A characterization of regular polygons*

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We will prove the following theorem, which is Theorem 4.14 on page 197 of Rational Numbers to Linear Equations (**RNLE**).

Main Theorem. *A polygon whose sides have the same length and whose angles have the same degree can be inscribed in a circle if and only if it is convex.*

This theorem is stated in Section 4.2 of RNLE purely for the benefit of readers' conceptual understanding of a **regular polygon**, which by definition is a polygon which has equal sides and equal angles and is *inscribed in a circle* in the sense that all its vertices lie on that circle. In a technical sense, it does not belong to RNLE because its proof requires some facts about triangles and circles that will not be available until Chapter 6 of Algebra and Geometry (the companion volume that follows RNLE). Since this theorem will not be put to use until Chapter 7 of Algebra and Geometry, there is no fear of circular reasoning in the proof below.

Preliminaries

For the understanding of a *convex polygon*, we will need the following *Jordan Curve Theorem for Polygons*, which is Theorem 4.13 on page 195 of RNLE.

Theorem 4.13. *The complement of a polygon $\mathcal{P}$ consists of two non-empty planar regions, B and E with the following properties:*

*I wish to thank Larry Francis for his excellent editorial assistance.

(i) B and E are both connected, B is bounded and E is unbounded, and $\mathcal{P}$ is their common boundary. Moreover, the three sets B, E, and $\mathcal{P}$ are disjoint and their union is the whole plane.

(ii) A segment joining a point of B to a point of E must intersect the polygon $\mathcal{P}$.

In addition, suppose we have two nonempty planar regions B' and E' so that $\mathcal{P}$ is their common boundary and so that the plane is the disjoint union of the three sets B', E', and $\mathcal{P}$. Then, after a change of notation if necessary, we have $B' = B$ and $E' = E$.

We take this opportunity to make a correction in RNLE. In lines 4 and 5 of page 196 in RNLE, the phrase "the three sets B, E, and $\mathcal{P}$" *should be* "the three sets B', E', and $\mathcal{P}$".

We recall the definition of a region being *connected* (RNLE, page 195). A **polygonal segment** is a finite collection of segments A_1A_2, A_2A_3, A_3A_4, $\ldots$, $A_{n-2}A_{n-1}$, $A_{n-1}A_n$, with the understanding that these segments need not be noncollinear and that there may be intersections among them. Then a region $\mathcal{R}$ in the plane is said to be **connected** if any two points in $\mathcal{R}$ can be joined by a polygonal segment *that lies completely in $\mathcal{R}$.* The *complement* of a subset $\mathcal{S}$ in the plane is the collection of all the points in the plane not lying in $\mathcal{S}$. A point Q is a **boundary point** of a region $\mathcal{S}$ in the plane if in every disk (no matter how small) around Q, there is a point in $\mathcal{S}$ and a point not in $\mathcal{S}$. The **boundary** of $\mathcal{S}$ consists of all the boundary points of $\mathcal{S}$.

Referring to Theorem 4.13, the region E is called the **exterior** of $\mathcal{P}$ and the union of $\mathcal{P}$ and B is called the **polygonal region** of $\mathcal{P}$. Then we say P is a **convex polygon** if its polygonal region is convex.

Recall that, without further notice, an angle means the *convex* angle (RNLE, page 182), and that the **convexity** of a polygon refers to the fact that the polygonal region of the polygon is convex.

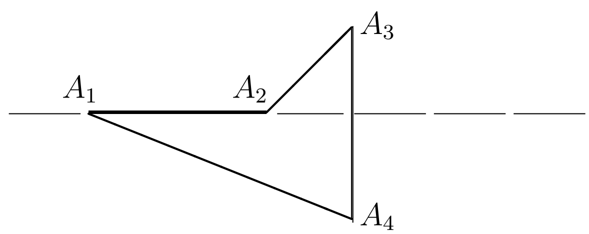
Proof: First part

We first prove half of the Main Theorem: *A convex polygon whose sides have the same length and whose angles have the same degree can be inscribed in a circle.*

We will show that the angle bisectors of all the angles of the given convex polygon meet at a common point that is equidistant from all of the vertices. To this end, the first step is to prove the following lemma.

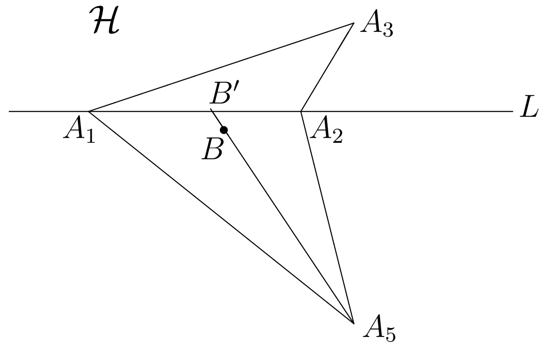
Lemma 1. *Let L be a line that contains a side of a convex polygon. Then the polygon lies in a closed half-plane of L.*

The fact that this is a special property of *convex* polygons can be seen from the following picture where the quadrilateral $A_1A_2A_3A_4$ fails to lie entirely in either closed half-plane of the line containing the side A_1A_2.



Proof of Lemma 1. Let the convex polygon $\mathcal{P}$ have vertices $A_1A_2\cdots A_n$. Consider one side A_1A_2 of $\mathcal{P}$, and let the line containing A_1A_2 be denoted by L. Also denote the half-plane of L containing A_3 by $\mathcal{H}$ and its *closed* half-plane by $\overline{\mathcal{H}}$. We claim: $\overline{\mathcal{H}}$ *contains all the vertices A_1, A_2, $\ldots$, A_n of $\mathcal{P}$.*

To see this, suppose (let us say) A_5 lies in the opposite half-plane of $\mathcal{H}$. Then A_3 and A_5 lie in opposite half-planes of L and the segment A_1A_2 lies in both convex angles $\angle A_1A_3A_2$ and $\angle A_1A_5A_2$.



By Theorem 4.13, $\mathcal{P}$ is the boundary of its polygonal region and, by the definition of a *boundary point* (see page 194 of RNLE), around each point Q of A_1A_2, there are points from the exterior E that are as close to Q as we please. Thus, let B be a point in E sufficiently near the midpoint of A_1A_2 so that it stays in both of the convex angles $\angle A_1A_3A_2$ and $\angle A_1A_5A_2$. Now B cannot lie in L because, if it does, it would lie in A_1A_2 which is part of $\mathcal{P}$ and therefore disjoint from E. Therefore B lies in one of the two half-planes of L. For definiteness, let us say B lies in the opposite half-plane of $\mathcal{H}$. In particular, B and A_5 lie in the same half-plane of L. By the crossbar axiom (see page 250 of RNLE), the ray R_{A_5B} will intersect A_1A_2 at a point $B' \in A_1A_2$ and B is between A_5 and B'. Since A_5 and B' are points in $\mathcal{P}$, the convexity of $\mathcal{P}$ implies that the segment A_5B' lies entirely in the polygonal region of $\mathcal{P}$. By the disjointness of the polygonal region of $\mathcal{P}$ from its exterior E (see Theorem 4.13(i), the segment A_5B' is disjoint from E. But $B \in A_5B'$, so B does not belong to E. This contradiction proves the claim.

Since the closed half-plane $\overline{\mathcal{H}}$ is convex, the vertices of $\mathcal{P}$ being in $\overline{\mathcal{H}}$ implies that the segments $A_1A_2, \ldots, A_{n-1}A_n, A_nA_1$ also lie in $\mathcal{P}$. Thus $\overline{\mathcal{H}}$ also contains $\mathcal{P}$.

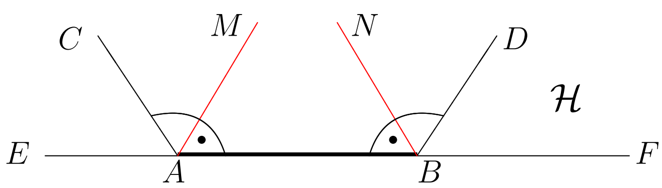
Of course, there is nothing special about the side A_1A_2. The preceding reasoning therefore proves Lemma 1.

In view of Lemma 1, the following lemma now makes sense. Anticipating the resulting notational complexity in its proof, we adopt an *ad hoc* notational scheme for the statement of this lemma.

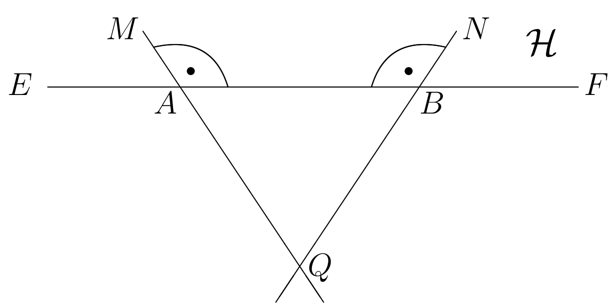
Lemma 2. *Let AB be one side of a convex polygon and let $\overline{\mathcal{H}}$ be the closed half-plane of L_{AB} containing the polygon. Then the angle bisectors of the adjacent angles $\angle A$*

and $\angle B$ *intersect* in the half-plane $\mathcal{H}$.

Proof of Lemma 2. Let C and D be points in $\mathcal{H}$ so that $\angle A = \angle CAB$ and $\angle B = \angle ABD$. Also let M and N be points in $\mathcal{H}$ so that the rays R_{AM} and R_{BN} are the angle bisectors of $\angle A$ and $\angle B$, respectively. We have to prove that the rays R_{AM} and R_{BN} intersect in $\mathcal{H}$.

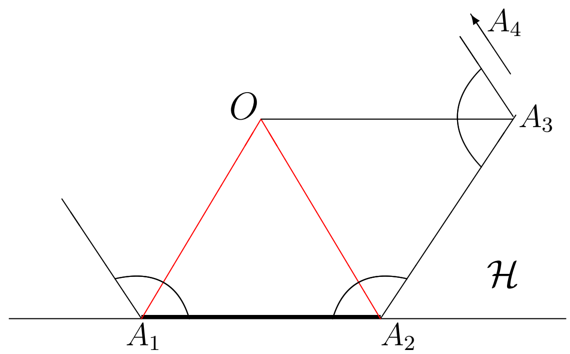


Consider the two lines L_{AM} and L_{BN} and their transversal L_{AB}. On L_{AB}, choose points E and F so that A is between E and B and the point B is between A and F, as shown. Because the two angles $\angle BAM$ and $\angle FBN$ both lie in the closed half-plane $\overline{\mathcal{H}}$ of L_{AB}, they are corresponding angles with respect to L_{AM} and L_{BN}. Now $|\angle BAM| = \frac{1}{2}|\angle BAC| < \frac{1}{2} \cdot 180° = 90°$, therefore $\angle BAM$ is acute. On the other hand, $\angle NBA$ is also acute for the same reason, and therefore its supplementary angle $\angle FBN$ is obtuse. It follows that $\angle BAM$ and $\angle FBN$ are not equal. By Theorem G18 on page 277 of RNLE, L_{AM} and L_{BN} are not parallel and hence must intersect. It remains to show that their point of intersection lies in $\mathcal{H}$. If not, then let their point of intersection Q lie in the opposite half-plane of $\mathcal{H}$ with respect to L_{AB}.

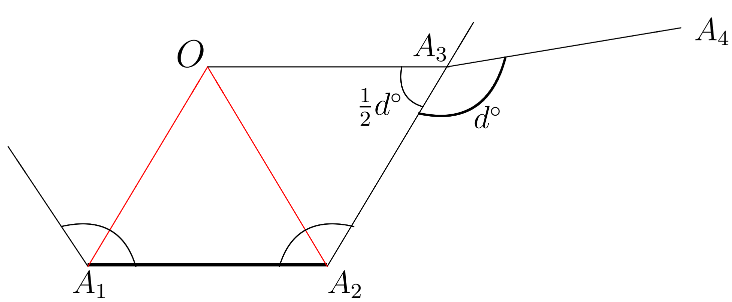


As we have observed above, $\angle BAM$ and $\angle NBA$ are both acute, and therefore their respective supplementary angles, $\angle QAB$ and $\angle ABQ$, must be obtuse. This implies that the angle sum of $\triangle QAB$ exceeds $180°$, a contradiction (see Theorem G32 in Section 6.5 of Algebra and Geometry). Therefore Q has to lie in $\mathcal{H}$ and Lemma 2 is proved.

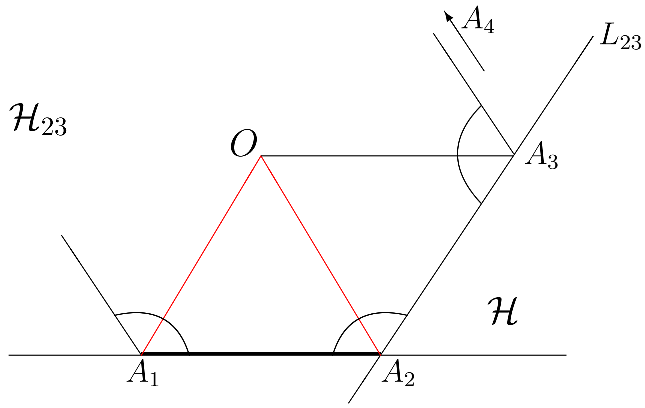
We can now finish the proof of the first half of the Main Theorem. Since the angles of the polygon $\mathcal{P}$ all have equal degrees, we may denote this common degree by d°. So let $\mathcal{P} = A_1A_2, \cdots A_n$ as before. Let the angle bisectors of $\angle A_1$ and $\angle A_2$ meet at a point O. Let L be the line containing the side A_1A_2. By Lemma 2, O lies in the half-plane $\mathcal{H}$ of L that contains $\mathcal{P}$. Join OA_3, as shown.



Consider the two triangles, $\triangle OA_1A_2$ and OA_3A_2. We claim: $\triangle OA_1A_2 \cong OA_3A_2$. This follows from SAS because $|\angle OA_2A_1| = |\angle OA_2A_3| = \frac{1}{2}\,d^\circ$, $|A_1A_2| = |A_3A_2|$ (by the hypothesis on $\mathcal{P}$), and the two triangles have the side OA_2 in common. Hence $|\angle OA_3A_2| = |\angle OA_1A_2| = \frac{1}{2}\,d^\circ$. Since $|\angle A_3| = d^\circ$ by the hypothesis on $\mathcal{P}$, this suggests that OA_3 is the angle bisector of $\angle A_3$. This will be true as soon as we can show that O lies in $\angle A_3$. *A priori*, however, this need not be the case because O and A_4 could conceivably lie in opposite half-planes of the line containing A_2 and A_3, as the following picture shows.



To show that this anomaly doesn't happen, we observe that the ray A_2O is the angle bisector of $\angle A_2$ and is, in particular, in the convex $\angle A_2$. Let L_{23} be the line containing A_2A_3. Then O and A_1 lie in the same half-plane of L_{23} (see page 236 of RNLE). Call this half-plane $\mathcal{H}_{23}$. Since A_1 lies in $\mathcal{H}_{23}$, by Lemma 1, the polygon $\mathcal{P}$ itself lies in the closed half-plane $\overline{\mathcal{H}_{23}}$ and therefore A_4 also lies in $\mathcal{H}_{23}$.



Let O' be a point in $\angle A_3$ ($= \angle A_2A_3A_4$) so that A_3O' is the angle bisector of $\angle A_3$. Observe that O' being in $\angle A_3$ means that it lies in the half-plane of L_{23} that contains A_4 and therefore O' lies in $\mathcal{H}_{23}$. Thus, $|\angle A_2A_3O'| = \frac{1}{2}\, d^\circ$. Now consider the two convex angles, $\angle A_2A_3O'$ and $\angle A_2A_3O$: they have one side in common (the ray from A_3 to A_2), O and O' lie in the same half-plane $\mathcal{H}_{23}$ of L_{23}, and $|\angle A_2A_3O| = |\angle A_2A_3O'| = \frac{1}{2}\, d^\circ$. By Lemma 4.10 on page 190 of RNLE, the other sides of the angles coincide, i.e., A_3O is the angle bisector of $\angle A_3$.

We may now look at O as the point of intersection of the angle bisectors of $\angle A_2$ and $\angle A_3$. A similar reasoning then shows that OA_4 is the angle bisector of $\angle A_4$, etc. In summary, the angle bisectors of all the angles $\angle A_i$, for $i = 1, 2, \ldots, n$ pass through the point O.

It remains to observe that O is equidistant from all the vertices A_i for $i = 1, 2, \ldots, n$. This is because, for example, in $\triangle OA_1A_2$, the angles $\angle OA_1A_2$ and $\angle OA_2A_1$ are equal as they have the same degree, $\frac{1}{2}\, d^\circ$. Thus, $|OA_1| = |OA_2|$ (Theorem G29 in Section 6.2 of Algebra and Geometry). Similarly, $|OA_1| = \cdots = |OA_n|$, and therefore the circle with center O and radius $|OA_1|$ passes through all the vertices of $\mathcal{P}$. The proof of the first half of the main theorem is complete.

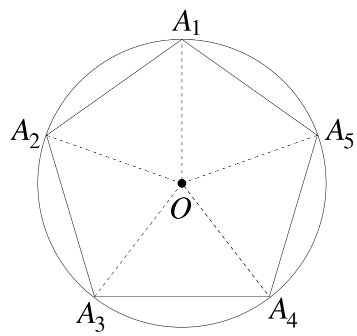
Proof: Second part

We next prove the converse: *If a polygon whose sides have the same length and whose angles have the same degree can be inscribed in a circle, then it is convex.*

We will prove something more general, which is of independent interest. .

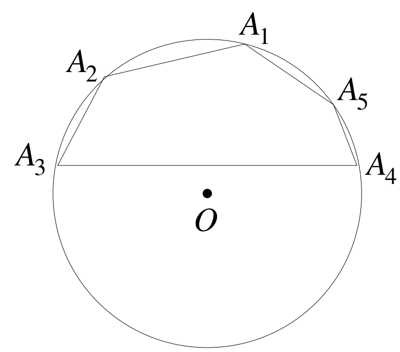
Lemma 3. *A polygon inscribed in a circle is convex.*

Before embarking on the proof of Lemma 3, we would like to give a heuristic argument (not a proof) for a special case: If a polygon (pictured below as a pentagon $A_1A_2A_3A_4A_5$) is inscribed in a circle O and all its sides have the same length and all its angles have the same degree, how can we see intuitively that it has to be convex?



If we join all the vertices to the center O, then all the triangles, $\triangle A_iOA_{i+1}$ for $i = 1, 2, 3, 4, 5$ (with $A_6 = A_1$ understood), are congruent because of SSS. Therefore the central angles $\angle A_iOA_{i+1}$ for $i = 1, 2, 3, 4, 5$ are all equal and, together, they fill up the full angle of 360° at O (remember that an angle is a region in the plane, not two rays). The vertices are thus "evenly distributed" on the circle O. Denote the line that contains the side A_iA_{i+1} by $L_{i\,i+1}$ (recall $A_6 = A_1$). Then each of the five lines, L_{12}, L_{23}, . . . , L_{51}, has the special property that its closed half-plane that contains the center O also contains all the vertices of the polygon.

To appreciate the last statement, observe that it is false for a general pentagon. For example, the half-plane of the line L_{34} in the picture to the right that contains the center O does not contain the other vertices A_1, A_2, and A_5. If these vertices were "evenly distributed", then none of the vertices A_2, A_1, and A_5 could have been squeezed into the "upper" arc $\overset{\frown}{A_3A_4}$.



Now back to our heuristic argument about a polygon inscribed in a circle with equal sides and equal angles. Denote the closed half-plane of $L_{i\,i+1}$ that contains the center O by $\mathcal{H}_{i\,i+1}$ for $i = 1, 2, 3, 4, 5$ (with $\mathcal{H}_{56} = \mathcal{H}_{51}$ understood). Glancing at the previous picture of a pentagon with vertices evenly distributed on the circle, one would be inclined to believe that the polygonal region enclosed by the pentagon is exactly the intersection of the closed half-planes $\mathcal{H}_{12}$, $\mathcal{H}_{23}$, $\mathcal{H}_{34}$, $\mathcal{H}_{45}$, and $\mathcal{H}_{51}$. But a closed half-plane of a line is convex, and intersections of convex sets are convex. So the polygonal region enclosed by $A_1A_2A_3A_4A_5$ is convex, and $A_1A_2A_3A_4A_5$ is, by definition, a convex polygon.

Proof of Lemma 3. Let $\mathcal{P}$ be the polygon $A_1A_2 \cdots A_n$ inscribed in a circle O (O being the center). The idea is to make use of the idea in the preceding heuristic argument: show that the polygonal region of $\mathcal{P}$ is equal to the intersection of a finite number of closed half-planes.

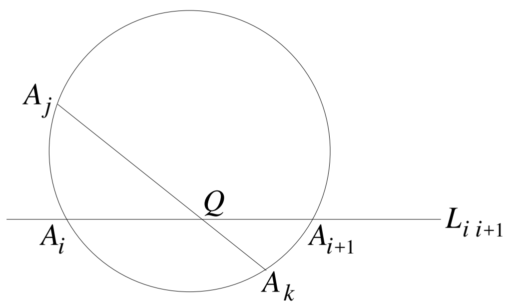
Let the closed disk inside circle O be denoted by $\overline{\mathcal{D}}$ as usual. Note that $\overline{\mathcal{D}}$ is convex (Theorem G47 in Algebra and Geometry). Let the line containing the chord A_iA_{i+1} be denoted by $L_{i\,i+1}$ as before. Then the intersections of the circle O with the two *closed* half-planes of $L_{i\,i+1}$ are called the **opposite arcs** determined by the chord A_iA_{i+1} (see Section 6.8 in Algebra and Geometry). We will need the following two observations.

Observation 1. Let $1 \leq i \leq n$. Of the two *opposite arcs* determined by the chord A_iA_{i+1} (always with $A_{n+1} = A_1$ understood), one of them contains no vertex of $\mathcal{P}$ other than A_i and A_{i+1} while the other arc contains all the vertices of $\mathcal{P}$.

Observation 1 allows us to introduce a notation: we will use $\overparen{A_iA_{i+1}}$ to denote the arc determined by the chord A_iA_{i+1} that contains no vertex of $\mathcal{P}$ other than A_i and A_{i+1}.

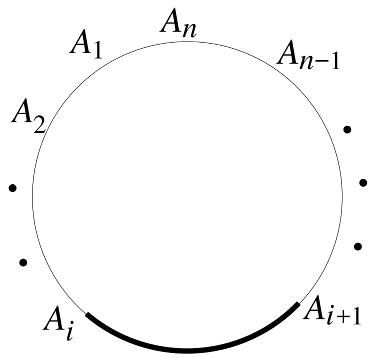
Observation 2. Any point on the circle O lies in $\overparen{A_iA_{i+1}}$ for some $i = 1, 2, \ldots n$ (again, by definition, $A_{n+1} = A_1$).

The reason for Observation 1 is as follows. Suppose A_j and A_k lie on opposite arcs of the chord A_iA_{i+1} for some j and k (neither j nor k being equal to i or $i+1$) as shown below. By the definition of opposite arcs, A_j and A_k lie in opposite half-planes of the line $L_{i\,i+1}$ containing the chord A_iA_{i+1}. By assumption (L4), the segment A_jA_k intersects the line $L_{i\,i+1}$ at some point to be called Q.



Since $\overline{\mathcal{D}}$ is convex, A_jA_k lies in $\overline{\mathcal{D}}$ and therefore $Q \in \overline{\mathcal{D}}$. Since also $Q \in L_{i\,i+1}$, we have $Q \in L_{i\,i+1} \cap \overline{\mathcal{D}}$. By Lemma 6.4 in Section 6.8 of Algebra and Geometry, $L_{i\,i+1} \cap \overline{\mathcal{D}}$ is the chord A_iA_{i+1}. Thus $Q \in A_iA_{i+1}$, and we see that Q is the intersection of A_iA_{i+1} and A_jA_k. But the two sides of a polygon cannot intersect except for adjacent sides at a common vertex, so this contradiction proves Observation 1.

Next, Observation 2 is a simple consequence of the fact that, with the vertices A_1, A_2, ..., A_n of $\mathcal{P}$ lying on the circle O, the union of the arcs $\overparen{A_1A_2}$, $\overparen{A_2A_3}$, ... $\overparen{A_nA_{n-1}}$, and $\overparen{A_nA_1}$ is the circle O, as shown in the following picture.



We can now get serious about the proof of Lemma 3. According to Observation 1, a half-plane of $L_{i\,i+1}$—which, we recall, does not contain $L_{i\,i+1}$ itself—either contains no vertex of $\mathcal{P}$ or contains all the vertices of $\mathcal{P}$ except A_i and A_{i+1}. Thus we can define for each $i = 1, 2, \ldots, n$,

$$\begin{aligned} \mathcal{H}^-_{i\,i+1} &= \text{the half-plane of } L_{i\,i+1} \text{ that contains no vertex of } \mathcal{P} \\ \mathcal{H}^+_{i\,i+1} &= \text{the half-plane of } L_{i\,i+1} \text{ opposite to } \mathcal{H}^-_{i\,i+1} \end{aligned}$$

Again, it is understood that $\mathcal{H}^+_{n\,n+1} = \mathcal{H}^+_{n1}$ and $\mathcal{H}^-_{n\,n+1} = \mathcal{H}^-_{n1}$. Note that in this notation, we have

$$\widehat{A_i A_{i+1}} = \{\text{the closed half-plane of } \mathcal{H}^-_{i\,i+1}\} \cap \overline{\mathcal{D}} \tag{1}$$

Because of Observation 1, $\mathcal{H}^+_{i\,i+1}$ contains every vertex of $\mathcal{P}$ except A_i and A_{i+1}. Now let

$$\begin{aligned} B_0 &= \bigcap_i \mathcal{H}^+_{i\,i+1} \ (= \text{ the intersection of } \mathcal{H}^+_{12},\, \mathcal{H}^+_{23},\, \ldots,\, \mathcal{H}^+_{n1}) \\ E_0 &= \bigcup_i \mathcal{H}^-_{i\,i+1} \ (= \text{ the union of } \mathcal{H}^-_{12},\, \mathcal{H}^-_{23},\, \ldots,\, \mathcal{H}^-_{n1}) \end{aligned}$$

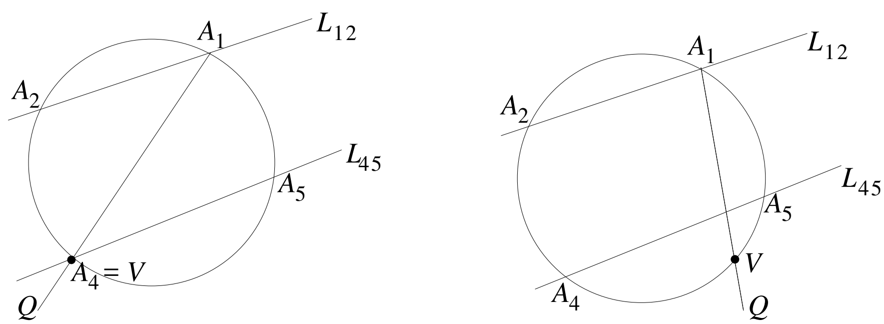
It is clear that B_0 and E_0 are both nonempty and that B_0, E_0, and $\mathcal{P}$ are disjoint; the latter is the consequence of $\mathcal{H}^+_{i\,i+1}$, $L_{A_iA_{i+1}}$, and $\mathcal{H}^-_{i\,i+1}$ being disjoint for every i (see Assumption (L4)(i) on page 176 of RNLE). Because each $\mathcal{H}^+_{i\,i+1}$ (respectively, $\mathcal{H}^-_{i\,i+1}$) has the line $L_{i\,i+1}$ as its boundary, it is also easy to see that B_0 (resp., E_0) has $\mathcal{P}$ as its boundary and that the plane is the disjoint union of B_0, E_0, and $\mathcal{P}$.

We claim that B_0 is bounded. In fact, we will show more:

$$B_0 \subset \text{the closed disk } \overline{\mathcal{D}} \tag{2}$$

We will prove this by contradiction. Suppose it is false, then there is a point $Q \in B_0$ in the exterior of the circle O. Since Q is in B_0, we have $Q \in \mathcal{H}_{12}^+$. Then Q is in the half-plane of L_{12} that contains A_3, A_4, ..., A_n. Let the segment A_1Q intersect the circle O at a point V.[1] By Observation 2, V lies in $\widehat{A_iA_{i+1}}$ for some i.

Suppose V lies in $\widehat{A_1A_2}$. From the assertion in (1), V lies in $\mathcal{H}_{12}^-$, contradicting the fact that Q lies in $\mathcal{H}_{12}^+$. So $i \neq 1$. The reasoning for all the cases where $i \geq 2$ is similar, so let us say V lies in $\widehat{A_4A_5}$. There are two possibilities. First, assume V is equal to one of the endpoints of the arc, say $V = A_4$, as in the picture on the left.



Then Q and A_1 lie in opposite half-planes of the line L_{45} since the segment A_1Q intersects L_{45} at A_4. Recall: $\mathcal{H}_{45}^+$ is the half-plane of L_{45} containing all the vertices of $\mathcal{P}$ except A_4 and A_5. Therefore A_1 has to be $\mathcal{H}_{45}^+$. Consequently, Q, being in the opposite half-plane of L_{45}, must lie in $\mathcal{H}_{45}^-$. This contradicts the fact that Q, being in B_0, lies in $\mathcal{H}_{45}^+$. So V cannot be an endpoint of the arc $\widehat{A_4A_5}$.

It remains to consider the case of V lying in $\widehat{A_4A_5}$ but not equal to A_4 or A_5, as in the above picture on the right. By the assertion in (1), V is in the half-plane $\mathcal{H}_{45}^-$. But as before, since the half-plane of L_{45} that contains A_1 is by definition the half-plane $\mathcal{H}_{45}^+$, A_1 and V lie in opposite half-planes of the line L_{45}. It follows that the segment A_1V intersects L_{45} and, *a fortiori*, the segment A_1Q also intersects the line L_{45}. Therefore A_1 and Q lie in opposite half-planes of L_{45}. As A_1 lies in $\mathcal{H}_{45}^+$, Q has to be in $\mathcal{H}_{45}^-$. Again, this contradicts the fact that Q, being in B_0, lies in $\mathcal{H}_{45}^+$.

Altogether, we see that there can be no such point Q in the exterior of the closed disc $\overline{\mathcal{D}}$. This proves the assertion in (2) and it follows that B_0 is bounded (i.e., B_0 is

[1] We will assume the existence of this point of intersection V without proof, as the proof requires an understanding of the real numbers that is beyond the level of school mathematics. However, see the discussion near the top of page 195 in RNLE.

contained in some closed disk).

By the last part of Theorem 4.13, the union of B_0 and $\mathcal{P}$ is the polygonal region of $\mathcal{P}$. It is now easy to see that the union of B_0 and $\mathcal{P}$ is equal to the intersection of the closed half-planes of $\mathcal{H}^+_{i\,i+1}$ for $i = 1, 2, \ldots, n$. Since closed half-planes are convex (Exercise 9 in Exercises 4.1 on page 180 of RNLE) and the intersections of convex sets are convex (Exercise 7 in Exercises 4.1 on page 180 of RNLE), the polygonal region enclosed by $\mathcal{P}$ is convex. By definition, $\mathcal{P}$ is a convex polygon and the proof of Lemma 3 is complete.

We have proved the Main Theorem.

Exercises

1. Let $\mathcal{P}$ be a convex polygon $A_1A_2\cdots A_n$ and let L_i be the line containing the side A_iA_{i+1}, $i = 1, 2, \ldots, n$ *(with* $A_{n+1} = A_1$ *understood).* Prove that the polygonal region of $\mathcal{P}$ is equal to the intersection of all the closed half-planes of L_i containing the polygon $\mathcal{P}$ for $i = 1, 2, \ldots, n-1$

2. If $\mathcal{C}$ is a circle with center O, let A_1, A_2 be two points on $\mathcal{C}$ so that $|\angle A_1OA_2| = \frac{360}{n}$ degrees for a positive integer n. Let ϱ be the rotation of $\frac{360}{n}$ degrees around the center O so that $\varrho(A_1) = A_2$. Now let $A_3 = \varrho(A_2)$, $A_4 = \varrho(A_3)$, $\ldots$, $A_n = \varrho(A_{n-1})$. Prove that $A_1 = \varrho(A_n)$, and that $A_1A_2\cdots A_n$ is a regular n-gon.