A theorem about similarity*

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Our goal is to prove the following theorem that was stated on page 286 of Rational Numbers to Linear Equations (**RNLE**). Recall that a **similarity** is by definition the composition of a finite number of dilations and congruences (page 284 of RNLE).

Main Theorem. *For a transformation F of the plane, the following three conditions are equivalent:*

(i) *F is the composition of a finite number of dilations and congruences.*

(ii) *F is equal to the composition of a dilation followed by a congruence.*

(iii) *F is equal to the composition of a congruence followed by a dilation.*

We have to first clarify the linguistic ambiguities. By "the composition of a dilation followed by a congruence" in (ii), we mean $\varphi \circ D$, where φ is a congruence and D is a dilation (so the dilation D moves a point first before the congruence φ). Similarly, by "the composition of a congruence followed by a dilation" (as in (iii)), we mean $D \circ \varphi$, where D is a dilation and φ is a congruence.

Before giving the proof of the main theorem, we should point out the pedagogical significance of the theorem itself in the context of school mathematics. What this theorem says is that the concept of *similarity* can be defined in three different—but equivalent—ways as in (i)–(iii) above. Without a doubt, either (ii) or (iii) is much more intuitive than (i). For example, if we introduce students to the concept of

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similarity by using (i) as the definition, then it raises the specter that there may be similar figures that require the use of a composition of at least 50 dilations and congruences to map one on the other. This would suggest that similarity is a very complicated concept. However, if one thinks of similarity in terms of (ii), for instance, then the concept becomes very simple: to check that a figure $\mathcal{S}_1$ is similar to another figure $\mathcal{S}_2$, just look for a suitable dilation D to see if $D(\mathcal{S}_1)$ will be congruent to $\mathcal{S}_2$. For this reason, when students are introduced to *similarity* as a precise concept for the first time in middle school, either (ii) or (iii) would be the preferred definition. This explains why (ii) is used as the definition of similarity in the author's text, **Teaching School Mathematics: Pre-Algebra**[1], for middle school teachers and educators, because it is more appropriate for the middle school curriculum.

Obviously, we are now obliged to explain why we use (i) of the main theorem as the definition of similarity in RNLE. This is because when we put emphasis on proving theorems about similarity, both (ii) and (iii) are too clumsy for this purpose. For illustration, suppose we have (ii) as the definition of similarity and we know two figures $\mathcal{A}$ and $\mathcal{B}$ are each similar to $\mathcal{S}$. In the usual notation, we are given $\mathcal{A} \sim \mathcal{S}$ and $\mathcal{B} \sim \mathcal{S}$. Naturally we want to say that $\mathcal{A}$ is similar to $\mathcal{B}$, i.e., we want $\mathcal{A} \sim \mathcal{B}$. Let us see how we can prove this. By hypothesis, we have dilations D and D' and congruences ψ and ψ' so that

$$\psi'(D'(\mathcal{A})) = \mathcal{S} \quad \text{and} \quad \psi(D(\mathcal{B})) = \mathcal{S} \tag{1}$$

To show that $\mathcal{A} \sim \mathcal{B}$, we must find a dilation Δ and a congruence φ so that

$$\varphi(\Delta(\mathcal{A})) = \mathcal{B} \tag{2}$$

From (1), we get

$$\psi'(D'(\mathcal{A})) = \psi(D(\mathcal{B}))$$

so that

$$(D^{-1} \circ \psi^{-1} \circ \psi' \circ D')(\mathcal{A}) = \mathcal{B}$$

Since $\psi^{-1} \circ \psi$ is obviously a congruence, let us denote it by ψ_1. Then we have

$$(D^{-1} \circ \psi_1 \circ D')(\mathcal{A}) = \mathcal{B} \tag{3}$$

[1]American Mathematical Society, Providence, RI, 2016. Note that its **index**, missing from the book, is available at `http://tinyurl.com/zjugvl4`.

For (2) to be true, we must be able to find a dilation Δ and a congruence φ so that

$$D^{-1} \circ \psi_1 \circ D' \;=\; \varphi \circ \Delta \tag{4}$$

It is not at all clear from (4) what φ and Δ should be.

If we are allowed to make use of Lemmas 1 and 2 immediately following, then we can define φ and Δ as follows. By Lemma 2, we know there exists a dilation D_1 so that $D^{-1} \circ \psi_1 = \psi_1 \circ D_1$. Thus the left side of (4) becomes

$$D^{-1} \circ \psi_1 \circ D' \;=\; \psi_1 \circ D_1 \circ D' \tag{5}$$

Now applying Lemma 2 to $D_1 \circ D'$ on the right side of (5), we get a translation T and a dilation Δ so that

$$\psi_1 \circ D_1 \circ D' \;=\; \psi_1 \circ T \circ \Delta \tag{6}$$

Letting φ be the congruence $\psi_1 \circ T$ and combining (5) and (6), we get (4), i.e.,

$$D^{-1} \circ \varphi \circ D' \;=\; \varphi \circ \Delta$$

Inasmuch as the fact that "$\mathcal{A} \sim \mathcal{S}$ and $\mathcal{B} \sim \mathcal{S}$ imply $\mathcal{A} \sim \mathcal{B}$" is something that should be known to all students at the outset of their introduction to similarity, we cannot afford to define similarity by making use of (*ii*) of the main theorem, for at least two reasons. The first is that, *with* (*ii*) *as the definition of similarity,* the proof of "$\mathcal{A} \sim \mathcal{S}$ and $\mathcal{B} \sim \mathcal{S}$ imply $\mathcal{A} \sim \mathcal{B}$" using Lemmas 1 and 2 is mathematically much too sophisticated for high school, and the second reason is that the proofs of these two lemmas—which will occupy us for most of the remainder of this article—are simply too long for use in the normal high school math classroom. For analogous reasons, we should not define similarity by making use of (*iii*) of the main theorem.

If we define similarity by using (*i*) of the main theorem, then technically speaking, it is much easier to show that two figures are similar. For example, the proof of "$\mathcal{A} \sim \mathcal{S}$ and $\mathcal{B} \sim \mathcal{S}$ imply $\mathcal{A} \sim \mathcal{B}$" becomes almost trivial (see Lemma 5.3 on page 285 of RNLE). An additional reason for making use of (*i*) to define similarity in RNLE is that, since high school students have already been exposed to transformations and their compositions in middle school, and since they have also been exposed to a more intuitive version of similarity, they are in a much better position, intellectually, to accept (i) as the definition of similarity. But of course we would not have the luxury of such a pedagogical option if we did not have the assurance that the main theorem is mathematically valid. As often happens in school mathematics, content dictates pedagogy.

We can now turn to the proof of the Main Theorem. It needs the following two technical lemmas, already alluded to above.

Lemma 1. *Let D be a dilation and φ a congruence. Then there is a dilation D' so that $D \circ \varphi = \varphi \circ D'$, and there is also a dilation D'' so that $\varphi \circ D = D'' \circ \varphi$.*

Lemma 2. *Given two dilations D_1 and D_2, there is a translation T and a dilation D so that $D_2 \circ D_1 = T \circ D$, and there is also a translation T' and a dilation D' so that $D_2 \circ D_1 = D' \circ T'$.*

***Proof of Lemma* 1**

We preface the proof with a few remarks. The lemma is *almost* the statement that D and φ commute, but D' and D'' are not going to be equal to D in general. In this case, arriving at the correct statement of the lemma is probably more difficult than proving it[2] because, once the statement of Lemma 1 is known, there is no mystery as to what the dilations D' and D'' would have to be: if $D \circ \varphi = \varphi \circ D'$, then necessarily $D' = \varphi^{-1} \circ D \circ \varphi$; and if $\varphi \circ D = D'' \circ \varphi$, then necessarily $D'' = \varphi \circ D \circ \varphi^{-1}$. This reasoning also tells us how to prove the lemma: define D' and D'' as above and then prove that D' and D'' so defined must be dilations.

Here is the *proof of the first part of Lemma* 1. Let the given dilation D have center O and scale factor r, and let φ be a congruence. We claim: *the transformation D' defined by $D' = \varphi^{-1} \circ D \circ \varphi$ is a dilation.*

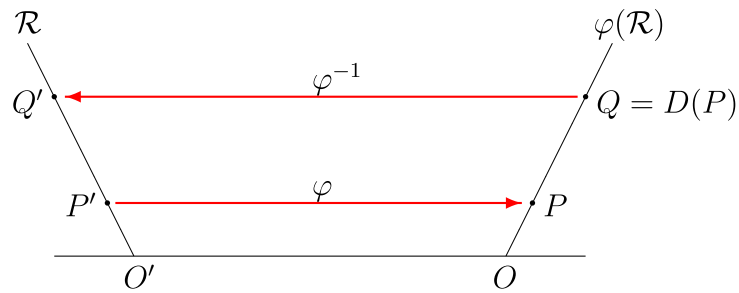
Since the congruence φ is a bijection, there is a point O' in the plane so that $\varphi(O') = O$. We will prove that D' is a dilation with center O' and scale factor r. We first show that $D'(O') = O'$. This is because $D(O) = O$ (by the definition of the

[2]This should remind you of a similar situation regarding the Pythagorean theorem.

center of a dilation) and $\varphi^{-1}(O) = O'$ (by the definition of O'). Therefore,

$$D'(O') \;=\; \varphi^{-1}(D(\varphi(O'))) \;=\; \varphi^{-1}(D(O)) \;=\; \varphi^{-1}(O) \;=\; O'$$

Next, we show that for a point $P' \neq O'$, the point $D'(P')$ lies on the ray $\mathcal{R} = R_{O'P'}$.



Since congruences and dilations map rays to rays, $\varphi(\mathcal{R})$ is the ray containing $\varphi(O') = O$ and the point $\varphi(P')$, to be denoted by P. Let $Q = D(P)$). Since D is a dilation centered at O, we have $Q \in \varphi(\mathcal{R})$. Finally, let $Q' = \varphi^{-1}(Q)$, then Q' lies in the ray $\mathcal{R}$, and, furthermore,

$$Q' \;=\; \varphi^{-1}(Q) \;=\; \varphi^{-1}(D(P)) \;=\; \varphi^{-1}(D(\varphi(P'))) \;=\; D'(P')$$

We have now proved that for any point $P' \neq O'$, $D'(P')$ $(= Q')$ lies on the ray $\mathcal{R} = R_{O'P'}$.

To show that D' is a dilation with scale factor r, it remains to verify that

$$|O'Q'| \;=\; r\,|O'P'| \tag{7}$$

Since $\varphi(O') = O$ and $\varphi(Q') = Q$, and φ is a congruence, we see that

$$|O'Q'| \;=\; |OQ|$$

But by the definition of Q, $Q = D(P)$, and since D is a dilation with center O and scale factor r, we have

$$|OQ| \;=\; r\,|OP|$$

But φ is a congruence and $\varphi(O'P') = OP$, therefore

$$|OP| \;=\; |O'P'|$$

Putting these three facts together, we get $|O'Q'| = r\,|O'P'|$, which is (7). This proves that D' is a dilation centered at O' with scale factor r. Since $D' = \varphi^{-1} \circ D \circ \varphi$ by definition, clearly $D \circ \varphi = \varphi \circ D'$. The first part of Lemma 1 is proved.

The proof of the second part is similar. Given a congruence φ and a dilation D with center O and scale factor r as before. Define a transformation D'' by $D'' = \varphi \circ D \circ \varphi^{-1}$. Then the preceding *claim* shows that D'' is a dilation with center $\varphi(O)$ and scale factor r. To see this, we rephrase the preceding claim as follows:

> If D is a dilation and ψ is a congruence, then the transformation F defined by $F = \psi^{-1} \circ D \circ \psi$ is a dilation.

Now letting $\psi = \varphi^{-1}$ and observing that $\psi^{-1} = (\varphi^{-1})^{-1} = \varphi$ (see Exercise 1), we see that D'' is a dilation. From the definition of $D'' = \varphi \circ D \circ \varphi^{-1}$, we see immediately that $\varphi \circ D = D'' \circ \varphi$. The proof of Lemma 1 is complete.

Proof of Lemma 2

Let O_1 and O_2 be the centers of the dilations D_1 and D_2, respectively. Let the scale factors of D_1 and D_2 be r_1 and r_2, respectively. The proof breaks up into two cases.

Case I. *Suppose $r_1 r_2 = 1$. Then $D_2 \circ D_1$ is a translation.*

This means that if $r_1 r_2 = 1$, then Lemma 2 is true in a stronger form: we can take the dilations D and D' in the lemma to be the identity map of the plane.

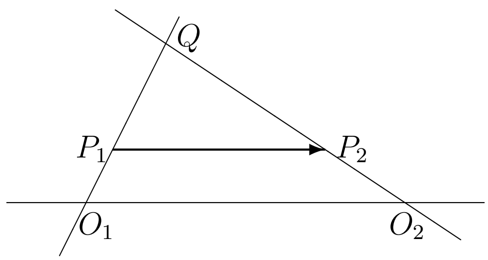
Case II. *Suppose $r_1 r_2 \neq 1$. Then there is a translation T and a dilation D so that $D_2 \circ D_1 = T \circ D$, and there is also a translation T' and a dilation D' so that $D_2 \circ D_1 = D' \circ T'$.*

We first prove Case I. Thus, $r_1 r_2 = 1$. We can ignore the trivial case where $r_1 = r_2 = 1$ because then $D_1 = D_2 =$ the identity transformation of the plane, and we can also ignore the other trivial case where $O_1 = O_2$ because then $D_2 \circ D_1$ is obviously a dilation with the same center O_1. Thus, for the rest of the proof of Case I, we assume $r_1 \neq 1$, $r_2 \neq 1$, and $O_1 \neq O_2$.

Let us first take up the case of $r_1 > 1$ (so that $r_2 < 1$). We claim that in this case,

$D_2 \circ D_1$ is the translation along a vector pointing in the same direction[3] *as $\overrightarrow{O_1O_2}$ with length equal to $(1-r_2)\,|O_1O_2|$.*

For the proof of the claim, let P_1 be a point in the plane, and let $(D_2 \circ D_1)(P_1) = P_2$. Then we have to show that the vector $\overrightarrow{P_1P_2}$ has length $(1-r_2)\,|O_1O_2|$ and points in the same direction as $\overrightarrow{O_1O_2}$. Let $D_1(P_1) = Q$, then $D_2(Q) = P_2$. Because $r_1 > 1$, $P_1 \in O_1Q$, and because $r_2 < 1$, $P_2 \in O_2Q$.



Since D_1 and D_2 are dilations with centers O_1 and O_2, respectively, and with scale factors r_1 and r_2, respectively, we have

$$|O_1Q| = r_1\,|O_1P_1| \quad (8)$$

$$|O_2P_2| = r_2\,|O_2Q| \quad (9)$$

Therefore,

$$\frac{|P_1Q|}{|O_1Q|} = \frac{|O_1Q| - |O_1P_1|}{|O_1Q|} = 1 - \frac{1}{r_1} = 1 - r_2$$

where the last equality is because $r_1r_2 = 1$. Similarly,

$$\frac{|P_2Q|}{|O_2Q|} = 1 - r_2$$

It follows that

$$\frac{|P_1Q|}{|O_1Q|} = \frac{|P_2Q|}{|O_2Q|}$$

because both are equal to $1 - r_2$. By FTS (Theorem G10 on page 256), we have

$$|P_1P_2| = (1-r_2)\,|O_1O_2| \quad (10)$$

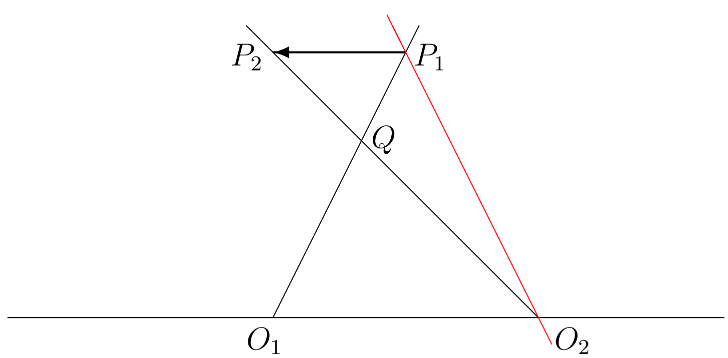
and

$$P_1P_2 \parallel O_1O_2 \quad (11)$$

[3]See page 231 of RNLE for the definition of *pointing in the same direction*.

In view of (11), we can now prove that $\overrightarrow{P_1P_2}$ and $\overrightarrow{O_1O_2}$ point in the same direction by proving that the closed half-plane of L_{O_1Q} that contains P_2 also contains both of the rays $R_{O_1O_2}$ and $R_{P_1P_2}$. Indeed, O_2P_2 contains no point of L_{O_1Q} because if it did—let us say O_2P_2 contains a point Q' of L_{O_1Q}—then the two lines L_{O_1Q} and $L_{O_2P_2}$ would have two distinct points Q and Q' in common and therefore must coincide (by Lemma 4.2 on page 165 of RNLE). A contradiction. Hence the segment O_2P_2 does not intersect L_{O_1Q}, and O_2 and P_2 belong to the same half-plane of L_{O_1Q} (see assumption (L4)(ii) on page 176 of RNLE). It is now straightforward to show that both rays $R_{O_1O_2}$ and $R_{P_1P_2}$ also belong to this closed half-plane (consider using Lemma 4.6 on page 174 of RNLE). So $\overrightarrow{P_1P_2}$ and $\overrightarrow{O_1O_2}$ point in the same direction. This proves the claim and therefore also Case I when $r_1 > 1$.

To complete the proof of Case I, it remains to consider the case of $r_1 < 1$ (so that $r_2 > 1$). We claim that in this case, $D_2 \circ D_1$ *is the translation along a vector pointing in the same direction as* $\overrightarrow{O_2O_1}$ *with length equal to* $(r_2 - 1)\,|O_1O_2|$. Let P_1 be a point in the plane and let $(D_2 \circ D_1)(P_1) = P_2$. Then we must prove that $\overrightarrow{P_1P_2}$ and $\overrightarrow{O_2O_1}$ point in the same direction and that $|P_1P_2| = (r_2 - 1)\,|O_11O_2|$. To this end, let $D_1(P_1) = Q$; then $D_2(Q) = P_2$. Now, because $r_1 < 1$, we have $Q \in O_1P_1$, and because $r_2 > 1$, we have $Q \in O_2P_2$, as shown.



A similar computation as before yields

$$\frac{|QP_1|}{|QO_1|} = \frac{|QP_2|}{|QO_2|} = r_2 - 1 \tag{12}$$

and of course $\angle O_1QO_2$ and $\angle P_1QP_2$ are equal on account of opposite angles at the point Q. Therefore $\triangle QO_1O_2 \sim \triangle QP_1P_2$ because of SAS for similarity (Theorem G21 on page 287 of RNLE).

This triangle similarity has two consequences. First, the angles $\angle QO_1O_2$ and $\angle QP_1P_2$ are equal, and this implies that the alternate interior angles of the transversal $L_{O_1P_1}$ with respect to the lines $L_{P_1P_2}$ and $L_{O_1O_2}$ are equal. By Theorem G19 on page 281 of RNLE, we have $P_1P_2 \parallel O_1O_2$. We are now in a position to show that the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{O_2O_1}$ point in the same direction. To this end, we will show that the closed half-plane of the line $L_{P_1O_2}$ containing P_2 also contains O_1. This is because the line L_{P_2Q} already intersects $L_{P_1O_2}$ at O_2 and therefore the segment P_2Q cannot contain another point of $L_{P_1O_2}$. In other words, P_2Q does not intersect $L_{P_1O_2}$ and therefore P_2 and Q lie in the same half-plane of $L_{P_1O_2}$. A similar reasoning also shows that O_1 and Q lie in the same half-plane of $L_{P_1O_2}$. Thus P_2 and O_1 lie in the same half-plane of $L_{P_1O_2}$, and it follows that the rays $R_{O_2O_1}$ and $R_{P_1P_2}$ lie in the same closed half-plane of $L_{P_1O_2}$ (again, consider using Lemma 4.6 on page 174 of RNLE). Together with the fact that $P_1P_2 \parallel O_1O_2$, this proves that the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{O_2O_1}$ point in the same direction.

A second consequence of the triangle similarity $\triangle QO_1O_2 \sim \triangle QP_1P_2$ is that, because of (12), we have

$$\frac{|P_1P_2|}{|O_1O_2|} \;=\; r_2 - 1$$

Equivalently, $|P_1P_2| = (r_2 - 1)\,|O_1O_2|$. This proves the claim and, therewith, completes the proof of Case I.

Next, we consider Case II of Lemma 2, where $r_1r_2 \neq 1$. As usual, let O_1 and O_2 be the centers of the dilations D_1 and D_2, respectively. We will first prove that there is a translation T and a dilation D so that $D_2 \circ D_1 = T \circ D$.

The proof of Case II hinges on a trivial *observation: if two dilations F and G have the same center O and their scale factors are r and s, respectively, then their composition $F \circ G$ is also a dilation, whose center is O and whose scale factor is rs.* This follows immediately from the definition of a dilation.

Back to the proof of Case II. Let D_0 be the dilation with center O_1 and scale factor $\frac{1}{r_1r_2}$. By the preceding observation, the composition $D_1 \circ D_0$ is a dilation with center O_1 and scale factor

$$r_1 \cdot \frac{1}{r_1r_2} \;=\; \frac{1}{r_2}$$

Therefore, $D_2 \circ (D_1 \circ D_0)$ is a composition of two dilations so that their scale factors are r_2 and $1/r_2$, respectively. Because the product of the scale factors is equal to 1, Case I implies that this composition is a translation T, i.e.,

$$D_2 \circ (D_1 \circ D_0) \;=\; T \tag{13}$$

The left side of (13) can be expressed differently. We have

$$D_2 \circ (D_1 \circ D_0) \;=\; (D_2 \circ D_1) \circ D_0, \tag{14}$$

because what this means (see page 208 of RNLE for the concept of equal transformations) is that for every point P in the plane,

$$\big(D_2 \circ (D_1 \circ D_0)\big)(P) \;=\; \big((D_2 \circ D_1) \circ D_0\big)(P).$$

But this is true because both sides are equal to $D_2(D_1(D_0(P)))$. Therefore (13) implies

$$(D_2 \circ D_1) \circ D_0 \;=\; T \tag{15}$$

This is *almost* what we want, and to get to exactly what we want, we introduce the dilation D whose center is O_1 (the center of D_0) and whose scale factor $r_1 r_2$. The virtue of D is that the composition $D_0 \circ D$ is the dilation with center O_1 and scale factor $\frac{1}{r_1 r_2} \cdot r_1 r_2 = 1$, i.e., $D_0 \circ D$ is the identity map I of the plane. We make use of this fact in the following way: from (15), we get

$$\big((D_2 \circ D_1) \circ D_0\big) \circ D \;=\; T \circ D \tag{16}$$

By the reasoning of (14), the left side of (16) is equal to $(D_2 \circ D_1) \circ (D_0 \circ D)$. Since $D_0 \circ D = I$, we see that the left side of (16) is equal to $(D_2 \circ D_1)$. Hence, (16) leads to $D_2 \circ D_1 = T \circ D$, as desired.

Finally, we prove that there is a translation T' and a dilation D' so that $D_2 \circ D_1 = D' \circ T'$. The proof is similar to the preceding one, so we will only sketch it and leave the details to Exercise 2. Let $\overline{D}$ be the dilation with center O_2 and scale factor $\frac{1}{r_1 r_2}$. Then $\overline{D} \circ D_2$ is a dilation with center O_2 and scale factor $\frac{1}{r_1}$, so that $(\overline{D} \circ D_2) \circ D_1$ is a composition of two dilations whose scale factors have a product equal to 1. By Case I, this composition is equal to a translation T', or

$$(\overline{D} \circ D_2) \circ D_1 \;=\; T'$$

By the reasoning of (14), the left side is equal to $\overline{D} \circ (D_2 \circ D_1)$. Thus we have

$$\overline{D} \circ (D_2 \circ D_1) \;=\; T' \tag{17}$$

Now let D' be the dilation with center O_2 and scale factor equal to $r_1 r_2$ so that $D' \circ \overline{D} = I$. Then (17) implies that

$$D' \circ \left(\overline{D} \circ (D_2 \circ D_1)\right) \;=\; D' \circ T'$$

By the reasoning of (14), the left side is equal to $(D' \circ \overline{D}) \circ (D_2 \circ D_1)$. Since $D' \circ \overline{D} = I$, the left side becomes $D_2 \circ D_1$. Therefore we conclude $D_2 \circ D_1 = D' \circ T'$. The proof of Lemma 2 is complete.

Proof of the Main Theorem

Observe that the equivalence of (ii) and (iii) is a trivial consequence of Lemma 1. To prove the Main Theorem, it suffices to prove $(i) \Longleftrightarrow (ii)$. Recall the statements of (i) and (ii):

(i) *F is the composition of a finite number of dilations and congruences.*

(ii) *F is equal to the composition of a dilation followed by a congruence.*

Obviously, (ii) implies (i). We have to prove (i) implies (ii). So suppose a transformation F is the composition of a finite number of congruences and dilations, and we have to show that there is a congruence φ and a dilation D so that $F = \varphi \circ D$. Let us first look at a special case. Suppose

$$F \;=\; \varphi_3 \circ D_3 \circ \varphi_2 \circ D_2 \circ D_1 \circ \varphi_1 \tag{18}$$

where the D's are dilations and the φ's are congruences. Notice that, in (18), there is a composition of two successive dilations, $D_2 \circ D_1$, but there is no composition of two *successive congruences*, and that is for a good reason. Suppose we have a composition of two successive congruences, e.g., $\varphi' \circ \varphi$ for congruences φ and φ'. Because a composition of congruences is always a congruence, we can more simply use a single congruence ψ to denote such a composition $\varphi' \circ \varphi$ in (18).

The first step of the proof is to "move" all the φ's in (18) to the left (thereby "moving" all the D's to the right). This can be accomplished very efficiently by moving the rightmost congruence (in case of (18), this would be φ_1) all the way to the left until there are no more dilations to its left. This statement will make more sense when we illustrate it with a specific case like (18). So start with φ_1 in (18). By Lemma 1, there is a dilation D_4 so that $D_1 \circ \varphi_1 = \varphi_1 \circ D_4$. Now (18) becomes

$$F \;=\; \varphi_3 \circ D_3 \circ \varphi_2 \circ D_2 \circ \varphi_1 \circ D_4 \tag{19}$$

One more application of Lemma 1 to $D_2 \circ \varphi_1$ in (19) shows that

$$F \;=\; \varphi_3 \circ D_3 \circ \varphi_2 \circ \varphi_1 \circ D_5 \circ D_4 \tag{20}$$

for a certain dilation D_5. Now the composition $\varphi_2 \circ \varphi_1$ in (20) is a congruence, so we may let $\varphi_4 = \varphi_2 \circ \varphi_1$ and keep in mind that φ_4 is a congruence. Thus, we have

$$F \;=\; \varphi_3 \circ D_3 \circ \varphi_4 \circ D_5 \circ D_4 \tag{21}$$

A final application of Lemma 1 shows that $D_3 \circ \varphi_4$ in (21) is equal to $\varphi_4 \circ D_6$ for a suitable dilation D_6. Then we obtain

$$F \;=\; \varphi_3 \circ \varphi_4 \circ D_6 \circ D_5 \circ D_4 \tag{22}$$

Now F has been expressed as the composition of congruences on the left and a sequence of dilations on the right.

The next step is to use Lemma 2 to reduce the sequence of dilations on the right of (22) to a single dilation, perhaps at the cost of introducing additional congruences into (22). We start with the first two D's on the left in (22), namely, $D_6 \circ D_5$. Thus, Lemma 2 says $D_6 \circ D_5$ in (22) is equal to $T_7 \circ D_7$ for a translation T_7 and a dilation D_7. So we get:

$$F \;=\; \varphi_3 \circ \varphi_4 \circ T_7 \circ D_7 \circ D_4 \tag{23}$$

Finally, a second application of Lemma 2 to (23) yields the fact that for some translation T_8 and some dilation D_8, we have $D_7 \circ D_4 = T_8 \circ D_8$. Therefore,

$$F \;=\; \varphi_3 \circ \varphi_4 \circ T_7 \circ T_8 \circ D_8 \tag{24}$$

Since each of φ_3, φ_4, T_7, and T_8 is a congruence, their composition is just a congruence, which we denote by φ. Hence we obtain from (24) that $F = \varphi \circ D_8$. This proves (ii) for the special case of F as in (18).

A little reflection will show that the preceding proof of "$(i) \Longrightarrow (ii)$" for the special case of an F as in (18) is in fact perfectly general. Indeed, what we have presented is an algorithm for transforming a composition of any finite number of congruences and dilations into the composition of a dilation followed by a congruence. More precisely, there are two steps:

> **Step 1.** Use Lemma 1 to move the rightmost congruence to the left and, in the process, combine any two neighboring congruences into one congruence as in the passage from (20) to (21). Keep doing so until there are no dilations to the left of a congruence, as in (22). This then leaves us with a composition of congruences on the left together with a composition of dilations on the right.
>
> **Step 2.** Now use Lemma 2 to reduce the composition of dilations on the right to the composition of a sequence of translations and a single dilation on the right (as in (24)).

It is not difficult to see that these two steps will suffice to prove that $(i) \Longrightarrow (ii)$ in general, but writing out the general proof will involve some gruesome notation with nothing new added to the reasoning. For this reason, we will stop while we are ahead. The proof of $(i) \Longrightarrow (ii)$ is complete. We have therefore proved the Main Theorem.

Exercises

1. If ψ is a bijection of the plane, prove that $(\psi^{-1})^{-1} = \psi$. (Hint: see Exercise 5 on page 216 of RNLE.)

2. Write out the details of the proof that, given two dilations D_1 and D_2, there is a translation T' and a dilation D' so that $D_2 \circ D_1 = D' \circ T'$.