2})\ = -\frac{3}{2}) \\ (-s)(-t) & = & st & (\text{e.g.,}\ \ (-\frac{1}{2})(-\frac{1}{5}) = \frac{1}{10}) \end{array}$$

Since we already know how to multiply the fractions s and t, we have completely described the product of rational numbers.

The last item, that if s and t are fractions then $(-s)(-t) = st$, is such a big part of school mathematics education that it is worthwhile to go over at least a special case of it. When students are puzzled by this phenomenon, the disbelief centers on how *the product of two negatives can make a positive*. The pressing need in this situation is most likely that of winning the psychological battle. So we propose to use a simple example to demonstrate why *such a phenomenon is inevitable*. Thus we will give the reason why

$$(-1)(-1) = 1$$

Let us concentrate on the part of this assertion that says $(-1)(-1)$ is a *positive* number. It would be nice if we can demonstrate this though a direct computation of the following type: *we know* $(-3) - (-8)$ *is positive* because we can use the definition of subtraction and the above rules for adding rational numbers to conclude that

$$(-3) - (-8) = (-3) + (-8)^* = 3^* + (8^*)^* = 3^* + 8 = 8 + 3^* = (8 - 3) = 5$$

This is a most satisfying proof because we see explicitly how the answer "5" comes out of a direct computation. The proof leaves no room for doubt. However, this kind of proof is not always around, and we are sometimes forced to use an indirect method to find the answer. To give this line of thinking some context, you may remember what you learned in your school chemistry: if you have to find out whether a bottle of liquid is acidic or alkaline, the best scenario would be that there is clear label on the bottle stating it is HCl or ammonia. If not, then you would have to resort to an indirect method by dipping a blue litmus strip in the liquid: if the strip turns red, then it is an acid. So you have to trust the litmus paper and allow it to give you the needed information indirectly. It is the same with $(-1)(-1)$. There is no known explicit computation with $(-1)(-1)$ so that a positive number pops out at the end of the computation, but a possible "litmus test" in this case is to add to $(-1)(-1)$ a negative number. If the answer is either 0 or positive, then you'd have to agree that $(-1)(-1)$ is a positive number. This is exactly what we are going to do.

So we are going to test the positivity of $(-1)(-1)$ by adding to it the negative number -1. Why -1 and not some other negative number? This comes from experience and some common sense; one way is to ask yourself *why not* -1 *?* After all, it is natural to think of -1 in this particular context. In any case, we are going to apply the distributive law (assumption (M1)) to this sum and get:

$$(-1)(-1) + (-1) = (-1)(-1) + 1 \cdot (-1) = \{(-1) + 1\}(-1) = 0 \cdot (-1) = 0,$$

where the last equality is by (M3). This therefore shows that, *if we believe in the distributive law for rational numbers*, it must be that $(-1)(-1)$ is positive. In fact, we know a bit more: if $(-1)(-1)$ added to -1 is 0, then $(-1)(-1)$ has to be 1. In other words, $(-1)(-1) = 1$.

The fact that the distributive law holds for rational numbers is also responsible for the general assertion that $(-s)(-t) = st$.

The concept of the division of rational numbers is the same as that of dividing whole numbers or dividing fractions. For two rational numbers x and y, with $y \neq 0$, $\frac{\boldsymbol{x}}{\boldsymbol{y}}$ is by definition the rational number z so that $x = zy$. As in the case of fractions, the existence and uniqueness of such a z must be proved. Assuming this, we can now clear up a standard confusion in the study of rational numbers mentioned above, namely, the reason why the following equalities are true:

$$\frac{3}{-7} = \frac{-3}{7} = -\frac{3}{7}.$$

First let $C = -\frac{3}{7}$. We want to prove that $\frac{3}{-7} = C$. This would be true, by definition, if we can prove $3 = C \times (-7)$, and this is so because

$$C \times (-7) = (-\frac{3}{7}) \times (-7) = (\frac{3}{7})(7) = 3$$

where we have made use of $(-a)(-b) = ab$ for all fractions. Of course this proves $\frac{3}{-7} = -\frac{3}{7}$. In a similar manner, we can prove $\frac{-3}{7} = -\frac{3}{7}$.

More generally, the same reasoning supports the assertion that if k and ℓ are whole numbers and $\ell \neq 0$, then

$$\frac{-k}{\ell} = \frac{k}{-\ell} = -\frac{k}{\ell} \quad \text{and} \quad \frac{-k}{-\ell} = \frac{k}{\ell}.$$

We may also summarize these two formulas in the following statement: *for any two integers a and b, with $b \neq 0$,*

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}.$$

This formula is well-nigh indispensable in everyday computations with rational numbers. In particular, it implies that

every rational number can be written as the quotient of two integers.

Thus, the rational number $-\frac{9}{7}$ is equal to $\frac{-9}{7}$ or $\frac{9}{-7}$.

The concept of complex fractions has a counterpart in rational numbers, of course. For lack of a better name, we call them **rational quotients**, and as in the case of complex fractions, rational quotients can be added, subtracted, multiplied, and divided as if their numerators and denominators were whole numbers.

Finally, to compare rational numbers, recall the definition of $x < y$ between two rational numbers x and y: it means x is to the left of y on the number line.

The following inequalities are basic to any discussion of rational numbers and therefore belong to middle school mathematics. Here, x, y, z are rational numbers, and the symbol " $\Longleftrightarrow$" stands for "is equivalent to":

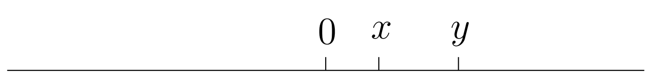
(i) For any x, y, $x < y \iff -x > -y$.

(ii) For any x, y, z, $x < y \iff x + z < y + z$.

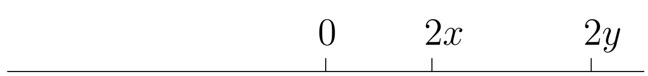
(iii) For any x, y, z, if $z > 0$, then $x < y \iff xz < yz$.

(iv) For any x, y, z, if $z < 0$, then $x < y \iff xz > yz$.

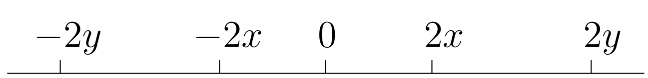
Of these, (iv) is the most intriguing. We give an intuitive argument of "$z < 0$ and $x < y$ imply $xz > yz$" that can be refined to be a correct argument. Consider the special case where $0 < x < y$ and $z = -2$. So we want to understand why $(-2)y < (-2)x$. We know that $(-2)y = -2y$ and $(-2)x = -2x$. Thus we want to see, intuitively, why $-2y < -2x$. From $0 < x < y$, we get the following picture:



Then the relative positions of $2x$ and $2y$ do not change, but each is pushed further to the right of 0:



If we reflect this picture across 0, we get the following:



We see that $-2y$ is now to the left of $-2x$, so that $-2y < -2x$, as claimed.

Obviously, this consideration is essentially unchanged if the number 2 is replaced by any negative number z.

(D) **Geometry** Our teachers are generally ill-prepared on the subject of geometry (cf. [IMAGES]). They are often misled into believing that introductory geometry is nothing more than one big vocabulary test, and not a very precise vocabulary at that. We have to impress on them, first of all, that there is a need for this vocabulary to be precise, and secondly, that geometry is about the *reasoning* underlying the precise study of spatial figures rather than just the vocabulary. Precision in the vocabulary is necessary because it is only through this vocabulary that we can transcribe intuitive spatial information into precise mathematics, and it is entirely on this vocabulary that we base our reasoning. The definitions of common concepts such as "angle", "convex figures", "polygons", etc., are anything but obvious. We also have to impress on them the fact that we draw a distinction here between what they *as teachers* should know and what they teach their students in K-8, especially in grades 4-5. The precision that they as teachers should learn may not literally translate into suitable classroom material in upper elementary and middle school as it might overwhelm students at this stage of their mathematical development. But it behooves teachers to learn such precision, because they must know the whole truth before they can judiciously hide unpleasant details in the service of good teaching. Part of the professional development in geometry would ideally include lots of drawings-by-hand and some hands-on activities such as the construction of regular polyhedra. There has to be an integration of the direct spatial input with the verbal-analytic output in a geometric lesson.

An important component of K–8 geometry (one that has not yet been fully implemented in the classroom) is familiarity with the three **basic rigid motions** in the plane: **rotation, translation**, and **reflection** (teachers should be steered away from the uncivilized terminology of "turn, glide, and flip"). Certainly, professional development must give precise definitions of these concepts, but in this case, the professional development may include the information that, with the availability of (overhead projector) transparencies, these concepts can be graphically demonstrated so that, in the school classroom, the precise definitions may be soft-pedaled in exchange for a tactile and intuitive understanding. The said demonstration consists of making identical drawings on two pieces of transparencies, preferably in different colors. By moving one against the other, the effects of basic rigid motions, and compositions thereof, can be tellingly displayed, and students get a sound conception of what these rigid motions do. It is in this context that we recommend that in a middle school classroom, the quite sophisticated precise definitions of basic rigid motion be soft-pedaled. The hope is that students will reprise these concepts in a high school geometry course, so that if they attain at least a good intuitive knowledge in the middle grades, they will gain a better understanding the second time around.

In terms of these basic rigid motions, **congruence** can now be defined as a finite composition of such. (See Example 4 of Part I.) Teachers should learn that, while "same size, same shape" is a good sales pitch about congruence for the general public, it should not be offered as a mathematical definition because it does not conform to the basic characteristic of precision. There is an urgent need in school mathematics to replace "same size, same shape" with the above definition of congruence that is correct and is also something that students can directly experiment with.

The next important topic is **dilation**. First of all, teachers should be convinced of the feasibility of teaching this concept in middle school. For example, it should be pointed out that a precise definition of dilation would allow students to magnify any picture by any scale factor, if enough sampling points of the original picture are chosen. (See Example 5 of Part I.) Such magnification (or contraction) activities have never ceased to impress students. Therefore teaching the correct definition of a dilation would not be a hard sell. Professional development can build on this fact. Of course the reason one needs dilation is that the concept of **similarity** can now be correctly defined as the composition of a dilation and a congruence. The error in school textbooks, defining similarity as "same shape but not necessarily same size", must be corrected as soon as possible.

The reason for the critical need of a definition of similarity is that a working knowledge of similar triangles is absolutely essential for students to achieve algebra. Without this knowledge, they would have no hope of understanding the interplay between a linear equation of two variables and its graph, which is a major topic in beginning algebra. This means our teachers of middle school must be completely at ease with similar triangles and know how to exploit the same, *and* they must know the underlying reasoning. The professional development on this topic may include the information that, while teachers should know the reasoning (i.e., proofs) behind the AAA (three pairs of equal angles) and SAS (two pairs of proportional sides and a pair of equal included angles) criteria for similarity, students in middle school can get by with less. Students can afford to learn to use these theoretical tools first and wait for their explanations later. This is standard practice in mathematics education (e.g., the teaching of calculus without epsilon-delta). *Professional development should assure teachers of this fact so that they do not feel overwhelmed by the need to teach all the proofs about similarity*, something that even our high school teachers may find difficult. The goal is to equip middle school teachers with this knowledge so that they can better instruct their students about similarity.

Another important topic in the teaching of elementary geometry is the concept of measurement, which leads to the standard mensuration formulas about area of triangles, circumference of circles, etc. Conceptually, there is no difference between length, area, or volume. If we let "measurement" stand for any of these three concepts, then on the basis of the following three entirely reasonable statements, all the standard mensuration formulas can be proved:

(1) Measurement is the same for congruent sets.

(2) Measurement is **additive**, in the sense that if two sets are disjoint except at their respective boundaries, then the measurement of the union is the sum of the measurements of the two sets.

(3) If a set S is the limit of a sequence $\{S_i\}$ in an appropriate sense, then the measurement of S is the limit of the measurements of the $\{S_i\}$.

Of course (3) has to be carefully qualified, and any discussion of "limit" has to remain intuitive, but if experience is any guide, such an approach to limit at the middle school level does not seem to be a hindrance to learning. As is well-known, the introduction of limit at this juncture is necessary if the circumference and area of a circle are to be meaningfully computed. The noteworthy feature of these three assumptions lies not in (3), but, rather, in how congruence enters the discussion of measurement through (1). The fact that the concept of congruence underlies the concept of measurement has not been sufficiently emphasized in school mathematics, but it should be. This is another example of the coherence of mathematics. Incidentally, the important role played by congruence in the study of measurements is one reason why congruence must be correctly defined.

It should be mentioned that while the number π can be defined in many ways, *a **strong recommendation** for school mathematics is to define it, not as circumference divided by the diameter, but as the area of the unit circle.* The former does not lend itself to any hands-on experiments to determine its value with any precision, whereas the latter does. Using accurate grid papers (with small grids), one can approximate the area of a circle by counting the number of grids in it together with elementary estimation of those only partially in it, and the value of π estimated by this method usually comes out to be within 0.05 of the exact value.

(E) **Algebra** It can be argued that the most basic aspect of the learning of algebra is the fluent use of symbols. Unfortunately, if textbooks are any guide, students' attempt to learn about symbols is at present hindered by the need to master the concept of a *variable*. There are two reasons why the concept of a variable unnecessarily obstructs learning. The first one is that the mathematics education literature, including textbooks, does not make explicit what a "variable" is. It is sometimes described as a quantity that changes or varies. At other times, it is asserted that students' understanding of this concept should be beyond recognizing that letters can be used to stand for unknown numbers in equations, but it does not say what exactly lies "beyond" this recognition. A second reason is that while mathematicians use the terminology of a "variable" informally and often, there is no *mathematical* concept called a "variable". The closest that comes mind is "an element in the domain of definition of a function", or "the indeterminate of the polynomial ring $\mathbf{R}[x]$", but certainly *nothing varies in mathematics*. The first task in the professional development of algebra therefore has to be to disabuse prospective teachers of this notion of a "variable" that they acquired in their K–12 education. There is absolutely no need for it. For further discussions of how to handle the pedagogical issue of the use of symbols and what a "variable" is, see pp. 3–6 of [Schmid-Wu].

In summary, the important thing at the beginning algebra is to get used to using symbols to represent numbers and to compute with them. It is not necessary to worry about what a "variable" means. However, it should be also pointed out that, the language of "variable" being entrenched in mathematics as it is, it would be to our advantage to follow the common usage and use it *informally* when it is convenient. But each time we do, we will be explicit about what the word stands (though it will not be anything that "varies").

Let a letter x stand for a number, in the same way that the pronoun "he" stands for a man. Then any (algebraic) expression in x is a number, and all the knowledge accumulated about rational numbers can be brought to bear on such expressions. There is a caveat, however. Because all we know about such an x is a number without any knowledge of its exact value, computations with expressions in x must then be done using only all the rules we know to be true for *all* numbers, namely, the associative and commutative laws and the distributive law. Doing computations not with specific numbers but with an arbitrary number brings into focus the concept of **generality**. For this reason, beginning algebra is *generalized arithmetic*. Nevertheless, arithmetic it is, and despite students' initial unfamiliarity with the presence of a large number of symbols, they will soon get used to computing with polynomials or rational expressions as ordinary numbers. Note that students who are uncomfortable with ordinary number computations to begin with may be made even more uncomfortable at this juncture. *This underscores the importance of a firm grasp of rational numbers for the learning of algebra.* (Cf. [Wu] 2001.) For example, the following addition of rational expressions in a number x can be carried out as with rational *quotients*,

$$\frac{x^2}{(3x^4+x+2)}+\frac{6}{(x^2+5)} \quad = \quad \frac{x^2(x^2+5)+6(3x^4+x+2)}{(3x^4+x+2)(x^2+5)} \ ,$$

because if k, ℓ, m, n stand, respectively, for the numbers x^2, $(3x^4+x+2)$, 6, and (x^2+5), then each of these is a rational number and the equality becomes nothing more that the usual formula for the addition of rational quotients:

$$\frac{k}{\ell}+\frac{m}{n}=\frac{kn+m\ell}{\ell n}$$

(See the comments on *rational quotients* in item (C) on **Rational numbers** above.)

Three remarks about the preceding paragraph should be made in the context of professional development. The first is that consideration of the arithmetic of rational expressions confirms why the arithmetic of complex fractions and rational quotients are indispensable to the learning of algebra. Teachers need to be aware of this fact for their own teaching of fractions and rational numbers. A second one is the reference above to x as a number. In school mathematics, the only kind of numbers treated with any thoroughness are the rational numbers. Irrational numbers are basically no more than a name. Unfortunately, it is not in the tradition of school mathematics to be explicit about the restriction to only rational numbers in mathematical discussions about real numbers. For example, the preceding paragraph implicitly assumes that *even if x is an irrational number, the addition of the two rational expressions above will continue to hold.* This is indeed correct on account of advanced considerations about the "extension of continuous functions from rational numbers to real numbers". The explanation of the phrase in quotes is beyond the level of normal professional development for middle school teachers, but we are nevertheless obligated to make teachers aware of this extrapolation from rational numbers to all numbers. One can succinctly formulate this extrapolation as **FASM**, the **Fundamental Assumption of School Mathematics** (see [Wu2002b]):

> All the information about the arithmetic operations on fractions can be extrapolated to all real numbers.

A third and final remark is that if we let x be a whole number, then the above addition

$$\frac{x^2}{(3x^4+x+2)} + \frac{6}{x^2+5}$$

becomes an addition of two (ordinary) fractions because the numerators and denominators are whole numbers. Notice therefore how the addition was carried out, which is to use the basic formula

$$\frac{k}{\ell} + \frac{m}{n} = \frac{kn+m\ell}{\ell n}$$

without worrying about the *LCM* of the whole numbers $3x^4+x+2$ and x^2+5. If elementary school teachers can take note of this fact in algebra, then they will realize how misguided it really is to teach the addition of fractions using the LCM of the denominators, which is how most school textbooks still teach it. If we do not want to mislead students with this kind of defective information, it would be most helpful if teachers can see through the defect. Incidentally, this is one reason why we want teachers to know the mathematics of several grades beyond what they teach (cf. [NMPa], Recommendation 19 on page xxi.)

It should be pointed out that, if the letters x and y are just numbers, then the distributive law gives

$$x^{n+1} - y^{n+1} = (x-y)(x^n + x^{n-1}y + x^{n-2}y^2 + x^{n-3}y^3 + \cdots + xy^{n-1} + y^n)$$

for *any* two numbers x and y, and *any* positive integer n.

Because this equality of these two expressions in x and y is valid for all numbers x and y, we call the equality an **identity**. Letting $y = 1$, we get another identity:

$$x^{n+1} - 1 = (x-1)(x^n + x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1)$$

for all numbers x and for all positive integers n. If $x \neq 1$, multiplying both sides by the number $\frac{1}{x-1}$ and switching the left and the right sides give:

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

for any number $x \neq 1$, and for any positive integer n. This is of course the so-called **summation of the finite geometric series**. In a school classroom, one might teach this summation formula by first doing a few concrete cases such as $n = 2$, $n = 3$ and $n = 4$, before doing it for a general n. This summation formula is usually taken up near the end of the study of algebra in high school. We have seen that there is no reason for the delay, all the more so because this formula is important in so many areas of mathematics.

A major topic in beginning algebra is the relationship between a linear equation in two variables $ax + by = c$ and its graph. To the extent that our teachers learned from their K–12 school textbooks, there would be many gaps in teachers' knowledge about these equations. The first is a correct definition of the **slope** of a line L. It needs to be shown that the slope of L defined by two chosen points P and Q on L is in fact independent of the choice of P and Q. In this case, it is not merely correctness for its own sake. Knowing this independence leads to the awareness that, in each situation, one can choose the two points most suitable to one's purpose for the computation of the slope. Sometimes, being able to make such a choice is the difference between success and failure. But this independence cannot be proved without knowing the AAA criterion for the similarity of triangles (i.e., two triangles with three pairs of equal angles are similar), and this is the reason similarity must be taught correctly before taking up algebra.

A second gap is the precise definition of the **graph of $\boldsymbol{ax+by=c}$** as the set of *all* the points (x', y') whose coordinates satisfy the equation, i.e., $ax' + by' = c$. Without explicitly invoking this definition, it would be impossible to prove the basic theorem of linear equations in two variables, to the effect that *the graph of $ax + by = c$ is a line, and any line is the graph of some linear equation of two variables.* The lack of emphasis in enunciating the definition of the graph of an equation goes hand-in-hand with the absence of this proof in most algebra textbooks. Such a proof depends strongly on knowing the precise definition of the graph of an equation and on knowing when two triangles are similar. Students who understand the details of this proof will have a good grasp of the genesis of the many forms of the equation of a line (point-slope form, slope-intercept form, etc.) that satisfies some prescribed conditions, e.g., passing through two given points; they will have no need to memorize these different forms by brute force. At the moment, anecdotal evidence suggests that the relationship between a linear equation and its graph remains a black box to many teachers and students; if this can be verified by research, it would mark a significant progress in the teaching and learning of algebra.

Associated with linear equations in two variables are linear inequalities. Again, one must first give a precise definition of the **graph of an inequality** and then prove that such a graph is a **half-plane**. There are many ways to handle this theorem. A drastic way to cut through the subtleties is to simply define a half-plane to be the graph of a linear inequality and then give many examples and ample discussions to make this drastic step reasonable. A more reasonable alternative is to define a half-plane of a line L which is not vertical to be all the points *above* L or all the points below L, and then prove the theorem. (The definition of the half-planes of a vertical line is trivial.) However, no matter which approach is adopted, it is not an acceptable way to teach mathematics by *not* defining either the graph of an inequality or a half-plane, and then engaging in the wishful thinking that, by drawing a few pictures, the reader would automatically understand how to define these terms and also buy into the theorem. The uphill battle one must fight in the professional development of algebra therefore includes the need to convince teachers not to engage in such practices and to learn a new way to deal with this topic.

Many algebra curricula now take up linear programming as an application of linear inequalities. The fact that a linear function assumes it maximum and minimum in a convex polygonal region at a vertex then requires a careful explanation. For this purpose, it is all the more reason to have a precise definition of the graph of a linear inequality and that of a half-plane.

The subject of simultaneous linear equations (to be called **linear systems**) is a straightforward study of the interplay between two linear equations, or what is the same thing, the interplay between two lines, but its simplicity is often compromised in school texts. There, the meaning of the **solution** of a linear system is almost never *explicitly* given, with the result that it is not used to explain why the point of intersection of the lines defined by the individual equations provides a solution of the linear system then cannot be given. Students are told to use graphing calculators to get the solution of a linear system, but they are not told why the graphing calculator gives the right answer. This is a new paradigm of learning-by-rote, and is one that we should strive to eliminate from the school classrooms. To this end, our teachers must receive careful instruction on the definitions in question as well as the associated explanations.

There is another misconception associated with linear systems. The usual method of solution of a linear system **by substitution**, due to misinformation from textbooks, has been interpreted as an exercise in the *symbolic* manipulation of variables. When done correctly, however, the method of substitution is strictly a computation with numbers, no more and no less, and there is no need in this context to worry about the purported complex meaning of the equal sign when applied to variables. Indeed, what the substitution method does is not to produce a solution, but rather, starting with the assumption that ***a solution*** $\boldsymbol{(x_0, y_0)}$ ***exists***, it shows what the values of x_0 and y_0 must be. Then one substitutes these values into the original linear system to verify easily that they are solutions. Explications of such subtleties have to be a necessary component of any reasonable professional development in algebra.

The next major topic in algebra is the concept of a function, its definition and the study of linear and quadratic functions. Again, one should insist on explicit definition of the **graph** of a function; for real-valued function f of one variable, its graph is the subset of the plane consisting of all ordered pairs $\{(x, f(x))\}$ where x is a member of the domain of f. The graphs of linear functions, $f(x) = cx + k$, are lines, and this follows from the work on graphs of linear equations in two variables because the graph of the function f is seen to be the graph of the linear equation in two variables $y = cx + k$. A special class of linear functions, those **without constant term** k, are especially important in middle school mathematics. They underlie all considerations of *constant rate*. Constant speed, for example, is the statement that there is a constant v, so that if the distance traveled from time 0 to time t is $f(t)$, then $f(t) = vt$. It also underlies all the problems connected with *proportional reasoning*. This then calls for a little soul-searching in this connection.

At the moment, there appears to be some misconception about the formulation of mathematical problems. Consider a prototypical proportional-reasoning problem such as:

> A group of 8 people are going camping for three days and need to carry their own water. They read in a guide book that 12.5 liters are needed for a party of 5 persons for 1 day. How much water should they carry?

It should be clear that this problem cannot be done without first assuming that *everybody drinks the same amount of water each day*. There is nothing obvious about this assumption, because even young kids can see that some people drink lots of water and others very little. If we believe that mathematics is precise, then precision demands that this assumption be made explicit. An alternative to the question, "How much water should they carry?", is to ask how to make a rough estimate of the amount of water they should carry *if we simplify matters by assuming that everybody drinks the same amount everyday.* Once we have this assumption, let f be the function *defined on the whole numbers* so that $f(n)$ is the amount of water n people drink each day. Then the assumption gives $f(n) = cn$ for some constant c, where c is the amount of water each person is assumed to drink per day. A common practice is to now allow the symbol n to stand for *any* number, and not just a whole number, so that $f(n)$ becomes a linear function without constant term. With the given data that $f(5) = 12.5$, we want the number $3f(8)$. From the former, we get $c = 2.5$, so the answer is $3 \times 2.5 \times 8 = 60$ liters. The main point is, however, that if proportional reasoning is about "understanding the underlying relationships in a proportional situation and working with these relationships" ([NRC 2001], p. 241), then the proportional relationship *must be made explicit for students* as otherwise students would be groping in the dark without a clue. Professional development should make this point very clear: guesswork is not to be confused with conceptual understanding, and there is no ground for assuming that every student "*understands*" that all people drink the same amount of water everyday.

Once these ground rules are understood, a discussion of word problems related to proportional reasoning from the point of view of linear functions should be both revealing and rewarding.

From linear functions we go to quadratic ones. The graph of a linear function is a line, but what is the graph of a quadratic function? If by some good fortune we know that the quadratic function is presented to us in the form of $f(x) = a(x+p)^2+q$, where a and q are fixed numbers, then one can picture the graph of f without too much effort, as follows. From $f(-p+s) = f(-p-s)$, we see that for all numbers s, the point $(-p-s, f(-p-s))$ and the point $(-p+s, f(-p+s))$ are symmetric relative to the vertical line $x = -p$. Therefore the graph of f has an axis of reflection symmetry along the vertical line $x = -p$, and the graph has its lowest (resp., highest) point at $(-p, f(-p))$, if $a > 0$ (resp., $a < 0$). Of course, $f(-p) = q$. So at least for simple quadratic functions expressible as $f(x) = a(x+p)^2 + q$, the graph is completely understood, and therewith, the function itself is completely understood. In fact, we can trivially read off from the equality $f(x) = a(x+p)^2 + q$ where the function is equal to 0, namely,

$$-p \pm \sqrt{-\frac{q}{a}}$$

The fundamental theorem about quadratic functions is that, by the technique of *completing the square*, every quadratic function can be written in the form $f(x) = a(x+p)^2 + q$ for fixed numbers a, p, and q. Included in this statement is the **quadratic formula** for the **roots** (zeros) of f, but the whole discussion make it perfectly clear that the technique of completing the square is the key to the understanding of quadratic functions. One can then go on to discuss the relationship between roots and the factoring of a quadratic function, and also the relationship between the roots and the coefficients a, b, c in $f(x) = ax^2+bx+c$. In particular, teachers should be aware that the quadratic formula trivializes, at least in principle, the problem of factoring quadratic polynomials, because one proves (the far from obvious statement) that if r_1, r_2 are the roots of a quadratic polynomial f, then $f(x) = a(x-r_1)(x-r_2)$ for some fixed number a, and this is a factoring of f. Since the quadratic formula provides the values of the roots, the factoring immediately follows. The implication of this discussion for professional development is therefore quite clear: our algebra teachers have to understand this aspect of the quadratic formula in order for them to teach the factoring of trinomials with the proper perspective.

The availability of quadratic functions enlarges the range of word problems. Their discussion should be an integral part of professional development.

If we look ahead into the high school curriculum, we see that introductory algebra is inextricably tied to the materials in high school mathematics as a whole. We have already emphasized the important role of similar triangles in the discussion of linear equations of two variables. For teachers to be comfortable teaching about such linear equations, they would have to learn the proofs of the basic criteria for similar triangles. As is well-known, similarity is the deepest part of plane geometry. Thus a teacher who wants to teach introductory algebra well should be at least familiar with high school geometry. Furthermore, we have seen that linear and quadratic functions are a staple of introductory algebra. Teachers cannot, however, teach about these functions if this is all they know. They need a reservoir of knowledge about a few other standard functions as well, e.g., higher degree polynomial functions, exponential functions, logarithmic functions, periodic functions. For polynomial functions in general, teachers need to be familiar with the most basic facts, such as the Fundamental Theorem of Algebra and its implications on the factorization of polynomials, especially the factorization of real polynomials. But these already comprise all the main topics in the school algebra curriculum. Finally, among the most profound applications of similarity is the definition of the trigonometric functions, and these are the basic building blocks of periodic functions. A teacher of beginning algebra should therefore also know something about trigonometry.

If we can draw any conclusion from the preceding discussion, it must be that a middle school teacher who teaches beginning algebra should have at least a good mastery of the high school mathematics curriculum.

(F) **Probability and statistics** It has long been recognized by some mathematicians and statisticians that K–6 is not the place to teach serious statistics. The recent publication of *Curriculum Focal Points* by the National Council of Teachers of Mathematics ([NCTM] 2006) has finally acknowledged this fact.

There is no harm in discussing the basic notions such as mode, mean, and median to liven up the discussion of number facts from time to time, but teachers should be made aware that meaningful statistics appears only in high school or beyond, when there is sufficient mathematical preparation to support such a discussion. They should also be alerted to the senseless practice in standardized tests of asking for the mode of a small number of items. The concept of mode is meaningful only when a large number of items is involved.

The basic concepts of probability can and should be discussed in middle school, of course, such as the fact that probabilities are numbers between 0 and 1, the concept of a sample space, the relation between theoretical probability and the relative frequency of an event. The teaching of simple facts concerning binomial coefficients and elementary combinations and permutations would also enrich the curriculum of middle school.

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