What can we do about our dysfunctional school mathematics curriculum?

(*Preface* and *To the Instructor* from Rational Numbers to Linear Equations)

Hung-Hsi Wu

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This is a collection of three excerpts from the author's book, **Rational Numbers to Linear Equations** (Amer. Math. Soc., Providence, RI, 2020):

Preface (pp. 2-21)

To the Instructor (pp. 22-37)

The Bibliography (pp. 38-42)

The first two excerpts are essentially a survey of the turbulence in the nation's school mathematics education over the past half-century. It is a survey from a unique perspective, namely, that our school mathematics curriculum can indeed be presented from kindergarten to grade 12 with *mathematical integrity*. The fact that such a presentation is possible has long been used as a working hypothesis, but it has not been verified until now in this volume and its five companion volumes. This perspective sheds new light on the 1989 NCTM mathematics education reform, the ensuing Math Wars, and the Common Core State Standards for Mathematics. It also suggests some new directions for research in school mathematics education.

Preface

The really vital importance of definition is not, I venture to think, sufficiently emphasized even in good textbooks.... they form the premises from which the rest of the algebraic theorems are to be derived by a process of logical deduction.
George A. Gibson ([Gibson], page 3)

A nation's mathematics education is only as good as its mathematics teachers. The ongoing crisis in school mathematics education (cf. [RAGS]) therefore raises the question: what have we done wrong in the preparation of mathematics teachers? The answer is *plenty:* our longstanding neglect of the *mathematical* education of teachers has come home to roost. This neglect manifests itself in K-12, where we fail to ensure that correct mathematics is taught to students—especially future teachers—and we compound this neglect by failing to provide the needed corrective measures in universities for pre-service teachers to repair their mathematical mis-education in K-12 (cf. [Wu2011b]). Thus, through no fault of their own, the mathematics teachers of our nation are put in the untenable position of teaching from a position of weakness: they do not possess the needed knowledge of mathematics to carry out their basic duties.

The present volume is the fourth of six volumes whose collective goal is to provide the needed mathematical backing for a full-scale attack on the crisis in the *mathematical* education of mathematics teachers in K-12. This volume is the first of three—the other two volumes being [Wu2020b] and [Wu2020c]—that give a systematic and grade-level-appropriate exposition of the mathematics of grades 9–12 (excluding probability[1] and statistics), together with some essential background information about rational numbers. This is the mathematical content knowledge that we believe, as of 2020, all high school mathematics teachers need for their teaching and all mathematics educators[2] interested in high school mathematics need for their research. The previous three volumes—the volume

[1]There is, however, an exposition of finite probability in Section 1.10 of [Wu2016a].
[2]We use the term "mathematics educators" to refer to university faculty in schools of education.

on the mathematics of grades K-6 ([Wu2011a]) and the two volumes on the mathematics of grades 6-8 ([Wu2016a] and [Wu2016b])—have already been published. We hope that these six volumes will serve the dual purpose of revamping the *mathematical* education in the universities of pre-service mathematics teachers and future mathematics educators on the one hand, and on the other, offering textbook publishers a *detailed* blueprint on how to introduce mathematics into school textbooks that is both correct and learnable. These six volumes will also shore up the critical mathematical backgrounds of supervisors of mathematics and mathematics professional developers.

There has been no lack of books on all or parts of **school mathematics**—the mathematics of K-12—in the education literature. We have chosen to add another 2500 pages (the approximate total length of these six volumes) to the already voluminous literature because we believe these volumes provide a first attempt at solving two of the central problems in mathematics education: whether school mathematics can be made to respect the integrity of mathematics, and how much mathematics a mathematics teacher or a mathematics educator needs to know.

School mathematics that respects mathematical integrity

We will address the former problem first. These six volumes give a *detailed* confirmation of the fact that school mathematics—while maintaining its fidelity to the progression of the standard school mathematics curriculum from kindergarten to grade 12—*can* be made to respect the integrity of mathematics. Such a confirmation has been a long time coming.

In the following pages, we will explain what mathematical integrity is and why it is important to have an exposition of school mathematics that respects mathematical integrity.[3]

At first glance, it seems absurd that there would be any need to discuss whether school mathematics respects mathematical integrity. Is not school mathematics, by its very name, part of mathematics and, as such, does it not follow that school mathematics carries the integrity inherent in the subject? This is a misconception about *school mathematics* that we must confront without delay. School mathematics is in fact *not* part of mathematics if *mathematics* is understood to be what working mathematicians do or what

[3] It would be legitimate to also inquire why it has taken so long for someone to try to meet this obvious need.

is taught to math majors in college mathematics departments. Rather, *school mathematics* is an *engineered* version of mathematics—in the sense of **mathematical engineering** introduced in [Wu2006]—in the same way that civil engineering is an *engineered* version of Newtonian mechanics. Mathematical engineering customizes the abstractions of mathematics for consumption by K-12 students. For example, a fraction in mathematics is a straightforward concept: it is an element of the quotient field of the integral domain of integers. Fortunately, no one suggests that we tell this to ten-year-olds. Mathematical engineering intervenes at this point to recast the concept of fractions so that fractions can be understood by elementary students (see [Wu1998]). There are many such examples all through the K-12 curriculum, e.g., negative numbers, slope of a line, geometric measurements (length, area, and volume), congruence, similarity, exponential functions, logarithms, axioms of plane geometry, etc. The engineering that is needed to make these abstract concepts learnable by school students is therefore substantial at times. Now there is good engineering, but there is also bad engineering, and the question is whether *good* mathematical engineering has been put in the service of school mathematics. Unhappily, the answer is *not always*. In fact, school mathematics and mathematical integrity parted ways at least five decades ago, and our schools have been plagued by products of very bad mathematical engineering ever since.

Before proceeding further, we first explain what *mathematical integrity* is because this concept is coming into focus,. We say a mathematical exposition has **mathematical integrity** if it embodies the following five qualities:

> (a) **Definitions**: Every concept is clearly and precisely defined so that there is no ambiguity about what is being discussed. (See the quote from Gibson at the beginning of this Preface.)
>
> (b) **Precision**: All statements are precise, especially the hypotheses that guarantee the validity of a mathematical assertion, the reasoning in a proof, and the conclusions that follow from a set of hypotheses.
>
> (c) **Reasoning**: All statements[4] other than the unavoidable basic assumptions are supported by reasoning.[5]
>
> (d) **Coherence**: The basic concepts and skills are logically interwoven to form a single fabric, and the interconnections among them are consistently revealed.

[4]With the exception of a few standard ones such as the fundamental theorem of algebra.

[5]Intuitively, reasoning supports even those assumptions because there are reasons why we want to assume them.

> (e) **Purposefulness**: The mathematical purpose behind every concept and skill is clearly brought out so as to leave no doubt about why it is where it is.

These we call the **Fundamental Principles of Mathematics.** A fuller discussion of these principles will be found on pp. xxviii–xxxv in the **To the Instructor** section on pp. xxvii ff. below, but two things need to be said right away. First, the role of definitions in school mathematics has been misunderstood, and misrepresented, in the education literature thus far, so that—to educators—the emphasis on definitions may seem to be misplaced. One will find a more balanced presentation about definitions on pp. xxx–xxxii. Next, there is no difference between *reasoning* and *proof* in a mathematical context, and what is generally called *problem solving* in the education literature is part of what is known as *theorem proving* in mathematics.[6] Overall, it should not be difficult to see—and these three volumes will bear witness to this fact—that these five fundamental principles are what make mathematics *transparent*, in the sense that everything is on the table and no guesswork or privileged knowledge is needed for its decoding. They are also the qualities that make mathematics *accessible* to all students and *learnable* by all students. If we want mathematics learning to take place in schools, it is incumbent on us to teach school mathematics that is consistent with these fundamental principles.

But to return to the discussion of school mathematics of the past five decades, we have to begin by asking what *is* school mathematics? This is in fact the question that these six volumes ([Wu2011a], [Wu2016a], . . . , [Wu2020c]) attempt to answer, but short of that, we will have to say school mathematics is the common content of most of the mathematics textbooks in K-12 and most of the *college textbooks* aimed at the professional development of mathematics teachers and mathematics educators (compare the review of school textbooks in Appendix B of Chapter 3 in [NMAP2]). If this strikes readers as too vague, they will be relieved to know that there is in fact an amazing consistency among these textbooks.[7] For example, a fraction is thought of as a piece of pizza or a part-of-a-whole, although neither conveys the message to students that a fraction is a *number* that they have to use for extensive computations. Consequently, with such a "definition" of a fraction, the arithmetic operations on fractions cannot be defined and their computational algorithms cannot be proved.[8] Another example, the concept of

[6] Compare the discussion on pp. xxxix-xli.

[7] When I first got to sample a wide range of the available K-12 textbooks for the first time around year 2000, I was convinced that the publishers were in collusion and simply agreed to copy each other.

[8] Remember: "Ours is not to reason why. Just invert and multiply."

the slope of a line in the coordinate plane is defined in most of these textbooks by taking two pre-assigned points on the line to form the *rise-over-run.* But why is this rise-over-run equal to the rise-over-run with respect to *another* pair of points on the same line? Almost all textbooks insinuate that this equality is obviously true and not worth fussing about. And so on. In general, school mathematics, as defined collectively by these textbooks, is antithetical to mathematical integrity in that it *lacks clarity* (due to a general absence of definitions and a pervasive lack of precision in its articulation), mostly asks for *rote memorization* as its default mode of learning (due to the pervasive absence of reasoning), is *incoherent* (due to its neglect of the inherent logical structure of mathematics), and *traverses the curriculum in a listless and pro forma manner* (due to its failure to recognize the mathematical purpose behind each topic). We call the content of these standard school mathematics textbooks **TSM (Textbook School Mathematics)**.[9] TSM is recognized, consciously or subconsciously, by teachers and educators to be unlearnable, and it is this unlearnability that emboldens countless sensible adults to proclaim, often with pride, "I am not good in math!".

There is a far more pernicious fallout from TSM, however, and it is the effect TSM has on mathematics teachers and educators. These teachers and educators have learned only TSM in K-12, but as of 2020, institutions of higher learning do not provide courses to help future teachers and educators to replace their knowledge of TSM with school mathematics with mathematical integrity. Consequently, all that most teachers can do when they go back to teach in K-12 is trot out the TSM they are familiar with, and all that most educators can do when they begin their research is to fall back on the TSM they were taught. So the next generation also learns TSM, and this is the vicious cycle that has rendered *school mathematics* synonymous with TSM for at least the past five decades. Most educators may have suspected that there must be more to school mathematics than TSM, but without access to an exposition of school mathematics with mathematical integrity, their suspicion remains just that, a suspicion.

Back in 1985, Lee Shulman lamented in his well-known address to the AERA about "the absence of focus on subject matter among the various research paradigms for the study of teaching" ([Shulman], page 6). Shulman was talking about all disciplines, but from the standpoint of these six volumes ([Wu2011a], [Wu2016a], ..., [Wu2020c]), we gain a clear perspective on how this neglect of the *subject matter* may have come about

[9]For a more extended discussion of TSM, see **To the Instructor** on pp. xxvii ff. as well as [Wu2014] and [Wu2018a]. Note that TSM provides a new window into the phenomenon known as *math phobia*.

in mathematics. We speculate that mathematics educators may have chosen not to pay any attention to mathematical content because, since school mathematics was apparently nothing more than TSM, they saw nothing in the *subject matter* of school mathematics worthy of their serious attention. To change mathematics educators' perception of the subject matter in mathematics, we have to give them access to a fully detailed exposition of school mathematics with mathematical integrity.

The omnipresence of TSM in the last half-century created the unmistakable impression that perhaps at least *some* of the travesties in TSM are necessarily endemic to school mathematics. Under the circumstances, it was not easy to imagine that school mathematics might have anything to do with mathematical integrity. But two things happened around 1990. In 1989, NCTM (National Council of Teachers of Mathematics) launched its *school mathematics education reform* by proclaiming that that school mathematics could be made to respect mathematical integrity. Without a detailed exposition of school mathematics that respects mathematical integrity to back up its claim, NCTM was of course going out on a limb. Then in 1994, Alan Schoenfeld made a scholarly statement with the clear implication that, while a school mathematics curriculum with mathematical integrity was certainly possible, we did not have it yet. What he wrote was, "Proof is not a thing separable from mathematics, as it appears to be in our curricula ... And I believe it can be imbedded in our curricula, at all levels." ([Schoenfeld1994], page 76). Schoenfeld's statement was prompted in part by the debates surrounding the NCTM reform. Note that beyond affirming his belief in the fundamental article of faith underlying the NCTM reform, he stated openly that, indeed, this article of faith had not yet been confirmed. We will return to Schoenfeld's statement below, but before proceeding further, we will make a few comments about the NCTM reform.

The foundational documents of the NCTM reform are the two sets of standards: the 1989 [NCTM1989] and the 2000 [PSSM]. Although NCTM did not have an explicit recognition of the concept of TSM, the 1989 reform was undoubtedly a revolt against the stranglehold of TSM on school mathematics education. NCTM declared in essence that mathematical integrity must be part of school mathematics. For example, [NCTM1989] states that one of the reform's goals is that students "become mathematically literate" (page 6). [PSSM] states that "a mathematics curriculum should be coherent" (page 15), "should focus on important mathematics" (page 15), and "reasoning and proof should be a consistent part of students' mathematical experience in prekindergarten through grade 12" (page 56). With the hindsight of thirty years, we can see all too clearly the obstacles that confronted the NCTM reform. With students, teachers, and educators completely immersed in TSM, the clarion call for *coherence*, *reasoning* and *proof* might as well have been stated in a foreign language. Most of them had no conception of what those words meant.

We have to remember that, for example, what little "proof" TSM has to offer resides only in the course in high school geometry, and even there, proofs are mainly taught by rote (see [Schoenfeld1988]). Back in 1989, there was no detailed point-by-point exposition of school mathematics that could provide a roadmap to show how mathematical integrity can be introduced into school mathematics. There were no school mathematics textbooks to replace the TSM-infested ones.[10] Most fatally, NCTM made no commitment to a massive and long-term professional development program to explain to teachers what mathematical integrity in their daily teaching could look like.[11] With these three strikes against the NCTM reform before it stepped up to the plate, the reform faced an insurmountable credibility crisis. Its visionary declaration about what school mathematics education *could be and ought to be* almost instantly became nothing more than appealing rhetoric. The need for a *detailed* exposition of school mathematics with mathematical integrity could not have been more urgent.

There is another way that having a detailed exposition of school mathematics with mathematical integrity would have helped with the reform. Both [NCTM1989] and [PSSM] did try to provide some mathematical details about the curriculum they envisioned and, in so doing, made some missteps. For example, page 96 of [NCTM1989] suggests that the addition of fractions—inscrutable as it is in TSM—has to be approached gingerly, and neither [NCTM1989] nor [PSSM] points out the profound error of using the *least common denominator* for the addition of fractions (see page 53 below for an explanation of this error). Students' difficulty with the multiplication and division of fractions is duly noted in [NCTM1989] and [PSSM], but again there is no substantive suggestion on how to get them out of the predicament. Either document could have pointed to the need for a *proof* of the area formula for a rectangle with *fractional* sides; such a proof would add immeasurably to students' knowledge and confidence in *reasoning and proof* about fraction multiplication *and* about the concept of area (see pp. 64ff. below for such a

[10]In year 2020, most of us can calmly look back and see that the reform curricula that were published post-1989 were essentially different incarnations of TSM.

[11]The absence of this commitment was no accident. To carry out this kind of professional development on a large scale, the need for something like these six volumes ([Wu2011a], . . . , [Wu2020c]) to serve as a guide would be absolute.

proof). But the fact remains that neither did. One suggestion on the division of fractions is made on page 219 of [PSSM], but it confuses the *division* of fractions with the concept of *division-with-remainder* (see Section 7.2 of [Wu2011a] for a careful discussion of the latter). The difficulties of the concept of slope for teachers (and students) are notorious (see, e.g., page 126 of [Stump] or [Newton-Poon2]), but neither [NCTM1989] nor [PSSM] seems to recognize that the concept of slope *as it is known in TSM* is not properly defined and therefore a new approach is called for (see pp. 437-457 in this volume). And so on. These and many other missteps could have been avoided had a detailed exposition of school mathematics with mathematical integrity been available.

Some twenty years later, 2010 saw the release of **CCSSM**, the Common Core State Standards for Mathematics ([CCSSM]). CCSSM calls for a *focused* and *coherent* curriculum that stresses both *conceptual understanding* and *procedural fluency* (pp. 3-4, and 6 of [CCSSM]). In addition, it also asks for *precision* and *clear definitions* (page 7, loc. cit.). The most pronounced difference between CCSSM and the NCTM reform lies in the specificity of the standards in CCSSM: they are much more explicit in specifying the progression of mathematical topics through the grades and, even more importantly, in steering the curriculum away (most of the time) from the defective practices abounding in TSM. Because of the latter, most of the standards in CCSSM look different from the traditional standards (including those briefly sketched out in the NCTM documents [NCTM1989] and [PSSM]). This is especially true for the standards on fractions, finite decimals, rational numbers (along the lines of Chapters 1 and 2 in [Wu2016a], similar to Chapters 1 and 2 in this volume), part of beginning algebra (along the lines of Chapters 1 and 4 in [Wu2016b], similar to Chapter 6 of this volume), and middle and high school geometry (along the lines of Chapters 4 and 5 of [Wu2016a], similar to Chapters 4 and 5 in this volume).

However, in the apparent absence of a *detailed* account of what a CCSSM curriculum would look like,[12] the specificity of the curricular deviations in CCSSM turns out to be more of a political liability than an asset. Many people immediately put CCSSM and the NCTM reform on the same footing. Their perception was that these two movements

[12]The situation surrounding the release of [CCSSM] is complicated. A detailed exposition of the part of the CCSSM curriculum mentioned above—fractions, finite decimals, rational numbers, parts of algebra, and middle school geometry—in the form of [Wu2010a] and [Wu2010b] in fact predated CCSSM (they were drafts for [Wu2016a] and [Wu2016b], respectively). However, the existence of these documents was not made widely known.

represented what happens when a bunch of wannabes pontificate about school mathematics education without knowing what they are talking about. A conspicuous example they cited is CCSSM's approach to the geometry curriculum in middle school and high school using reflections, rotations, and translations as the basic building blocks. Such a change is necessitated by the inherently flawed TSM geometry curriculum based on an *uninformed* interpretation of the work of Euclid some twenty-three centuries ago (see pp. 205–214 below for a more detailed explanation, and see Chapter 8 of [Wu2020b] for a comprehensive one). Instead, CCSSM calls for a nuanced two-step process to introduce reflections, rotations, and translations as the foundational building blocks of the school geometry curriculum. Standards 8.G on page 55 of [CCSSM] describe how, in grade 8, these transformations can be used informally (but correctly) in heuristic arguments to develop students' intuition about transformations (as detailed in Chapter 4 and 5 of [Wu2016a]). Then in high school, these transformations are precisely defined to be used for formal proofs (as in Chapters 4 and 5 of this volume and Chapters 6 and 7 in [Wu2020b]). But not having such details available back in 2010, many critics, educators, and teachers immediately predicted the impending doom of this effort by CCSSM by citing the failures of putative similar experiments in other nations. Moreover, they also predicted (not entirely incorrectly) the almost certain confusion among teachers who would try to implement this new curriculum, and they regarded as inevitable the disappearance of proofs from CCSSM high school geometry. In the absence of a detailed exposition that shows how to navigate and implement the Common Core geometry standards with mathematical integrity,[13] such misunderstanding led inevitably to harsh criticism (see, e.g., [Milgram-Wurman], pp. 4-5, and [Phelps-Milgram], page 10 and footnote 15 on page 41). So CCSSM ends up facing the same wide credibility gap that plagued the NCTM reform twenty years ago.

It did not help that the CCSSM agenda also left out the critical component of professional development for teachers, thereby creating the same sense of bewilderment in classrooms across the land (see [Education Week], [Loewus1], [Loewus2], and [Sawchuk]). It would seem that CCSSM is repeating the same mistake as the NCTM reform by not taking seriously *the need* to offer sustained, large scale professional development for teachers to help with its implementation. With the publication of these three volumes, at least one complete exposition of school mathematics with mathematical integrity—an exposition

[13]Note, however, that many curricular details on geometry were soon provided in [Wu2012], [Wu2013], [Wu2016a], and [Wu2016b], but they were not made widely known. A detailed CCSSM-aligned high school geometry curriculum, in existence since 2007, will appear in the second volume of this three-volume set, [Wu2020b].

that is also consistent in the main with CCSSM—will be available to provide the needed guidance for this kind of professional development, but will these volumes be too little too late? Only time can tell.

The recent drive towards mathematical integrity

It should be abundantly clear from the foregoing discussion that any real improvement in school mathematics education requires us to rethink the mathematical education of teachers and educators. In particular, the destructive presence of TSM can no longer be ignored. These six volumes have been written with the express intent of encouraging and supporting such rethinking.

In the last few years, several books have made a concerted effort to promote the introduction of mathematical integrity into school mathematics, e.g., [MET2], [NCTM2009], [MUST] and the sixteen volumes in the NCTM series *Developing Essential Understanding* (e.g., [Ellis-Bieda-Knuth]). In a book entitled, *We Reason & We Prove for All Mathematics*, Arbaugh et al. respond directly to Schoenfeld's belief in the possibility of imbedding proofs in all levels of K-12 (quoted on page xii) by flatly stating that, in their volume, they "will provide guidance about how to make reasoning-and-proving a reality in your classroom" ([Arbaugh et al.], page x). These developments are welcome because their willingness to directly address the *content* of K-12 mathematics represents a giant step forward in school mathematics education at a time when many are still clinging to the idea that integrating fun, engaging activities into the classroom—while leaving TSM intact—is the way to improve school mathematics education. Nevertheless, we must also add a word of caution at this juncture of school mathematics education concerning the effectiveness of "providing guidance" in small doses, quite apart from the *quality* of the guidance itself.

As of 2020, we have to face the unpleasant truth that, because of the longstanding malfeasance of the education establishment, most in school mathematics education have been immersed in TSM, and only TSM, for their entire lives. Consequently, most end up being deficient in a detailed knowledge of the *inner workings* of mathematics on the one hand, and in a *coherent view* of mathematics as a whole on the other. An example of the former is the chronic failure to recognize that, without precise definitions, correct reasoning (= proof) is unattainable. Another example of the same is the fact that a proof must not be confused with a heuristic argument, no matter how attractive that heuristic argument may be.

Examples of the lack of a global, coherent view of mathematics abound in TSM, but we will limit our discussion to only three of them. The first is the lack of awareness of the overall hierarchical structure of mathematics, e.g., in order to move forward *mathematically* in a mathematical development, one may only use results already proved earlier. There is no better illustration of this lack than the "proof" in TSM of equivalent fractions using fraction multiplication[14]—one that is universally taught in TSM. Such a "proof" should be recognized for what it is: *totally anti-mathematical*. Here, the details are, step-by-step, impeccable, but the flagrant mathematical error lies in using a fact—about fraction multiplication that can only be proved *later* in the development of fractions—to justify a foundational result about fractions that is needed almost as soon as a fraction is defined. (See pp. 270-271 in [Wu2011a] for further discussions of this error).

A second example is the role of congruence in school mathematics. In TSM, the concept of the congruence of two arbitrary figures is not well-defined, and only the congruence of *triangles* is used for proofs in high school geometry (ASA, SAS, SSS, etc.). Moreover, congruence seems to have little relevance to daily life. TSM does not mention that, without the fundamental assumption that lengths, areas, and volumes remain the same for congruent geometric figures, it is impossible to derive any area or volume formulas (in particular, not even the area formulas for parallelograms and triangles). This realization makes it imperative that, in teaching the area formula for a triangle in grade 6 or 7 (for example), teachers make an effort to bring out the important role that the concept of congruence plays in geometric measurements (see Section 5.3 in [Wu2016b]). The same realization also impacts the geometry curriculum in high school: the TSM treatment of *congruence* as the "congruence of triangles" à la Euclid will have to be upgraded so that it can make sense of the "congruence of any two geometric figures" (see the Overview of Chapters 4 and 5 on pp. 205ff.). Such an upgrade is needed, for example, for the study of quadratic functions (see Section 2.1 of [Wu2020b]). This glaring defect in the TSM treatment of a foundational concept like *congruence* is in fact one of the main reasons necessitating the overhaul of the TSM geometry curriculum in middle and high school (see Chapters 4 and 5 of [Wu2016a], Chapters 4 and 5 in this volume, and Chapters 6 and 7 of [Wu2020b]). This kind of longitudinal coherence of the school mathematics curriculum, so vital for students' mathematics learning and on such a detailed level, is unlikely to be brought up in the context of providing general guidance piecemeal.

[14]This is the reasoning that $\frac{2}{3} = \frac{2\times 4}{3\times 4}$ because $\frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{4}{4} = \frac{2\times 4}{3\times 4}$.

A final example of the lack of a global, coherent view of mathematics in TSM is the transition in the middle school curriculum from rational numbers to *real numbers* due to the emergence of numbers such as $\sqrt{2}$, π, etc. TSM makes believe that the introduction of irrational numbers and their arithmetic operations into the school curriculum can be done surreptitiously and informally, *without any explicit mathematical discussion.* The resulting mathematical errors and their consequences for teachers and educators are profound. See, the example on page xxxii below, among many such examples. Also see the discussion of the incorrectness of the equality, $\sqrt{2}+\sqrt{3}=\sqrt{5}$, on pp. 207ff. of [MUST] which makes no mention of the fact that the arithmetic operations on irrational numbers are never given a serious and explicit discussion with ninth-graders. What is needed for the purpose of helping students make this transition is something like the FASM (Fundamental Assumption of School Mathematics) stated on page 174, but unhappily, nothing like FASM has ever appeared in TSM.

To address such deficiencies at both ends of the school mathematics spectrum, it would be reasonable to argue that a systematic exposure of teachers and mathematics educators to a *complete* exposition of the mathematics—one that honors mathematical integrity—over several grades at the very least[15] will be the only effective cure (see [NMAP1], Recommendation 19 on page xxi).

We have just given a partial explanation of why these six volumes (this volume, together with [Wu2011a], [Wu2016a], [Wu2016b], [Wu2020b], and [Wu2020c]) require 2500 pages of detailed *mathematical* discussions to confirm the fact that school mathematics can be made to respect mathematical integrity. Because of the corrosive effects of TSM that have pervaded and degraded school mathematics for so long, we are obliged to rebuild school mathematics from the ground up. In these six volumes, we take nothing for granted. For example, we pay special attention to the need for correct definitions as the basis for reasoning and proofs; we want to drive home the point that once a definition of a concept (such as a *fraction*) is given, then *every* subsequent assertion about this concept has to be based on the definition, and *on the definition alone.* Every statement in these volumes, from whole numbers to calculus, is carefully proved.[16] The intended goal of this effort is to clarify, cumulatively, the mathematical meaning of the declarative statement, "A implies B", as a purely deductive process that begins with the hypothesis A and arrives at the conclusion B. This is in contrast with the common practice in

[15]Teachers need to know where their students come from and where they are headed, curriculum-wise.

[16]With the usual disclaimer that there are a very few theorems that we must intentionally assume without proof.

TSM of "explaining" something by telling a story, by drawing an analogy, or by offering an attractive pattern or heuristic argument. These six volumes take an entirely different tack: they show, *consistently*, how to verify "A implies B" in mathematics by moving from A to B on the basis of definitions, explicit assumptions, or theorems with the help of logic. These volumes do so—we emphasize—from the first page to the last because we believe that the way to teach is not to pontificate but to lead by example. This process of acculturating teachers (and ultimately their students) to reasoning and proof does not have to be rigid or formal, especially in the early grades (see, e.g., Sections 4.2 or Section 6.2 of [Wu2011a]), but the essential elements of logical deduction must be put in place and maintained *ab initio* to preserve the integrity of mathematics. We also go into extensive detail about such seemingly pedestrian topics as the proper use of symbols (Sections 6.1 and 6.2 on pp. 385ff.), the meaning of an equation, and what it means to solve an equation (see pp. 416-419), with the hope that the long years of obfuscation in TSM with such jargon as "variables" and "symbolic manipulations for solving an equation" will be brought to a merciful end.

We hope that the foregoing discussion has made the case for the critical need for a thorough-going exposition of school mathematics with mathematical integrity. Incidentally, the only reason we have made repeated references in this whole discussion to the same six volumes by the present author is that there is no comparable exposition at the moment. It is in fact our hope that the publication of these six volumes will encourage others to come up with their own ways of replacing TSM across K-12 with a development of school mathematics that respects mathematical integrity.

How much mathematics teachers need to know

Knowing what school mathematics with mathematical integrity looks like enables us to face up to the second problem in mathematics education that was mentioned on page viii: how much mathematics a mathematics teacher or a mathematics educator needs to know. For teachers, this problem has a long history; see, e.g., [Ball], [Ball-McDiarmid], [Begle], [Goldhaber-Brewer], and [Monk]. We can speculate that, because the school curriculum has been dominated by TSM for so long and the flaws in TSM are so pronounced and extensive,[17] mathematics educators were reluctant to prescribe the content knowledge teachers need in terms of TSM on the one hand, and uncertain about *what* to prescribe on the other. After all, there was simply no available exposition of school mathematics

[17] Regardless of the fact that the term TSM was coined only in 2011.

with mathematical integrity. Now that these six volumes are available, it is possible to make a first attempt at describing the *minimum* knowledge that teachers and educators in elementary, middle, and high school, respectively, need to be effective in their work (again, see Recommendation 19 on p. xxi of [NMAP1]).

> Those in elementary school mathematics education[18] should know the equivalent of [Wu2011a] minus Chapters 23, 31, 37, 41, and 42; they should also have some acquaintance with the equivalent of Chapters 4 and 5 of [Wu2016a] and Chapters 1 and 2 of [Wu2016b].
>
> Those in middle school mathematics education should know the equivalent of [Wu2016a] and [Wu2016b], and have some acquaintance with the equivalent of Part 1 of [Wu2011a] and Chapters 4 and 5 of this volume.
>
> Those in high school mathematics education should know the equivalent of this volume, [Wu2020b], and [Wu2020c]. In addition, because pre-service teachers and educators interested in high school mathematics are typically math majors in college, they are expected to know something about linear algebra, i.e., vector spaces and matrices. Those who intend to teach calculus or do research on the teaching of calculus should also know something about Taylor's theorem and the Taylor series expansions of standard elementary functions such as sine, cosine, exponential function, and logarithm; they should also know some multi-variable calculus.

Now consider the teaching and learning of fractions and (finite) decimals. While education researchers of the past five decades were no doubt aware of the simple treatment of fractions in abstract algebra, their uncritical acceptance of TSM misled them into believing that, *for elementary school students,* one can do no better than teaching fractions as pieces of pizzas or some variation thereof. Consequently, they focussed their research on the teaching and learning of fractions, for the most part, on tweaking the TSM model of fractions-as-pizzas—with no thought given to helping students learn about fractions *as numbers* or learn to reason their way through the arithmetic of fractions.[19] As a

[18] We strongly believe that the mathematics of elementary school should be taught by *mathematics teachers*. See [Wu2009].

[19] Unhappily, TSM also claims some professional mathematicians among its victims: these mathematicians have come to believe that teaching fractions in schools can lead to nothing more than "confusion and memorization". See, for example, [DeTurck].

result, education research on fractions has focussed on increasing children's *experiential and informal* familiarity with fractions based on the pizza model rather than on increasing children's *mathematical* knowledge of fractions based on a correct definition of a fraction. If it had tried to do the latter, it would have rejected the absurd pizza model from the outset (see, e.g., pp. 33–35 of [Wu2008] for a brief discussion of the relevant literature). The same body of education research has also tried to make sense—unsuccessfully of course—of other anti-mathematical practices, such as treating decimals as a different kind of number, adding and subtracting fractions using the least common denominator, or teaching the multiplication and division of fractions without precise definitions. Only recently have researchers become aware of a more reasonable foundation for fractions (initiated in [Wu1998] and expanded in [Wu2011a]; abbreviated versions are given in Chapter 1 of [Wu2016a] and Chapter 1 of this volume)[20] that puts the study of fractions on the number line, emphasizes the concept of a fraction as *a number* for arithmetic computations, and makes sense of (finite) decimals as a special collection of fractions. There is still some distance to go in this direction, such as honoring the definition of a fraction by using it in *every* situation, e.g., for multiplication, for division, for understanding ratios, etc. We eagerly look forward to a change along these lines in the education research on fractions and decimals in the years to come (cf. [Siegler et al.]).

School textbooks

Better school mathematics education requires not only more effective teachers but also textbooks that contain only school mathematics with mathematical integrity. Our discussion thus far has been all about getting more effective teachers but nothing about getting better textbooks. This is not because we believe textbooks are less important, but since most school textbooks are published by the major publishers, there is little that people in academia can do to convince publishers to abandon their bottom-line mentality and write better textbooks (cf. [Keeghan]). However, there are now several online curricula written more or less in accordance with CCSSM and, according to some reports, a few seem to be showing promise.[21]

[20] This approach to fractions and decimals—as presented in [Wu2016a]—served as a blueprint for the fractions and decimals standards of [CCSSM]. Because this volume is for consumption by high school teachers and mathematics educators, what is in Chapter 1 is more brief and slightly more sophisticated than its counterparts in [Wu2011a] and [Wu2016a].

[21] It is uncertain whether any of the textbook evaluation agencies is aware of the importance of having mathematical integrity in mathematics textbooks.

As we said at the beginning, we hope these six volumes under discussion can serve as a blueprint for better school textbooks. But let us add a few caveats in this regard. First of all, these six volumes are certainly not student textbooks: they are written specifically for adults (teachers and educators, maybe some curious parents). Nevertheless, their mathematical content has been carefully *customized* (i.e., engineered) for use in the appropriate grades, at least as far as the *mathematical level of sophistication is concerned,* so that after some straightforward pedagogical modifications and embellishments, they can be expanded into student textbooks. An example of how such an expansion may be realized will appear before long, *we hope,* in the form of a student textbook for grade 8 that will be posted on the author's homepage, `https://math.berkeley.edu/~wu/`. At the very least, we believe these six volumes taken together can serve as a *detailed guide* for textbook publishers on how to write school mathematics textbooks across K-12 that respect both the standard curricular sequence and mathematical integrity. For this purpose, textbook writers should take note that there are several major departures from the standard school curriculum in this volume and [Wu2020b] and [Wu2020c]. Briefly, they are the following:

> (1) The conversion of fractions to infinite decimals and *geometric measurements* (length, area, and volume) are two topics typically taught in middle school, but in these volumes they appear in the third volume, [Wu2020c], after the introduction of limits (see Chapters 3-5 of [Wu2020c]). Fortunately, the procedural aspect of the conversion of fractions to decimals is addressed (*and* partially explained) in Section 1.5 (pp. 70ff.) of this volume, and there is an intuitive discussion of geometric measurements in Chapter 5 of [Wu2016a] which is actually adequate for (a somewhat superficial) use in a high school classroom.
>
> The main reason for these two departures is that it is impossible to make sense of infinite decimals and geometric measurements without the use of limits. Our teachers' and educators' critical need for some real understanding of the subtleties of both topics accounts for this departure from the norm. In any case, any adaptation of Chapters 3-5 of [Wu2020c] for student textbooks will require selective omissions.
>
> (2) The presentation of high school geometry in these volumes deviates from the traditional one. The concept of congruence is defined in

terms of the *tangible, accessible* concepts of reflections, rotations, and translations in the plane, and similarity is defined in terms of congruence and the equally *tangible and accessible* concept of dilation. A detailed explanation is given in the Overview of Chapters 4 and 5 on pp. 205ff. as well as in Section 4.7 on pp. 328ff. and Chapter 8 of the second volume, [Wu2020b]. Because CCSSM has since adopted this approach to middle and high school geometry, no defense of this deviation will be necessary here.

(3) These three volumes propose a different progression of geometry from middle school to high school, as follows. In grade 8, teach enough *informal* geometry to get to the concept of similar triangles, the angle-angle similarity criterion, and the proof of the Pythagorean theorem before embarking on introductory algebra in high school. Then in the high school geometry course, revisit the topic of similar triangles, but this time from a more formal standpoint. Again, see the Overview of Chapters 4 and 5 on pp. 205ff. for an explanation. (This departure from the standard sequencing has also been adopted by CCSSM.)

The presentation of the curricular shift described in (3) will be given in Chapters 4 and 5 of this volume, *but with a mild twist*. Because the *informal* geometry (proposed for grade 8) has already been treated in detail in [Wu2016a], the geometry in Chapters 4 and 5 of the present volume will be the *formal* high school counterpart of the informal geometry in [Wu2016a]. The exposition of the main body of plane geometry (geometry of the triangle and the circle along with constructions with ruler and compass) then resumes in Chapters 6 and 7 of the second volume, [Wu2020b], *after* we have finished discussing the standard topics of second-year algebra.

Final thoughts

We call special attention to the fact that the third and last of these three volumes, [Wu2020c], is essentially an introduction to mathematical analysis, customized specifically for consumption by prospective mathematics teachers and educators.[22] It is likely that this material will also benefit beginning math majors in college.

[22]Here as well as elsewhere in these three volumes, we are engaging in serious *mathematical engineering* in the sense of [Wu2006].

We should also address an obvious question that probably has been on readers' minds all along, namely, why does this volume on high school mathematics begin with the middle school topics of fractions and rational numbers? Nothing need be said about the obvious relevance of these topics to mathematics educators, but we owe high school teachers an explanation of why we consider these topics to be an integral part of *their* content knowledge. It is a fact—though hidden in TSM—that rational numbers, *rather than* real numbers, are the backbone of the mathematics in grades 5–12. Unfortunately, because of TSM, students in all grades seem to have trouble with fractions and, consequently, with rational numbers. Given the hierarchical structure of mathematics, it is not surprising that students' inability to learn algebra can often be traced back to their weakness in the foundational subjects of fractions and rational numbers. This was pointed out in the National Mathematics Advisory Panel Report (see page 18 of [NMAP1]). Indeed, the story has been told many times that even students in honors sections of Algebra 2 plead with their teachers to give them instructions on fractions. So, to be effective in teaching the standard topics of high school mathematics, high school teachers must have a TSM-free working knowledge of fractions and rational numbers as well.

A final reflection: Earlier, we quoted Lee Shulman's lament about "the absence of focus on subject matter among the various research paradigms for the study of teaching" (see page xi). These six volumes have now redefined the meaning of this *subject matter* for school mathematics. We hope mathematics educators will discover through these volumes that the mathematics underlying school mathematics, when presented correctly, is no longer meaningless like TSM and is worthy of their best efforts to learn it. Moreover, the *subject matter*, thus redefined, will have repercussions on "the study of teaching". As school mathematics becomes more learnable by all students, and therefore more teachable by all teachers, pedagogy will have to focus—not on how to render the incomprehensible[23] palatable—but on how to facilitate the *normal process* of learning so that all students can learn how to reason critically and correctly.

But for all that, it will be necessary to first make school mathematics that respects mathematical integrity an integral part of mathematics education research. This then harks back to Lee Shulman's lament. It is our belief and our hope that school mathematics education will improve when mathematics education research begins to address, not TSM, but *school mathematics with mathematical integrity.*

[23] That is, TSM.

Acknowledgements

The drafts of this volume and its companion volumes, [Wu2020b] and [Wu2020c], have been used since 2006 in the mathematics department at the University of California at Berkeley as textbooks for a three-semester sequence of courses, Math 151–153, that was created for pre-service high school teachers.[24] The two people most responsible for making these courses a reality were the two chairs of the Mathematics Department in those early years: Calvin Moore and Ted Slaman. I am immensely indebted to them for their support. I should not fail to mention that, at one point, Ted volunteered to teach an extra course for me in order to free me up for the writing of an early draft of these volumes. Would that all of us had chairs like him! Mark Richards, then Dean of Physical Sciences, was also behind these courses from the beginning. His support not only meant a lot to me personally, but I suspect that it also had something to do with the survival of these courses in a research-oriented department.

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Section 1.9 of the third volume ([Wu2020c]) on the importance of sine and cosine could not have been written without special help from Professors Thomas Kailath and

[24]Since the Fall of 2018, this three-semester sequence has been pared down to a two-semester sequence. A partial study of the effects of these courses on pre-service teachers can be found in [Newton-Poon1].

[25]Sadly, Dick passed away on October 9, 2019.

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Last but not least, I have to single out two names for my special expression of gratitude. Larry Francis has been my editor for many years, and he has pored over every single draft of these manuscripts with the same meticulous care from the first word to the last. I want to take this opportunity to thank him for the invaluable help he has consistently provided me. Ole Hald took it upon himself to teach the whole Math 151–153 sequence—without a break—several times to help me improve these volumes. That he did, in more ways than I can count. His numerous corrections and suggestions, big and small, all through the last nine years have led to *many* dramatic improvements. My indebtedness to him is too great to be expressed in words.

Hung-Hsi Wu
Berkeley, California
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To the Instructor

These three volumes (the other two being [Wu2020b] and [Wu2020c]) have been written expressly for high school mathematics teachers and mathematics educators.[26] Their goal is to revisit the high school mathematics curriculum, together with relevant topics from middle school, to help teachers better understand the mathematics they are or will be teaching and to help educators establish a sound mathematical platform on which to base their research. In terms of mathematical sophistication, these three volumes are designed for use in upper division courses for math majors in college. Since their content consists of topics in the upper end of school mathematics (including one-variable calculus), these volumes are in the unenviable position of straddling two disciplines: mathematics and education. Such being the case, these volumes will inevitably inspire misconceptions on both sides. We must therefore address their possible misuse in the hands of both mathematicians and educators. To this end, let us briefly review the state of school mathematics education as of 2020.

The phenomenon of TSM

For roughly the last five decades, the nation has had a *de facto* national school mathematics curriculum, one that has been defined by the standard school mathematics textbooks. The mathematics encoded in these textbooks is extremely flawed.[27] We call the body of knowledge encoded in these textbooks **TSM** (**Textbook School Mathematics**; see page xi). We will presently give a superficial survey of some of these flaws,[28] but what matters to us here is the fact that institutions of higher learning appear to be oblivious to the rampant mathematical mis-education of students in K–12 and have done very little to address the insidious presence of TSM in the mathematics taught to K-12 students

[26] We use the term "mathematics educators" to refer to university faculty in schools of education.

[27] These statements about curriculum and textbooks do not take into account how much the quality of school textbooks and teachers' content knowledge *may* have evolved recently with the advent of **CCSSM** (Common Core State Standards for Mathematics) ([CCSSM]) in 2010.

[28] Detailed criticisms and explicit corrections of these flaws are scattered throughout these volumes.

over the last 50 years. As a result, mathematics teachers are forced to carry out their teaching duties with all the misconceptions they acquired from TSM intact, and educators likewise continue to base their research on what they learned from TSM. So TSM lives on unchallenged.

These three volumes are the conclusion of a six-volume series[29] whose goal is to correct the universities' curricular oversight in the *mathematical education of teachers and educators* by providing the needed mathematical knowledge to break the vicious cycle of TSM. For this reason, these volumes pay special attention to mathematical integrity (as defined on page ix) and transparency, so that every concept is precisely defined and every assertion is completely explained,[30] and so that the exposition here is as close as possible to what is taught in a high school classroom.

TSM has appeared in different guises; after all, the NCTM reform (see pp. xii ff.) was largely ushered in around 1989. But beneath the surface its essential substance has stayed remarkably constant (compare [Wu2014]). TSM is characterized by a lack of clear definitions, faulty or non-existent reasoning, pervasive imprecision, general incoherence, and a consistent failure to make the case about why each standard topic in the school curriculum is worthy of study. Let us go through each of these issues in some detail.

(1) Definitions. In TSM, *correct* definitions of even the most basic concepts are usually not available. Here is a partial list:

> fraction, multiplication of fractions, division of fractions, one fraction being bigger or smaller than another, finite decimal, infinite decimal, mixed number, ratio, percent, rate, constant rate, negative number, the four arithmetic operations on rational numbers, congruence, similarity, length of a curve, area of a planar region, volume of a solid, expression, equation, graph of a function, graph of an inequality, half-plane, polygon, interior angle of a polygon, regular polygon, slope of a line, parabola, inverse function, etc.

Consequently, students are forced to work with concepts whose mathematical meaning is at best only partially revealed to them. Consider, for example, the concept of division. TSM offers no precise definition of division for whole numbers, fractions, rational numbers, real numbers, or complex numbers. If it did, the division concept would become much

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[29] The earlier volumes in the series are [Wu2011a], [Wu2016a], and [Wu2016b].

[30] In other words, every theorem is completely proved. Of course there are a few theorems that cannot be proved in context, such as the fundamental theorem of algebra.

more learnable because it is in fact the same for all these number systems (thus we also witness the incoherence of TSM). The lack of a definition for division leads inevitably to the impossibility of reasoning about the division of fractions, which then leads to "ours is not to reason why, just invert-and-multiply". We have here a prime example of the convergence of the lack of definitions, the lack of reasoning, and the lack of coherence.

The reason we need precise *definitions* is that they create a level playing field for all learners, in the sense that each person—including the teacher—has *all the needed information* about a given concept from the very beginning and this information is the same for everyone. This eliminates any need to spend time looking for "tricks", "insider knowledge", or hidden agendas. The level playing field makes every concept accessible to all learners, and this fact is what the discussion of *equity* in school mathematics education seems to have overlooked thus far. To put this statement in context, think of TSM's definition of a fraction as a piece of pizza: even elementary students can immediately see that there is more to a fraction than just being a piece of pizza. For example, "$\frac{5}{8}$ miles of dirt road" has nothing to do with pieces of a pizza. The credibility gap between what students are made to learn and what they subconsciously recognize to be false disrupts the learning process, often fatally.

In mathematics, there can be no valid reasoning without precise definitions. Consider, for example, TSM's proof of $(-2)(-3) = 2 \times 3$. Such a proof requires that we know what -2 is, what -3 is, what properties these negative integers are assumed to possess, and what it means to multiply (-2) by (-3) so that we can use them to justify this claim. Since TSM does not offer any information of this kind, it argues instead as follows: $3 \cdot (-3)$, being 3 copies of -3, is equal to -9, and likewise, $2 \cdot (-3) = -6$, $1 \cdot (-3) = -3$, and of course $0 \cdot (-3) = 0$. Now look at the pattern formed by these consecutive products:

$$3 \cdot (-3) = -9, \quad 2 \cdot (-3) = -6, \quad 1 \cdot (-3) = -3, \quad 0 \cdot (-3) = 0$$

Clearly when the first factor decreases by 1, the product *increases* by 3. Now, when the 0 in the product $0 \cdot (-3)$ decreases by 1 (so that 0 becomes -1), the product $(-1)(-3)$ ceases to make sense. Nevertheless, TSM urges students to believe that *the pattern must persist no matter what* so that this product will once more increase by 3 and therefore $(-1)(-3) = 3$. By the same token, when the -1 in $(-1)(-3)$ decreases by 1 again (so that -1 becomes -2), the product must again increase by 3 for the same reason and $(-2)(-3) = 6 = 2 \times 3$, as desired. This is what TSM considers to be "reasoning".

TSM goes further. Using a similar argument for $(-2)(-3) = 2 \times 3$, one can show that $(-a)(-b) = ab$ for all *whole numbers* a and b. Now, TSM asks students to take another big leap of faith: if $(-a)(-b) = ab$ is true for whole numbers a and b, then it must also be true when a and b are *arbitrary numbers*. This is how TSM "proves" that *negative times negative is positive*.

Slighting definitions in TSM can also take a different form: the graph of a linear inequality $ax + by \le c$ is claimed to be a half-plane of the line $ax + by = c$, and the "proof" usually consists of checking a few examples. Thus the points $(0, 0)$, $(-2, 0)$, and $(1, -1)$ are found to lie below the line defined by $x + 3y = 2$ and, since they all satisfy $x + 3y \le 2$, it is believable that the "lower half-plane" of the line $x + 3y = 2$ is the graph of $x + 3y \le 2$. Further experimentation with other points below the line defined by $x + 3y = 2$ adds to this conviction. Again, no reasoning is involved and, more importantly, neither "graph of an inequality" nor "half-plane" is defined in such a discussion because these terms sound so familiar that TSM apparently believes no definition is necessary. At other times, reasoning is simply suppressed, such as when the coordinates of the vertex of the graph of $ax^2 + bx + c$ are peremptorily declared to be

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right).$$

End of discussion.

Our emphasis on the importance of definitions in school mathematics compels us to address a misconception about the role of definitions in school mathematics education. To many teachers and educators, the word "definition" connotes something tedious and nonessential that students must memorize for standardized tests. It may also conjure an image of cut-and-dried, top-down instruction that begins with a rigid and unmotivated definition and continues with the definition's formal and equally unmotivated appearance in a chain of logical arguments. Understandably, most educators find this scenario unappetizing. Their response is that, at least in school mathematics, the definition of a concept should emerge at the end—but not at the beginning—of an extended *intuitive discussion* of the hows and whys of the concept.[31] In addition, the so-called conceptual understanding of the concept is believed to lie in the intuitive discussion but not in the *formal definition* itself, the latter being nothing more than an afterthought.

These two opposite conceptions of definition ignore the possibility of a middle ground: one can state the precise definition of a concept at the beginning of a lesson to set the tone of the subsequent mathematical discussion and exploration, which is to show students

[31]Proponents of this approach to definitions often seem to forget that, after the emergence of a precise definition, *students are still owed a systematic exposition of mathematics using the definition* so that they can learn about how the definition fits into the overall logical structure of mathematics.

that *this is all they will ever need to know about the concept as far as doing mathematics is concerned*. Such transparency—demanded by the mathematical culture of the past century (cf. [Quinn])—is what is most sorely missing in TSM, which consistently leaves students in doubt about what a *fraction* is or might be, what a *negative number* is, what *congruence* means, etc. In this middle ground, a definition can be explored and explained in intuitive terms in the ensuing discussion on the one hand, and on the other, *put to use in proofs—in its precise formulation—to show how and why the definition is absolutely indispensable to any kind of reasoning concerning the concept.* With the consistent use of precise definitions, the line between what is correct and what is intuitive but maybe incorrect (such as the TSM-proof of *negative times negative is positive*) becomes clearly drawn. It is the frequent blurring of this line in TSM that contributes massively to the general misapprehension in mathematics education about what a proof is (part of this misapprehension is described in, e.g., [NCTM2009], [Ellis-Bieda-Knuth], and [Arbaugh et al.]).

These three volumes (this volume, [Wu2020b], and [Wu2020c]) will always take a position in the aforementioned middle ground. Consider the definition of a fraction, for example: it is one of a special collection of points on the number line (page 13). This is the only meaning of a fraction that is needed to drive the fairly intricate *mathematical development* of fractions, and, for this reason, the definition of a fraction as a certain point on the number line is the one that will be unapologetically used all through these three volumes. To help teachers and students feel comfortable with this definition, we give an extensive intuitive discussion of why such a definition for a fraction is necessary on pp. 5-13. This intuitive discussion, naturally, opens the door to whatever pedagogical strategy a teacher wants to invest in it. Unlike in TSM, however, this definition is not given to be forgotten. On the contrary, *all* subsequent discussions about fractions will refer to this *precise definition* (but not to the intuitive discussion that preceded it) and, of course, all the proofs about fractions will also depend on this formal definition because mathematics demands no less. Students need to learn what a proof is and how it works; the exposition here tries to meet this need by (gently) laying bare the fact that reasoning in proofs requires precise definitions. As a second example, we give the definition of the slope of a line only after an extensive intuitive discussion on pp. 437-448 about *what* slope is supposed to measure and *how* we may hope to measure it. Again, the emphasis is on the fact that this definition of slope is not the conclusion, but the *beginning* of a long logical development that goes from page 448 to the end of Chapter 6 on page

495, and into trigonometry (relation with the tangent function), calculus (definition of the derivative), and beyond.

(2) Reasoning. Reasoning is the lifeblood of mathematics, and the main reason for learning mathematics is to learn how to reason. In the context of school mathematics, *reasoning* is important to students because it is the tool that empowers them to explore on their own and verify for themselves what is true and what is false without having to take other people's words on faith. *Reasoning* gives them confidence and independence. But when students have to accustom themselves to performing one unexplained rote skill after another, year after year, their ability to reason will naturally atrophy. Many students find it more expedient to stop asking *why* and simply take any order that comes their way sight unseen just to get by.[32] One can only speculate on the cumulative effect this kind of mathematics "learning" has on those students who go on to become teachers and mathematics educators.

(3) Precision. The purpose of *precision* is to eliminate errors and minimize misconceptions, but in TSM students learn at every turn that they should not believe exactly what they are told, but must learn to be creative in interpreting it. For example, TSM preaches the virtue of using *the theorem on equivalent fractions* to simplify fractions and does not hesitate to simplify a rational expression in x as follows:

$$\frac{(x-1)(x^2+3)}{x(x-1)} \;=\; \frac{x^2+3}{x}$$

This looks familiar because "canceling the same number from top and bottom" is exactly what the theorem on equivalent fractions is supposed to do. Unfortunately, this theorem only guarantees

$$\frac{c\,a}{b\,c} \;=\; \frac{a}{b}$$

when a, b, and c are *whole numbers* (b and c understood to be nonzero). In the previous rational expression, however, none of $(x-1)$, (x^2+3), and x is necessarily a whole number because x could be, for example, $\sqrt{5}$. Therefore, according to TSM, students in algebra should look back at equivalent fractions and realize that the theorem on equivalent fractions—*in spite of what it says*—can actually be applied to "fractions" whose "numerators" and "denominators" are not whole numbers. Thus TSM encourages students to believe that "nothing needs to be taken precisely and one must be flexible in interpreting what one learns". This extrapolation-happy mindset is the opposite of what it takes to learn a precise subject like mathematics or any of the exact sciences. For example, we

[32] There is consistent anecdotal evidence from teachers in the trenches that such is the case.

cannot allow students to believe that the domain of definition of $\log x$ is $[0, \infty)$ since $[0, \infty)$ is more or less the same as $(0, \infty)$. Indeed, the presence or absence of the single point "0" is the difference between true and false.

Another example of how a lack of precision leads to misconceptions is the statement that "$\beta^0 = 1$", where β is a nonzero number. Because TSM does not use precise language, it does not—or cannot—draw a sharp distinction between a *heuristic argument*, a *definition*, and a *proof*. Consequently, it has misled numerous students and teachers into believing that the heuristic argument for defining β^0 to be 1 is in fact a "proof" that $\beta^0 = 1$. The same misconception persists for negative exponents (e.g., $\beta^{-n} = 1/\beta^n$). The lack of precision is so pervasive in TSM that there is no end to such examples.

(4) Coherence. Another reason why TSM is less than learnable is its incoherence. Skills in TSM are framed as part of a long laundry list, and the lack of definitions for concepts ensures that skills and their underlying concepts remain forever disconnected. Mathematics, on the other hand, unfolds from a few central ideas, and concepts and skills are developed along the way to meet the needs that emerge in the process of unfolding. An acceptable exposition of mathematics therefore tells a coherent story that makes mathematics memorable. For example, consider the fact that TSM makes the four standard algorithms for whole numbers four separate rote-learning skills. Thus TSM hides from students the overriding theme that the Hindu-Arabic numeral system is universally adopted because it makes possible a simple, algorithmic procedure for computations, namely, if we can carry out an operation ($+$, $-$, $\times$, or $\div$) for *single-digit numbers,* then we can carry out this operation for all whole numbers no matter how many digits they have (see Chapter 3 of [Wu2011a]). The standard algorithms are the vehicles that bridge operations with *single-digit* numbers and operations on *all* whole numbers. Moreover, the standard algorithms can be simply explained by a straightforward application of the associative, commutative, and distributive laws. From this perspective, a teacher can explain to students, convincingly, why the multiplication table is very much worth learning; this would ease one of the main pedagogical bottlenecks in elementary school. Moreover, a teacher can also make sense of the associative, commutative, and distributive laws to elementary students and help them see that these are vital tools for doing mathematics rather than dinosaurs in an outdated school curriculum. If these facts had been widely known during the 1990's, the senseless debate on whether the standard algorithms should be taught might not have arisen and the Math Wars might not have taken place at all.

TSM also treats whole numbers, fractions, (finite) decimals, and rational numbers as four different kinds of numbers. The reality is that, first of all, decimals are a special class of fractions (see pp. 18ff.), whole numbers are part of fractions, and fractions are part of rational numbers. Moreover, the four arithmetic operations ($+$, $-$, $\times$, and $\div$) in each of these number systems do not essentially change from system to system. There is a smooth conceptual transition at each step of the passage from whole numbers to fractions and from fractions to rational numbers; see Parts 2 and 3 of [Wu2011a], or Sections 2.2, 2.4, and 2.5 in this volume. This coherence facilitates learning: instead of having to learn about four different kinds of numbers, students basically only need to learn about one number system (the rational numbers). Yet another example is the conceptual unity between linear functions and quadratic functions: in each case, the leading term—ax for linear functions and ax^2 for quadratic functions—determines the shape of the graph of the function completely, and the studies of the two kinds of functions become similar as each revolves around the shape of the graph (see Section 2.1 of [Wu2020b]). Mathematical coherence gives us many such storylines, and a few more will be detailed below.

(5) Purposefulness. In addition to the preceding four shortcomings—a lack of clear definitions, faulty or non-existent reasoning, pervasive imprecision, and general incoherence—TSM has a fifth fatal flaw: it lacks *purposefulness*. Purposefulness is what gives mathematics its vitality and focus: the fact is that a mathematical investigation, at any level, is always carried out with a specific goal in mind. When a mathematics textbook reflects this goal-oriented character of mathematics, it propels the mathematical narrative forward and facilitates its learning by making students aware of where the discussion is headed, and *why*. Too often, TSM lurches from topic to topic with no apparent purpose, leading students to wonder why they should bother to tag along. One example is the introduction of the absolute value of a number. Many teachers and students are mystified by being saddled with such a "frivolous" skill: "just kill the negative sign", as one teacher put it. Yet TSM never tries to demystify his concept. (For an explanation of the need to introduce *absolute value,* see, e.g., the discussion on pp. 170ff.). Another is the seemingly inexplicable replacement of the square root and cube root symbols of a positive number b, i.e., $\sqrt{b}$ and $\sqrt[3]{b}$, by rational exponents, $b^{1/2}$ and $b^{1/3}$, respectively (see, e.g., Section 4.2 of [Wu2020b]). Because TSM teaches the laws of exponents as merely "number facts", it is inevitable that it would fail to point out the *purpose* of this change of notation, which is to shift focus from the operation of taking roots to the properties of the *exponential function* b^x for a fixed positive b. A final example is the way TSM teaches estimation completely by rote, without ever telling students why and when estimation is important and therefore worth learning. Indeed, we often *have to* make estimates, either because precision is unattainable or unnecessary, or because we purposely use estimation as a tool to help achieve precision (see [Wu2011a], Section 10.3).

To summarize, if we want students to be taught mathematics that is learnable, then we must discard TSM and replace it with the kind of mathematics that possesses these five qualities:

Every concept has a clear definition.
Every statement is precise.
Every assertion is supported by reasoning.
Its development is coherent.
Its development is purposeful.

We have come across them before on page x: these are the *Fundamental Principles of Mathematics* (also see Section 2.1 in [Wu2018a]).

TSM consistently violates all five fundamental principles. Because of the dominance of TSM for at least the past half-century, most students come out of K-12 knowing only TSM but not mathematics that respects these fundamental principles. To them, learning mathematics is not about learning how to reason or distinguish true from false but about memorizing facts and tricks to get correct answers. Faced with this crisis, what should be the responsibility of institutions of higher learning? Should it be to create courses for future teachers and educators to help them *systematically* replace their knowledge of TSM with mathematics that is consistent with the five fundamental principles? Or should it be, rather, to leave TSM alone but make it more palatable by helping teachers infuse their classrooms with activities that *suggest* visions of reasoning, problem solving, and sense making? As of this writing, an overwhelming majority of the institutions of higher learning are choosing the latter alternative.

At this point, we return to the earlier question about some of the ways both university mathematicians and educators might misunderstand and misuse these three volumes.

Potential misuse by mathematicians

First, consider the case of mathematicians. They are likely to scoff at what they perceive to be the triviality of the content in these volumes: no groups, no homomorphisms, no compact sets, no holomorphic functions, and no Gaussian curvature. They may therefore be tempted to elevate the level of the presentation, for example, by introducing the concept of a *field* and show that, when two fractions symbols m/n and k/ℓ (with whole numbers m, n, k, ℓ, and $n \neq 0$, $\ell \neq 0$) satisfying $m\ell = nk$ are identified, and when $+$ and $\times$ are defined by the usual formulas, the fraction symbols form a field. In this elegant manner, they can efficiently cover all the standard facts in the arithmetic of fractions in the school curriculum.[33] This is certainly a better way than *defining fractions as points on the number line* to teach teachers and educators about fractions, is it not? Likewise, mathematicians may find finite geometry to be a more exciting introduction to axiomatic systems than any proposed improvements on the high school geometry course in TSM. The list goes on. Consequently, pre-service teachers and educators may end up learning from mathematicians some interesting mathematics, but not mathematics that would help them overcome the handicap of knowing only TSM.

Mathematicians may also engage in another popular approach to the professional development of teachers and educators: teaching the solution of hard problems. Because mathematicians tend to take their own mastery of fundamental skills and concepts for granted, many do not realize that it is nearly impossible for teachers who have been immersed in thirteen years or more of TSM to acquire, *on their own*, a mastery of a *mathematically correct* version of the basic skills and concepts. Mathematicians are therefore likely to consider their major goal in the professional development of teachers and educators to be teaching them how to solve *hard problems*. Surely, so the belief goes, if teachers can handle the "hard stuff", they will be able to handle the "easy stuff" in K-12. Since this belief is entirely in line with one of the current slogans in school mathematics education about the critical importance of *problem solving*, many teachers may be all too eager to teach their students the extracurricular skills of solving challenging problems *in addition to* teaching them TSM day in and day out. In any case, the relatively unglamorous content of these three volumes (this volume, [Wu2020b], and [Wu2020c])—designed to replace TSM—will get shunted aside into supplementary reading assignments.

At the risk of belaboring the point, the focus of these three volumes is on showing how to *replace teachers' and educators' knowledge of TSM in grades 9-12* with mathematics that respects the fundamental principles of mathematics. Therefore, reformulating the mathematics of 9-12 from an advanced mathematical standpoint to obtain a more elegant presentation is not the point. Introducing novel elementary topics (such as Pick's theorem or the 4-point affine plane) into the mathematics education of teachers and educators is

[33] This is my paraphrase of a mathematician's account of his professional development institute around year 2000.

also not the point. Rather, the point in year 2020 is to do the essential spadework of revisiting the standard 9-12 curriculum—topic by topic, along the lines laid out in these three volumes—showing teachers and educators how the TSM in each case can be supplanted by mathematics that makes sense to them and to their students. For example, since most pre-service teachers and educators have not been exposed to the use of precise definitions in mathematics, they are unlikely to know that definitions are supposed to be used, *exactly as written, no more and no less*, in logical arguments. One of the most formidable tasks confronting mathematicians is, in fact, how to change educators' and teachers' perception of the role of definitions in reasoning.

As illustration, consider how TSM handles slope. There are two ways, but we will mention only one of them.[34] TSM pretends that, by defining the slope of a line L using the difference quotient with respect to two *pre-chosen* points P and Q on L,[35] such a difference quotient is a property *of the line itself* (rather than a property of the two points P and Q). In addition, TSM pretends that it can use "reasoning" based on this defective definition to derive the equation of a line when (for example) its slope and a given point on it are prescribed. Here is the inherent danger of thirteen years of continuous exposure to this kind of pseudo-reasoning: teachers cease to recognize that (a) such a definition of slope is defective and (b) such a defective definition of slope cannot possibly support the purported derivation (= proof) of the equation of a line. It therefore comes to pass that—as a result of the flaws in our education system—many teachers and educators end up being confused about even the meaning of the simplest kind of reasoning: "A implies B". They need—and deserve—all the help we can give so that they can finally experience *genuine mathematics*, i.e., mathematics that is based on the fundamental principles of mathematics.

Of course, the ultimate goal is for teachers to *use* this new knowledge to teach their own students so that those students can achieve a true understanding of what "A implies B" means and what real *reasoning* is all about. With this in mind, we introduce in Section 6.4 (pp. 436ff.) the concept of slope by discussing what slope is supposed to measure (an example of *purposefulness*) and how to measure it, which then leads to the formulation of a precise definition. With the availability of the AA-criterion for triangle similarity (Theorem G22 on page 372), we then show how this definition leads to the formula for

[34] A second way is to *define* a line to be the graph of a linear equation $y = mx + b$ and then *define* the slope of this line to be m. This is the definition of a line in advanced mathematics, but it is *so profoundly inappropriate* for use in K-12 that we will just ignore it.

[35] This is the "rise-over-run".

the slope of a line as the difference quotient of the coordinates of *any* two points on the line (the "rise-over-run"). Having this critical flexibility to compute the slope—plus an earlier elucidation of what an *equation* is (pp. 416–419)—we easily obtain the equation of a line passing through a given point with a given slope, *with correct reasoning* this time around (see pp. 462ff.). Of course the same kind of reasoning can be applied to similar problems when other reasonable geometric data are prescribed for the line.

By guiding teachers and educators systematically through the *correction of TSM errors on a case-by-case basis,* we believe they will gain a new and deeper understanding of school mathematics. Ultimately, we hope that, if we institutions of higher learning and the education establishment can persevere in committing themselves to this painstaking work, the students of these teachers and educators will be spared the ravages of TSM. If there is an easier way to undo thirteen years and more of mis-education in mathematics, we are not aware of it.

A main emphasis in using these three volumes should therefore be on providing patient guidance to teachers and educators to help them overcome the many handicaps inflicted on them by TSM. In this light, we can say with confidence that, for now, the best way for mathematicians to help educate teachers and educators is to firm up their mathematical foundations. Let us repair the damage TSM has done to their mathematics content knowledge by helping them to acquire a knowledge of school mathematics that is consistent with the fundamental principles of mathematics.

Potential misuse by educators

Next, we address the issue of how educators may misuse these three volumes. Educators may very well frown on the volumes' insistence on precise definitions and precise reasoning and their unremitting emphasis on proofs while, apparently, neglecting *problem solving*, *conceptual understanding*, and *sense making*. To them, good professional development concentrates on all of these issues plus contextual learning, student thinking, and communication with students. Because these three volumes never explicitly mention problem solving, conceptual understanding, or sense making *per se* (or, for that matter, contextual learning or student thinking), their content may be dismissed by educators as merely skills-oriented or *technical knowledge for its own sake* and, as such, get relegated to reading assignments outside of class. They may believe that precious class time can be put to better use by calling on students to share their solutions to difficult problems or by holding small group discussions about problem solving strategies.

We believe this attitude is also misguided because the critical missing piece in the contemporary mathematical education of teachers and educators is an exposure to a systematic exposition of the standard topics of the school curriculum that respects the fundamental principles of mathematics. Teachers' lack of access to such a mathematical exposition is what lies at the heart of much of the current education crisis. Let us explain.

Consider *problem solving.* At the moment, the goal of getting all students to be proficient in solving problems is being pursued with missionary zeal, but what seems to be missing in this single-minded pursuit is the recognition that the body of knowledge we call mathematics consists of nothing more than a sequence of problems posed, and then solved, by making logical deductions on the basis of precise definitions, clearly stated hypotheses, and known results.[36] This is after all the whole point of the classic two-volume work, [Pólya-Szegö], which introduces students to mathematical research through the solutions to a long list of problems. For example, the Pythagorean theorem and its many proofs are nothing more than solutions to the problem posed by people from diverse cultures long ago: "Is there any relationship among the three sides of a right triangle?" *There is no essential difference between problem solving and theorem proving in mathematics.* Each time we solve a problem, we in effect prove a theorem (trivial as that theorem may sometimes be).

The main point of this observation is that if we want students to be proficient in problem solving, then we must give them plenty of examples of grade-appropriate proofs all through (at least) grades 4-12 and engage them regularly in grade-appropriate theorem-proving activities. If we can get students to see, day in and day out, that problem solving is a way of life in mathematics, and if we also *routinely get them involved in problem solving* (i.e., *theorem-proving*), students will learn problem solving naturally through such a long-term immersion. In the process, they will get to experience that, to solve problems, they need to have precise definitions and precise hypotheses as a starting point, know the direction they are headed before they make a move (sense making), and be able to make deductions from precise definitions and known facts. Definitions, sense making, and reasoning will therefore come together *naturally* for students if they learn mathematics that is consistent with the five fundamental principles.

[36]It is in this light that the previous remark about the purposefulness of mathematics can be better understood: before solving a problem, one should know *why* the problem was posed in the first place. Note that, for beginners (i.e., school students), the overwhelming emphasis has to be on *solving* problems rather than the more elusive issue of posing problems.

We make the effort to put problem solving in the context of the fundamental principles of mathematics because there is a danger in pursuing problem solving *per se* in the midst of the TSM-induced corruption of school mathematics. In a generic situation, teachers teach TSM and only pay lip service to "problem solving", while in the best case scenario, teachers *keep TSM intact* while teaching students how to solve problems on a separate, parallel track *outside of TSM*. Lest we forget, TSM considers "out of a hundred" to be a correct definition of *percent*, expands the product of two linear polynomials by "FOILing", and assumes that in any problem about *rate*, one can automatically assume that the rate is constant ("Lynnette can wash 95 cars in 5 days. How many cars can Lynnette wash in 11 days?"), etc. In this environment, it is futile to talk about (*correct*) problem solving. Until we can rid school classrooms of TSM, the most we can hope for is having teachers teach, on the one hand, definition-free concepts with a bag of tricks-sans-reasoning to get correct answers and, on the other hand, reasoning skills for solving a separate collection of problems for special occasions. In other words, two parallel universes will co-exist in school mathematics classrooms. So long as TSM continues to reign in school classrooms, most students will only be comfortable doing one-step problems and any problem solving ability they possess will only be something that is artificially grafted onto the TSM they know.

If we want to avert this kind of **bipolar mathematics education** in schools, we must begin by providing teachers with a better mathematical education. Then we can hope that teachers will teach mathematics consistent with the fundamental principles of mathematics[37] so that students' problem solving abilities can evolve naturally from the mathematics they learn. It is partly for this reason that the six volumes under discussion[38] choose to present the mathematics of K-12 with explanations (= proofs) for all the skills. In particular, these three volumes on the mathematics of grades 9-12 provide proofs for every theorem. (At the same time, they also caution against certain proofs that are simply too long or too tedious to be presented in a high school classroom.) The hope is that when teachers and educators get to experience firsthand that every part of school mathematics is suffused with reasoning, they will not fail to teach reasoning to their own students as a matter of routine. Only then will it make sense to consider problem solving

[37]And, of course, to also get school textbooks that are unsullied by TSM. However, it seems likely as of 2020 that major publishers will hold onto TSM until there are sufficiently large numbers of knowledgeable teachers who demand better textbooks. See the end of [Wu2015].

[38]These three volumes, together with [Wu2011a], [Wu2016a], and [Wu2016b].

to be an integral part of school mathematics.

The importance of correct content knowledge

In general, the idea is that if we give teachers and educators an exposition of mathematics that *makes sense* and has built-in *conceptual understanding* and *reasoning,* then we can hope to create classrooms with an intellectual climate that enables students to absorb these qualities as if by osmosis. Perhaps an analogy can further clarify this issue: if we want to teach writing, it would be more effective to let students read good writing and learn from it directly rather than to let them read bad writing and simultaneously attend special sessions on the fine points of effective written communication.

If we want school mathematics to be suffused with reasoning, conceptual understanding, and sense making, then we must recognize that these are not qualities that can stand apart from mathematical details. Rather, they are firmly anchored to hard-and-fast mathematical facts. Take proofs (= reasoning), for example. If we only talk about proofs in the context of TSM, then our conception of what a proof is will be extremely flawed because there are essentially no correct proofs in TSM. For starters, since TSM has no precise definitions, there can be no hope of finding a completely correct proof in TSM. Therefore, when teaching from these three volumes,[39] it is imperative to first concentrate on getting across to teachers and educators the *details* of the mathematical reformulation of the school curriculum. Specifically, we stress the importance of offering educators a *valid* alternative to TSM for their future research. Only then can we hope to witness a reconceptualization—*in mathematics education*—of reasoning, conceptual understanding, problem solving, etc., on the basis of a solid mathematical foundation.

Reasoning, conceptual understanding, and *sense making* are qualities intrinsic to school mathematics that respects the fundamental principles of mathematics. We see in these three volumes a continuous narrative from topic to topic and from chapter to chapter to guide the reader through this long journey. The sense making will be self-evident to the reader. Moreover, when every assertion is backed up by an explanation (= proof), reasoning will rise to the surface for all to see. In their presentation of the natural unfolding of mathematical ideas, these volumes also routinely point out connections between definitions, concepts, theorems, and proofs. Some connections may not be immediately apparent. For example, in Section 6.1 of this volume (page 400), we explicitly point out the connection between Mersenne primes and the summation of finite geometric series.

[39] As well as from the other three volumes, [Wu2011a], [Wu2016a], and [Wu2016b])

Other connections span several grades: there is a striking similarity between the *proofs* of the area formula for rectangles whose sides are fractions (Theorem 1.7 on pp. 62ff.), the ASA congruence criterion (Theorem G9 on pp. 319ff.), the SSS congruence criterion (Theorem G28 in Section 6.2 of [Wu2020b]), the fundamental theorem of similarity (Theorem G10 in Section 6.4 of [Wu2020b]), and the theorem about the equality of angles on a circle subtending the same arc (Theorem G52 in Section 6.8 of [Wu2020b]). All these proofs are achieved by breaking up a complicated argument into two or more clear-cut steps, each involving simpler arguments. In other words, they demonstrate how to reduce the complex to the simple, so prospective teachers and educators can learn from such instructive examples about the fine art of problem solving (= reasoning).

The foregoing unrelenting emphasis on mathematical content should not lead readers to believe that these three volumes deal with mathematics at the expense of pedagogy. To the extent that these volumes are designed to promote better teaching in the schools, they do not sidestep pedagogical issues. Extensive pedagogical comments are offered whenever they are called for, and they are clearly displayed as such; see, for example, pp. 21, 29, 48, 52, 68, 156, 338, 341, etc. Nevertheless, our most urgent task—*the* fundamental task—in the mathematical education of teachers and educators as of 2020 has to be the *reconstruction of their mathematical knowledge base*. This is not about judiciously *tinkering* with what teachers and educators already know or *tweaking* their existing knowledge here and there. Rather, it is about the hard work of *replacing* their knowledge of TSM with mathematics that is consistent with the fundamental principles of mathematics *from the ground up.* The primary goal of these three volumes is to give a detailed exposition of school mathematics in grades 9–12 to help educators and teachers achieve this reconstruction.

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