# Teaching Geometry According to the Common Core Standards

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January 1, 2012.
**Third revision: October 10, 2013.**

## Contents

# Preface

*Le juge:* Accusé, vous tâcherez d'être bref.
*L'accusé:* Je tâcherai d'être clair.
—G. Courteline[1]

This document is a collection of grade-by-grade mathematical commentaries on the teaching of the geometry standards in the **CCSSM** (Common Core State Standards for Mathematics) from grade 4 to high school. The emphasis is on the *progression of the mathematical ideas* through the grades. It complements the usual writings and discussions on the CCSSM which emphasize the latter's Practice Standards. It is hoped that this document will promote a better understanding of the Practice Standards by giving them *mathematical* substance rather than adding to the *verbal* descriptions of what mathematics is about. Seeing (correct) mathematics in action is a far better way of coming to grips with these Practice Standards but, unfortunately, in an era of **Textbook School Mathematics (TSM)**,[2] one does not get to see mathematics in action too often. Mathematicians should have done much more to reveal the true nature of mathematics, but they didn't, and school mathematics education is the worse for it. Let us hope that, with the advent of the CCSSM, more of such efforts will be forthcoming.

The geometry standards in the CCSSM deviate from the usual geometry standards in at least two respects, one big and one small. The small one is that, for the first time, special attention is paid to the need of a proof for the area formula for rectangles when the side lengths are *fractions*. This is standard NF 4b in grade 5:

> Find the area of a rectangle with fractional side lengths by tiling it with [rectangles] of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

[1] Quoted in the classic, *Commutative Algebra*, of Zariski-Samuel. Literal translation: The judge: "The defendant will try to be brief." The defendant replies, "I will try to be clear."

[2] A turquoise box around a phrase or a sentence (such as Textbook School Mathematics ) indicates an active link to an article online.

The lack of explanation for the rectangle area formula when the side lengths are fractions is symptomatic of what has gone wrong in school mathematics education, or more precisely, in TSM. Often basic facts are not clearly explained, or if explained, it is done incorrectly. Because the explanation in this case requires a full understanding of fraction multiplication and the basic ingredients of the concept of *area* (see (a)–(d) on page 21 ), and because the reasoning is far from routine and yet very accessible to fifth graders, this explanation is potentially a high point in students' encounter with geometric measurements (length, area, volume) in K–12. Let us make sure that it is so this time around.

The major deviation of the CCSSM from the usual geometry standards occurs in grade 8 and high school. There is at present an almost total disconnect in TSM between the geometry of middle school and that of high school. Congruence and similarity are (vaguely!) defined in middle school as "same-size-and-same-shape" and "same-shape-but-not-necessarily-same-size", respectively, while middle-school geometric transformations (rotations, reflections, and translations) are taught seemingly only for the purpose of art appreciation, such as appreciating the internal symmetries of Escher's famous designs and medieval Islamic art without any reference to their mathematical relevance.

In the high school geometry of TSM, congruence and similarity are defined anew (e.g., ASA, SSS, etc.) but *only* for polygons, thereby creating the impression that mathematical precision can be achieved only for polygons. As for geometric transformations, well, they are relegated to the end of the year as an enhancement of the concepts of congruence, *if time allows*.

It goes without saying that such a disconnect is not acceptable mathematics education. In contrast, there is a seamless transition from the geometry of grade 8 to high school geometry in the CCSSM. The concepts of rotation, reflection, translation, and dilation taught in grade 8—basically on an intuitive level—become the foundation for the mathematical development of the high school geometry course. In the process, students get to see, perhaps for the first time, the *mathematical* significance of rotation, reflection, translation, and dilation as well as the precise meaning of congruence and similarity. Thus the latter are no longer seen to be some abstract and shadowy concepts but are, rather, concepts open to tactile investigations. In addition, it is only through the precise definition of congruence as a composition of rotations, reflections, and translations that students can begin to make sense of what is known in TSM as "CPCTC" (*corresponding parts of congruent triangles are congruent*).

Many have expressed reservations about the CCSSM geometry standards. Because rotation, reflection, translation, and dilation are now used for a serious mathematical purpose, there is a perception that so-called "transformational geometry" (whatever that means) rules the CCSSM geometry curriculum. "Transformational geometry" is perceived to be something quaint and faddish—not to say incomprehensible to school students.

The truth is that the school geometry curriculum in TSM has been dysfunctional for far too long and the needed restructuring is way overdue. The new course charted by the CCSSM will be seen to be not only mathematically defensible but also a conservative one, in the sense that it does not inject any new topics into the standard curriculum. Its main innovation lies in nothing more than exhibiting *new connections among the existing topics to clarify their mathematical relationships.* Let it be noted explicitly that

***the CCSSM do not pursue transformational geometry per se.***

Geometric transformations are merely a means to an end: they are used in a strictly utilitarian way to streamline and shed light on the existing school geometry curriculum. For example, once reflections, rotations, reflections, and dilations have contributed to the proofs of the standard triangle congruence and similarity criteria (SAS, SSS, etc.), the development of plane geometry can proceed in the usual way if one so desires.

At the moment, the introductory portion of such a development of geometry can be found, in greater detail than is given in this article, in Chapters 4–7 of H. Wu, Pre-Algebra. In fact, this approach to school geometry has been taught to teachers in three-week professional development institutes since 2004, and the document H. Wu, Pre-Algebra is nothing but a set of lecture notes from these institutes. This set of notes is referenced on page 92 of the CCSSM, but under a different title:

Wu, H., "Lecture Notes for the 2009 Pre-Algebra Institute,"
September 15, 2009.

A completely revised version of these notes will appear as part a textbook for middle school mathematics teachers:

H. Wu, *From Pre-Algebra to Algebra*, to appear in 2014.

Furthermore, a complete account of this treatment of school geometry will appear (probably in 2015) as part of a two-volume textbook for high school teachers:

H. Wu, *Mathematics of the Secondary School Curriculum, I and II.*

In the meantime, please see page 150 and page 154 for further comments on this issue.

The logical relationship between reflections, rotations, etc., on the one hand and congruence and similarity on the other in plane geometry, while completely routine to working geometers, is mostly unknown to teachers and mathematics educators because mathematicians have been negligent in sharing their knowledge. A successful implementation of the CCSSM therefore requires a massive national effort to teach mathematics to inservice and preservice teachers. To the extent that such an effort does not seem to be forthcoming as of October 2013, I am posting this document online in order to make a reasonably detailed account of this knowledge freely available.

If this mathematical restructuring of school geometry by the CCSSM can be backed up by a strong commitment to the professional development of teachers, there is no reason to doubt that greater student achievement would follow.

The reader will note that the present commentaries avoid the pitfall of most existing materials that treat school geometry as an exercise in learning new vocabulary or memorizing new formulas. The main focus will be on mathematical ideas with detailed reasoning given whenever it is feasible. In essence, this article gives substance to the Practice Standards in the CCSSM (recall the comments on the Practice Standards at the beginning of this Preface). In the case of the transition from middle school to high school geometry, the commentaries on the relevant geometry standards are uncommonly detailed for exactly the reason that there seems to be no such account in the education literature. Special attention is given to showing how,

> ***once some obvious properties of these transformations are assumed, one can give precise proofs of all the standard theorems in plane geometry.***

Because even that may not be enough to inform teachers and publishers, other references will be given in due course for further study (see page 64 and page 114).

**Acknowledgements.** I owe Angelo Segalla and Clinton Rempel an immense debt for their invaluable help in the preparation of this article. Over a period of nine months, they not only gave me mathematical and linguistic feedback about the exposition, but also told me how to adjust my writing to the realities of the school classroom. This would have been a lesser article without their intervention.

This revision was made possible only through the generosity of my friends: David Collins, Larry Francis, and Sunil Koswatta in addition to Angelo and Clinton. They pointed out many flaws of the original from different angles, and their critical comments spurred me to rewrite several passages. The resulting improvements should be obvious to one and all. It gives me pleasure to thank them warmly for their contributions. I also want to thank Larry and Sunil for creating animations at my request to go with this article; they can be found on:

pp. 77, 82, 102, and 144. (Larry)

pp. 68, 70, and 73. (Sunil)

*The Common Core geometry standards for each grade are recalled at the beginning of that grade in* sans serif *font.*

## GRADE 4

**Geometric measurement: understand concepts of angle and measure angles.**

5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:

a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a one-degree angle, and can be used to measure angles.

b. An angle that turns through n one-degree angles is said to have an angle measure of n degrees.

6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

7. Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

### **Geometry** 4.G

**Draw and identify lines and angles, and classify shapes by properties of their lines and angles.**

1. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

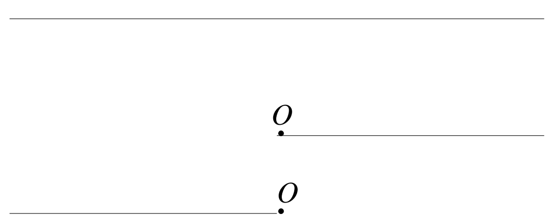
2. Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

3. Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

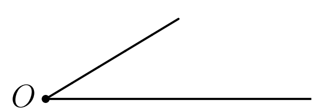
## Comments on teaching grade 4 geometry

The main topics of Grade 4 geometry are angles and their measurements, and the phenomena of perpendicularity and parallelism.

We know that a line goes on forever in two directions (first of the three figures below). When a point $O$ is chosen on a line, it creates two **rays**: one ray goes on forever from $O$ to the right, and the other goes on forever from $O$ to the left, as shown. Thus each ray goes on forever only in one direction. In each ray, the point $O$ is called the **vertex** of the ray.




An **angle** is the figure formed by two rays with a common vertex, as shown.




We are mainly interested in angles where the two rays are distinct. The case where the two rays coincide is called the **zero angle**. The case where the two rays together form a straight line is called a **straight angle**, as shown.

For an angle that is neither the a zero angle nor a straight angle, there is a question of *which part* of the angle we want to measure. Take an angle whose **sides** are the rays $OA$ and $OB$, as shown below. Then $\angle AOB$ could be either one of two parts, as indicated by the respective arcs below.

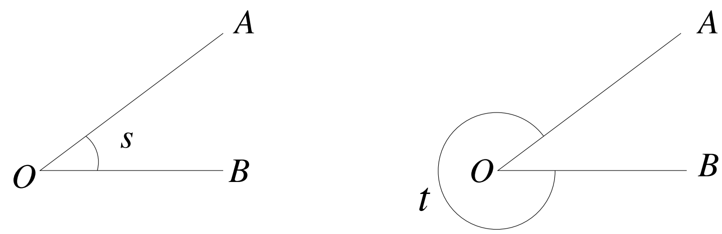
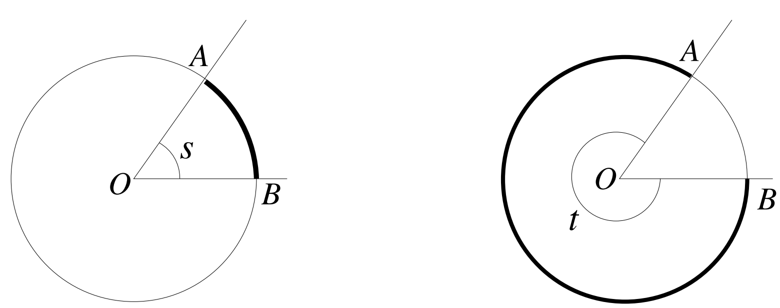



Figure 1

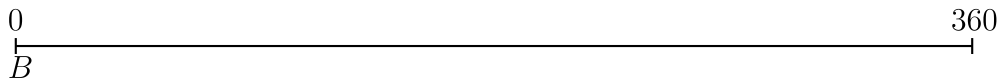
We will use the notation $\angle s$ to denote the part of the angle $\angle AOB$ indicated on the left, and $\angle t$ to denote the part of the angle $\angle AOB$ indicated on the right. If nothing is said, then $\angle AOB$ will be understood to mean $\angle s$.

Just as we measure the length of a line segment in order to be able to say which is longer, we want to also measure the "size" of an angle so that we can say which is "bigger". In the case of length, recall that we have to begin by agreeing on a *unit* (inch, cm, ft, etc.) so that we can say a segment has length 1 (respectively, inch, cm, ft, etc.), we likewise must agree on a unit of measurement for the "size" of an angle. A common unit is *degree*, and we explain what it is as follows.

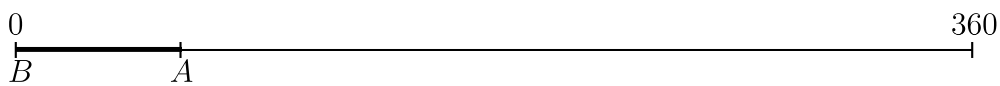
Given $\angle AOB$ as above, we draw a circle with the vertex $O$ as center. Then the sides of $\angle AOB$ intercept an **arc** on the circle: $\angle s$ intercepts the thickened arc on the left, and $\angle t$ intercepts the thickened arc on the right.



The length of this arc *when the length of the whole circle is taken to be 360* is what is meant by the **degree** of the angle. Let us explain this in greater detail. Think of the circle as a (very thin piece of round) steel wire. We may as well assume that the points $A$ and $B$ of the angle are points on the circle. Then imagine we cut the wire open at $B$ and we stretch it out as a line segment with $B$ as the left endpoint of the segment. Now divide the segment into 360 **equal parts** (i.e., parts of equal length) and let the length of one part to be the unit 1 on this number line (the unit is too small to be drawn below). Then this unit is called a **degree**. Relative to the degree, the length of the whole segment (i.e., wire) is 360 degrees.

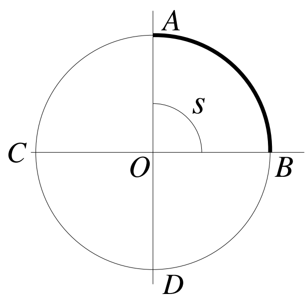


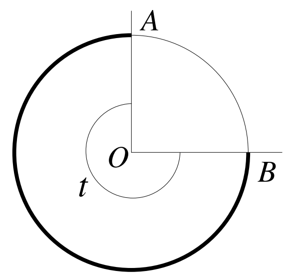

Recall that $A$ is now a point on the segment. If we consider $\angle s$, let the arc $\widehat{AB}$ correspond to the thickened segment from $B$ to $A$ shown below; the length of the latter (or what is the same thing, the number represented by $A$) is what is called the **degree of the angle $\angle s$**. We emphasize that this length of the segment from $B$ to $A$ is taken relative to the unit which is the degree.



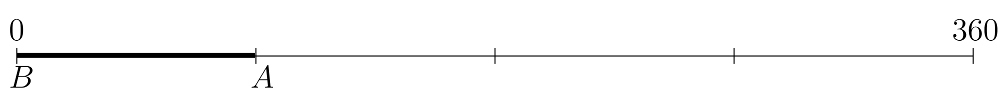

If we consider $\angle t$ instead, then the arc intercepted by $\angle t$ is the following thickened segment. As with $\angle s$, the **degree of $\angle t$** is by definition the length of this thickened segment (again taken relative to the degree):


What needs to be emphasized is that, *so long as the center of the circle is $O$, then no matter how "big" the circle is, the degree of $\angle s$ or $\angle t$ defined with respect to the circle will always be the same number, so long as the whole wire is declared to have length 360 degrees.* In intuitive language, the length of the intercepted arc by $\angle AOB$ always stays a fixed fraction of the total length of the circle (called the **circumference**) regardless of which circle is used. This assertion should fascinate fourth graders as indeed it is a remarkable fact. They should be encouraged to verify it by extensive experimentation, for example, using a string to model circles of many sizes. In fourth grade, the experimentation should mostly be done with "nice" angles such as what we call "90 degree" angles:




One way to do this is to draw a circle with center $O$ on a piece of paper and fold the paper to get a line $BOC$ through $O$ (see the above picture on the left). Notice that the paper folding identifies the two halves of the circle separated by line $BOC$ (see the discussion of the "symmetry of the circle" on page 12 immediately following). Now fold the circle once again across the center $O$ so that the point $B$ is on top of the point $C$, thereby getting a line $AOD$. We are interested in the part of angle $\angle AOB$ indicated by $\angle s$ in the picture, which intercepts the thickened arc on the circle between $A$ and $B$. Now notice that the paper folding, across line $AOD$ and also across line $BOC$, identifies the four arcs separated by the two lines. Therefore these four arcs have the same length, and therefore the whole circle is now divided into four parts of equal length. It follows that if we stretch out the circle into a line segment as before, then the point $A$ would be the first division point if the segment is divided into 4 parts of equal length:



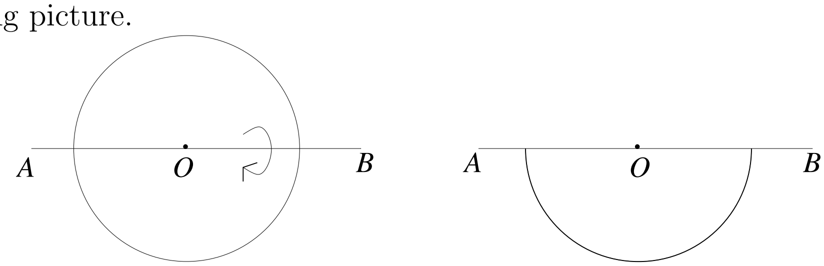

If the length of the whole segment is to be 360 degrees, then segment $OA$, being one part when 360 is divided into 4 equal parts, has length 90 degrees (but see page 13 for a more elaborate discussion of 90 degree angles). Since this discussion has nothing to do with how big the circle is, we see that $\angle s$ has 90 degrees no matter which circle is used to measure $\angle s$, so long as its center is at $O$.

We can give the same discussion to the angle $\angle t$ in the preceding picture on the right; it has 270 degrees because it is the totality of 3 parts when 360 is divided into 4 equal parts. Needless to say, we can do the same to other angles which have 45, 60, or 120 degrees.

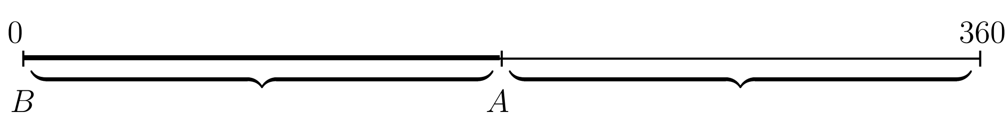
The notation for the measure of an angle $\angle t$ is $|\angle \boldsymbol{t}|$, or sometimes, $\boldsymbol{m(\angle t)}$. One also uses the notation $|\angle \boldsymbol{AOB}|$ or $\boldsymbol{m(\angle AOB)}$ for the same purpose, but keep in mind that this notation carries with it the uncertainty of whether $|\angle s|$ or $|\angle t|$ is being measured (in the notation of Figure 1 on page 9) unless it is clearly specified. Suppose the angle $\angle s$ in Figure 1 on page 9 has $d$ degrees, then a common terminology is that the side $OA$ is obtained from $OB$ by **turning $\boldsymbol{d}$ degrees counterclockwise**. Similarly, we say $OB$ is obtained from $OA$ by **turning $\boldsymbol{d}$ degrees clockwise**.

This definition of degree is the principle that underlies the construction of the protractor. Students should be given various angles to find their degrees with the help of a protractor. (For a simple demonstration, click here.)

A few special angles have degrees that are so striking that no protractor is needed for their determination. One of these is the straight angle, *A straight angle is* 180° (the ° is the abbreviation for "degree"), and the reason for this is equally interesting. To measure a straight angle $\angle AOB$, we draw a circle centered at $O$. Now the circle is **symmetric with respect to the line $\boldsymbol{AB}$**, which is a line passing through the center, in the sense that if the circle is drawn on a piece of paper and the paper is folded across the line $AB$, then the circle folds into itself, as shown on the right of the following picture.




This shows that the arclength of the upper **semi-circle** is equal to that of the lower semi-circle. By the above definition of degree, the degree of the straight angle $\angle AOB$ is half of 360°, which is, of course, 180°.

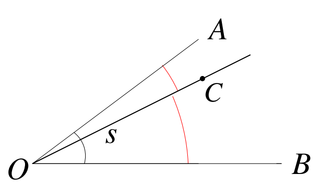


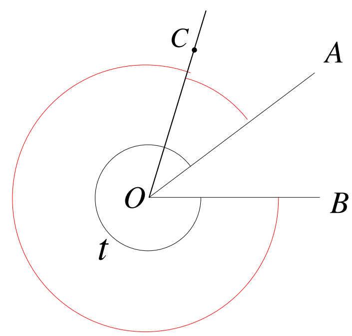
It would be instructive to explain to fourth graders why the converse statement is also true: if an angle is $180°$, then the two sides of the angle form a line.

Before we discuss angles of $90°$, we observe an important **additive property** of degree measurement: Given an angle $\angle AOB$, we first decide which part of $\angle AOB$ we wish to address, i.e., in the notation of Figure 1 on page 9, we decide at the outset whether we use $\angle AOB$ to refer to $\angle s$ or to $\angle t$. Once that is done, then it is entirely unambiguous to say whether **a point $\boldsymbol{C}$ is in $\boldsymbol{\angle AOB}$** or not. That done, then it is always the case that if $C$ is in $\angle AOB$,

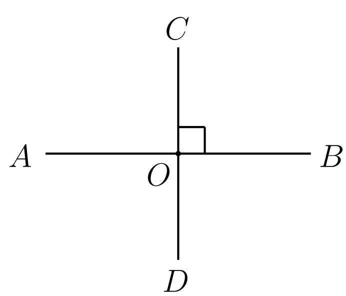
$$|\angle AOC| + |\angle COB| = |\angle AOB|.$$

as the following pictures show:



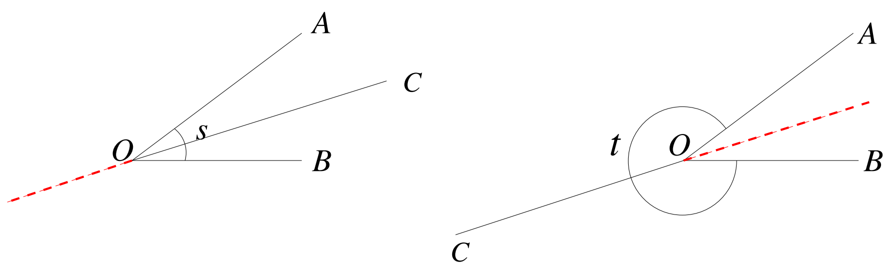




It follows from simple arithmetic using this additive property that, if two lines $CD$ and $AB$ meet at a point $O$ and one of the four angles, let us say $\angle COB$ at the point of intersection is $90°$, then all four angles are $90°$. See the picture below, where the broken segment $\urcorner$ is the standard notation to indicate an angle of $90°$.




In this case, we say **line $\boldsymbol{AB}$ is perpendicular to $\boldsymbol{CD}$**: in symbols, $\boldsymbol{CD \perp AB}$.

Notice also that in the case of $CD \perp AB$, the ray $OC$ divides the straight angle into two **equal parts** in the sense that the two angles $\angle AOC$ and $\angle COB$ have the same degree. The ray $OC$ is then called the **angle bisector** of $\angle AOB$. In general, if a ray $OC$ for a point $C$ in the angle $AOB$ has the property that $\angle AOC$ and $\angle COB$ have the same degree, then $OC$ is said to be the **angle bisector** of the angle $\angle AOB$. For $\angle s$ and $\angle t$ of Figure 1, here are their angle bisectors.

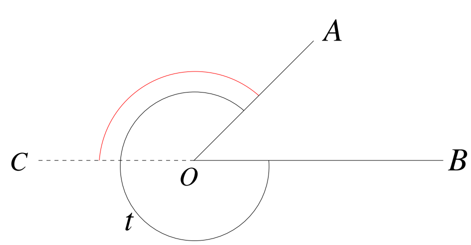



Observe that if the two angle bisectors above are drawn in the same figure, it is a straightforward computation to show that they form a straight line; in a fourth grade classroom, the simple explanation should be given for angles with whole-number degrees (but make sure that the degree is an even number because you want the degree of each half-angle to be a whole number too). Point out that every angle has an angle bisector. There should be plenty of exercises of using a protractor to find the (approximate) angle bisector of a given angle.

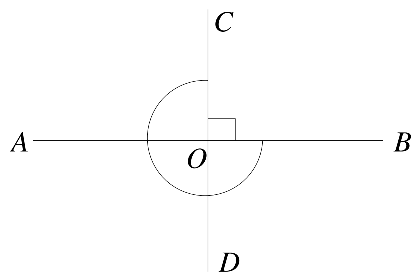
A protractor is designed to measure angles up to 180°. However, the additive property of degrees makes possible the measurement of angles bigger than a straight angle by the use of a protractor. For example, the degree of $\angle t$ below is

$$|\angle BOC| + |\angle COA| \;=\; 180^\circ + |\angle AOC|,$$

and we can measure the indicated $\angle AOC$ with a protractor.

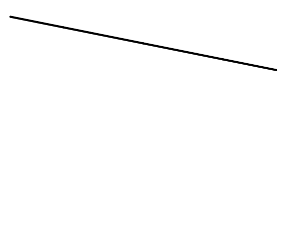



By the same token, we can easily recognize an angle of 270°, e.g., the part of $\angle COB$ below as indicated by the arc:

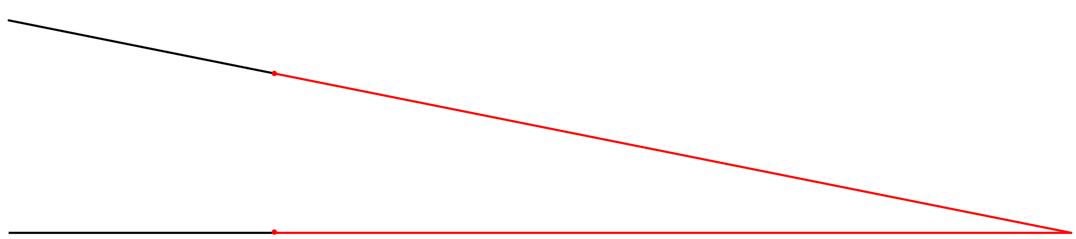



By definition, an **acute angle** is an angle that is $< 90°$, an **obtuse angle** is one that is $> 90°$, and a **right angle** is one that is $90°$.

Perpendicularity is one of two special relationships between two lines. The other is *parallelism*. Two lines are said to be **parallel** if they do not intersect. One should emphasize to students that the concept of parallelism applies only to *lines*, which extend in both directions *indefinitely*, rather than to *segments*. Thus while the following two segments do not intersect, they are *not* parallel,



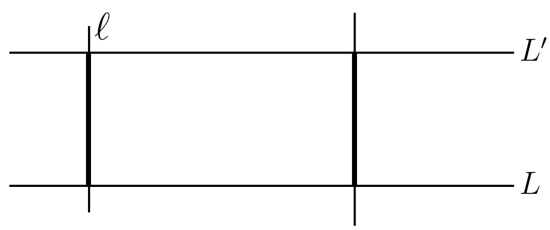
because the *lines* containing these segments do intersect. More precisely, when the same segments are extended sufficiently far to the right, they intersect, as shown:



Students should be alerted, early on, to the fact that we can never represent a complete line pictorially on a finite piece of paper, only a part of a line. So classroom instruction should be careful to distinguish between what a picture *suggests* and what a picture *says literally*.

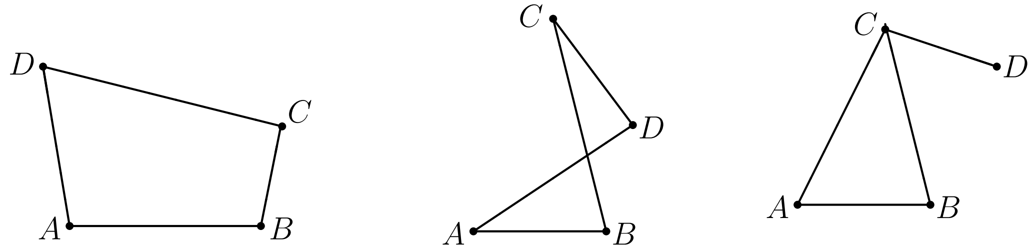
It would help students to develop geometric intuition if they can verify the following by hands-on experiments:

- If a line is perpendicular to one of two parallel lines, then it is perpendicular to both.
- Let $L$ and $L'$ be parallel lines, and let another line $\ell$ be perpendicular to both. Then the length of the segment intercepted on $\ell$ by $L$ and $L'$ is always the same, independent of where $\ell$ is located (so long as it is perpendicular to both $L$ and $L'$. This length is called the **distance** between $L$ and $L'$.

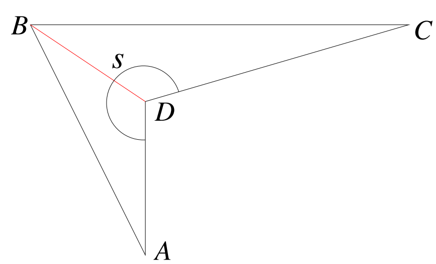



- If two distinct lines are perpendicular to the same line, then they are parallel to each other.
- Define a **triangle** to be a figure consisting of the segments joining three **non-collinear points** (i.e., they do not lie on a line). Then the sum of the degrees of the angles of a triangle is 180°.

We can make use of these facts as follows. A **quadrilateral** is a figure consisting of four distinct points $A$, $B$, $C$, $D$ (called **vertices**) together with the four segments $AB$, $BC$, $CD$, $DA$ (called **edges** or **sides**) so that the only intersections allowed between the edges are at the vertices, namely, $AB$ and $BC$ intersect at $B$, $BC$ and $CD$ intersect at $C$, $CD$ and $DA$ intersect at $D$, and $DA$ and $AB$ intersect at $A$. The segments $AC$ and $BD$ are called the **diagonals** of the quadrilateral. The idea behind the definition of a quadrilateral is that we do not want either figure in the middle or on the right below to be a quadrilateral:




Assuming the last bullet above, one can give the simple reasoning as to why the sum of the angles of a quadrilateral is 360° (draw a diagonal to separate the quadrilateral into two triangles). However, one has to be careful in the case of a quadrilateral that looks like this:



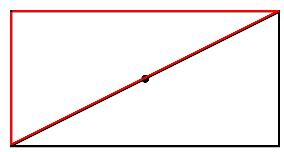

In this situation, when we sum all the angles of this quadrilateral, the angle to use at the vertex $D$ will be understood to be the angle $\angle s$ rather than the other part of $\angle ADC$. Furthermore, one should use the diagonal $BD$ in this case for the verification that the sum of the angles of this quadrilateral is 360°. (One may also notice that the diagonal $AC$ lies outside the quadrilateral $ABCD$.)

A quadrilateral is called a **parallelogram** if opposite sides are parallel, and is called a **rectangle** if all four angles are 90°. It follows from the third bullet on page 16 that a rectangle is a parallelogram, and from the second bullet that the opposite sides of a rectangle are **equal** (i.e., of the same length). It is valuable to impress on students that as soon as a quadrilateral has four angles equal to 90°, then its opposite sides must be equal (see page 140 for a proof).[3] In addition, because the sum of the angles of a quadrilateral is 360°, a quadrilateral with three right angles is a rectangle.

A rectangle with four equal sides is called a **square**.

[3] Of course, the usual definition of a rectangle is that it is a quadrilateral with *both* properties that all four angles are 90° and each pair of opposite sides are equal. We avoid this definition because it makes it difficult to recognize whether a quadrilateral is a rectangle or not.

A triangle is called a **right triangle** if one of its angles is a right angle, an **acute triangle** if all three angles are acute, and an **obtuse triangle** if it contains an obtuse angle. It follows from the last bullet on page 16 that a triangle cannot have more than one obtuse angle or more than one right angle. The easiest way to get a right triangle is to draw a diagonal of the rectangle; one gets two right triangles. One can verify, by hands-on experiments such as cutting papers, for example, that the 180 degree **rotation** around the midpoint of the diagonal brings one of the right triangles exactly on top of the other. We say in this case that a rectangle has **rotational symmetry.**



An **isosceles triangle** is a triangle with two equal sides. We usually refer to the common vertex of the equal sides as the **top vertex** of the isosceles triangle, and the angle at the top vertex the **top angle**. Again, verify through hands-on activities that an isosceles triangle is symmetric with respect to (the line containing) the angle bisector of the top angle. It follows that the angles facing the equal sides are **equal** (i.e., same degree).

A triangle with three equal sides is called an **equilateral triangle**. Thus all three angles of an equilateral triangle are equal.

## GRADE 5

### **Number and Operation — Fractions** 5.NF

4. Apply and extend previous understandings of multiplication [of fractions] to multiply a fraction or whole number by a fraction.

b. Find the area of a rectangle with fractional side lengths by tiling it with [rectangles] of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

**Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.**

3. Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

a. A cube with side length 1 unit, called a unit cube, is said to have one cubic unit of volume, and can be used to measure volume.

b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.

4. Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

5. Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.

a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

b. Apply the formulas $V = \ell \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.

c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

## **Geometry** 5.G

**Graph points on the coordinate plane to solve real-world and mathematical problems.**

1. Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., $x$-axis and $x$-coordinate, $y$-axis and $y$-coordinate).

2. Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation. Classify two-dimensional figures into categories based on their properties.

3. Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.

4. Classify two-dimensional figures in a hierarchy based on properties.

### **Comments on teaching grade 5 geometry**

The main objectives of this grade are the computation of the area formula of a rectangle whose side lengths are fractions, the introduction of the concept of the volume of a rectangular prism, the setting up of a coordinate system in the plane, and the classification of common triangles and quadrilaterals according to their properties.

The computation of the area formula of a rectangle with fractional sides should be a high point in school geometry. The result is not in doubt: it is length times width. But *it is the reasoning that reveals the essence of the concept of area*, and this reasoning is of course based on the basic properties of fractions and area. In advanced mathematics, we simply *prove* that there is a way to assign an area to a region in the plane so that the area so obtained enjoys these desirable properties.[4] However, this is a torturous process that is entirely unsuitable for use in schools, much less in grade 5. So we start from the opposite end by assuming that such an assignment is possible, and concentrate instead of finding out, *if the assignment of area possesses the following obvious properties,* what the area of each geometric figure must be:

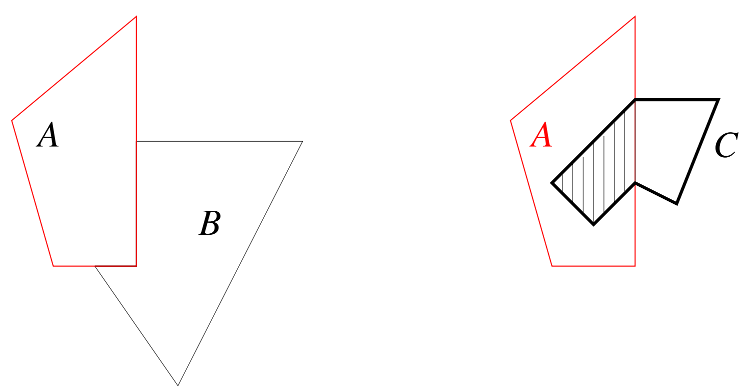
(a) The area of a planar region is always a number $\geq 0$.

(b) The area of a **unit square** (a square whose sides have length 1) is 1 square unit.

(c) If two regions are *congruent*, then their areas are equal.

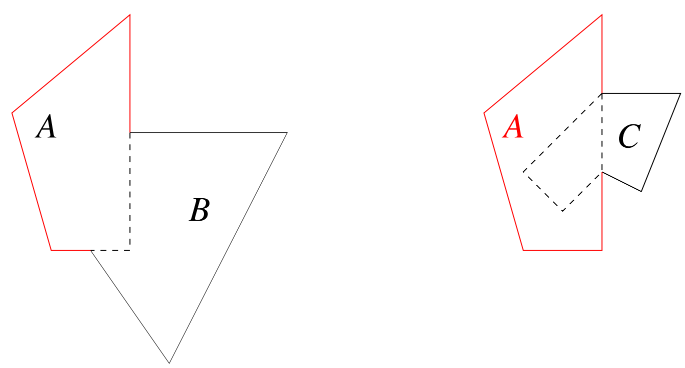
(d) (*Additivity*) If two regions have at most (part of) their boundaries in common, then the area of the region obtained by combining the two is the sum of their individual areas.

We can amplify on the meaning of (d) by the following pictures.




[4]Strictly speaking, we can only assign an area to *some*, but not all regions.

On the left, the regions $A$ and $B$ intersect only in a horizontal segment and a vertical segment along their common boundary, so it is intuitively clear that the area of the combined region of $A$ and $B$ is the sum of the areas of $A$ and $B$. This is exactly what (d) says. See the left figure below. On the right, the regions $A$ and $C$ have more than part of their boundaries in common as they overlap in a sizable region. So the area of the combined region of $A$ and $C$ is clearly strictly smaller than the sum of the areas of $A$ and $C$ because, in adding the areas of $A$ and $C$ together, we count the area of the overlapped region twice. See the right figure below.




We make a simple observation: (d) easily implies that if a region $\mathcal{R}$ is the combined region of several smaller regions (i.e., more than two) that have at most their boundaries in common, then the area of $\mathcal{R}$ is the sum of the areas of these smaller regions.

Regarding (c), it suffices to define "congruent regions" in a fifth grade classroom as "same shape and same size"; one can also check congruence of figures by cutting out cardboard drawings and moving one on top of another. More precisely, the only fact we need for the computation of the area of a rectangle is that rectangles with the same length and width are "congruent" and therefore have the same area.

We can now compute the area of a rectangle with sides $\frac{3}{4}$ and $\frac{2}{7}$. (The proof is taken from the proof of Theorem 2 on pp. 63–64 of H. Wu, Pre-Algebra.[5]) We give the side lengths these explicit values because in a fifth grade classroom, one has to begin with simple cases like this. Moreover, it will be seen that the reasoning is perfectly general. Also observe how the computation is guided at every turn by (a)–(d).

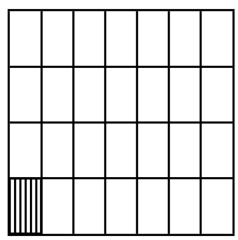
[5]Recall: The turquoise box indicates an active link to an article.

We break up the computation into two steps.

($i$) The area of a rectangle with sides $\frac{1}{4}$ and $\frac{1}{7}$.

($ii$) The area of a rectangle with sides $\frac{3}{4}$ and $\frac{2}{7}$.

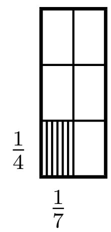
We begin with ($i$). To get a rectangle with sides $\frac{1}{4}$ and $\frac{1}{7}$, divide the vertical sides of a fixed unit square into 4 equal parts and the horizontal sides into 7 equal parts. Joining the corresponding division points, both horizontally and vertically, leads to a partition of the unit square into $4 \times 7$ (= 28) congruent rectangles, and therefore 28 rectangles of equal areas, by (c). Observe that each small rectangle in this division has vertical side of length $\frac{1}{4}$ and horizontal side of length $\frac{1}{7}$, and is congruent to the shaded small rectangle in the lower left corner, as shown.




What Step (i) asks for is the area of this shaded rectangle. The unit square is now divided into 28 small rectangles each of which is congruent to this shaded rectangle. By (c) of page 21, the unit square has been divided into 28 parts of equal area. Consider the number line where the unit is the area of the unit square; then we have divided the unit into 28 equal parts. By the definition of a fraction, each one of these 28 areas represent $\frac{1}{28}$, which is equal to $\frac{1}{4\times 7}$. In other words, the area of the shaded rectangle in the lower left corner (with side lengths $\frac{1}{4}$ and $\frac{1}{7}$ ) is equal to $\frac{1}{4\times 7}$. Therefore:

$$\text{Area of rectangle with sides } \frac{1}{4} \text{ and } \frac{1}{7} \quad = \quad \frac{1}{4 \times 7} \tag{1}$$

To perform the computation in Step ($ii$), *we change strategy completely.* Instead of partitioning the unit square, we use small rectangles of sides $\frac{1}{4}$ and $\frac{1}{7}$ to build a rectangle of sides $\frac{3}{4}$ and $\frac{2}{7}$. By the definition of $\frac{3}{4}$, it is the combination of 3 segments each of length $\frac{1}{4}$. Similarly, the side of length $\frac{2}{7}$ consists of 2 combined segments each of length $\frac{1}{7}$. Thus we create a large rectangle consisting of 3 rows of small rectangles each of sides $\frac{1}{4}$ and $\frac{1}{7}$, and each row has two columns of these small rectangles. This large rectangle then has side lengths $\frac{3}{4}$ and $\frac{2}{7}$.




By equation (1), each of the small rectangles has area $\frac{1}{4\times 7}$. Since the big rectangle contains exactly $3 \times 2$ such congruent rectangles, its area is (by (d) above):

$$\underbrace{\frac{1}{4\times 7} + \frac{1}{4\times 7} + \cdots + \frac{1}{4\times 7}}_{3\times 2} \quad = \quad \frac{3\times 2}{4\times 7} \; = \; \frac{3}{4} \times \frac{2}{7}$$

Therefore the conclusion of (*ii*) is:

$$\text{Area of rectangle with sides } \tfrac{3}{4} \text{ and } \tfrac{2}{7} \quad = \quad \frac{3}{4} \times \frac{2}{7}$$

In other words, the area of the rectangle is the product of (the lengths of) its sides.

The general case follows this reasoning word for word. In most fifth grade classrooms, it would be beneficial to *state* the general formula, as follows. If $m$, $n$, $k$, $\ell$ are nonzero whole numbers, then:

$$\text{Area of rectangle with sides } \tfrac{m}{n} \text{ and } \tfrac{k}{\ell} \quad = \quad \frac{m}{n} \times \frac{k}{\ell} \tag{2}$$

Instead of giving an explanation of equation ( 2 ) directly in terms of symbols, it would probably be more productive to compute the areas of several rectangles whose sides have lengths equal to reasonable fractions, e.g., $\frac{3}{2}$ and $\frac{1}{5}$, $\frac{7}{3}$ and $\frac{5}{6}$, etc., and direct students' attention to the fact that the reasoning in each case follows a fixed pattern which then affirms the truth of the general case.[6] Of course one should give the

[6]There is much talk about using "patterns" to ease students' entry into algebra. Unfortunately, it is not often recognized that it is the "thought patterns" like the proofs just described rather than visual patterns that truly matter in this pedagogical strategy. When all is said and done, Content Dictates Pedagogy in mathematics education.

symbolic proof (computation) if the students are up to it.

If students have a firm mastery of the preceding computation, then the concept of the volume of a rectangular prism will be almost anti-climactic: it is more of the same (see equation (3) below). First, the assignment of a number to a (3-dimensional) solid, called its **volume**, is qualitatively identical to the case of area. We will make analogous assumptions on how volumes are assigned to solids:

(a) The volume of a solid is always a number $\geq 0$.

(b) The volume of a *unit cube* (a rectangular prism whose *edges* all have length 1) is 1.

(c) If two solids are *congruent*, then their volumes are equal.

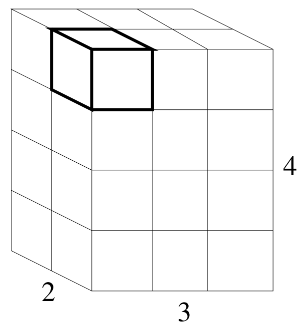
(d) (*additivity*) If two solids have at most (part of) their boundaries in common, then the volume of the solid obtained by combining the two is the sum of their individual volumes.

Again, we will not define "congruent solids" precisely in 5th grade except to appeal to the intuitive idea that congruent solids are those with the "same size and same shape".

We want to show that, if a rectangular prism $\mathcal{R}$ has edge lengths equal to $\ell$, $w$ and $h$, *and* $\ell$*,* $w$*,* $h$ *are all whole numbers*, then the volume of $\mathcal{R}$ is the product of these three numbers, i.e.,

$$\text{volume } \mathcal{R} \;=\; \ell \times w \times h \tag{3}$$

In the interest of clarity, we will prove the special case $\ell = 2$, $w = 3$, and $h = 4$. The reason for this expository decision is that the reasoning in this special case is in fact completely general. So let $P$ be a rectangular prism with edge lengths 2, 3, and 4. Divide each of these edges into segments of unit length. Pass a plane through corresponding division points of each group of parallel edges, and these planes give rise to a partition of $P$ into $2 \times 3 \times 4$ unit cubes because each horizontal layer of unit cubes has two rows and each row has three columns and therefore each layer has $2 \times 3$ unit cubes; moreover, there are 4 such horizontal layers.



Each unit cube has volume 1 (by (a)), and since there are $2 \times 3 \times 4$ of them, the additivity of volume (i.e., (d) above) implies that the volume of $P$ is

$$\underbrace{1 + 1 + \cdots + 1}_{2\times 3\times 4} \;=\; 2 \times 3 \times 4,$$

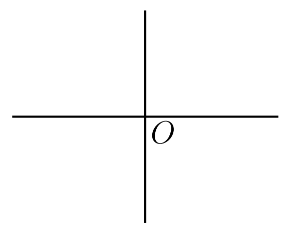
as desired. It is clear that since the explicit values of 2, 3, 4 played no role in the preceding argument, we see that equation ( 3 ) is correct.

The next topic—coordinatizing the plane—may be regarded as an extension of the idea of the number line to the plane. The essence of the number line is to assign a number to each point on the line, *provided 0 and 1 have been fixed on the line.* (In grade 5, we recognize that the "number" in question can only be a fraction, but once negative numbers have been introduced in grade 6, "number" will refer to all rational numbers.) Of course, once 0 and 1 have been fixed, it would follow that, conversely, every number corresponds to a unique point on the line. With this in mind, we are going to show how to associate *an ordered pair of numbers* to each point in the plane *provided a pair of perpendicular axes has been fixed in the plane.*[7] Conversely, once such a pair of axes has been fixed, each ordered pair of numbers will correspond to a unique point of the plane. We will begin by explaining how to associate an ordered pair of numbers to a point in the plane. The fundamental idea is very simple, and it is not unlike the way we associate to each house in an rectangular array of streets its street address: a number and a street name.

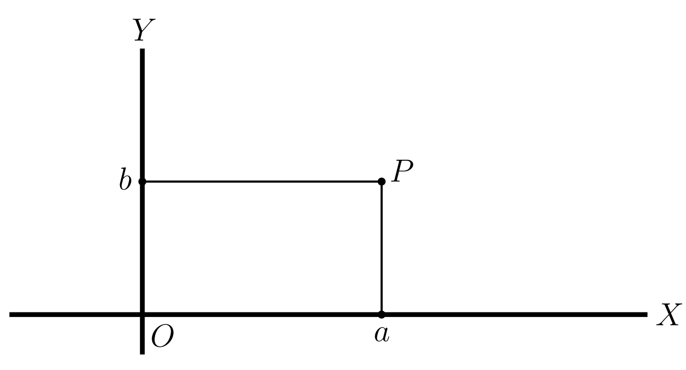
Choose two perpendicular lines in the plane which intersect at a point to be denoted by $O$. $O$ is called the **origin** of the coordinate system. Let one of them

[7]For the gifted students, they can be assigned the task of coordinatizing the plane using *any* two intersecting lines.

be horizontal (i.e., extending from left to right) and let the other be vertical. The horizontal line is traditionally designated as the **$x$-axis**, and the vertical one the **$y$-axis**. We will regard both as number lines and will henceforth identify each point on these **coordinate axes** (as the $x$ and $y$ axes have come to be called) with a number. As expected, we choose the fractions on the $x$-axis to be on the right of $O$ so that $O$ is the zero of the $x$-axis; we also choose the fractions on the $y$-axis to be above $O$ on the $y$-axis so that $O$ is also the zero of the $y$-axis. These axes divide the plane into four parts (called **quadrants**): upper left, upper right, lower left, and lower right.




We will associate with each point $P$ in the plane an *ordered* pair of numbers, but because we do not as yet have negative numbers, we will limit ourselves to the upper right portion of the plane, which is usually called the **first quadrant**. So let $P$ be a point in the first quadrant. Let us agree to call any line parallel to the $x$-axis a **horizontal line**, and also any line parallel to the $y$-axis a **vertical line**. Then through $P$ draw two lines, one vertical and one horizontal, so that they intersect the $x$-axis at a number $a$ and the $y$-axis at a number $b$, respectively. Then the ordered pair of numbers $(\boldsymbol{a}, \boldsymbol{b})$ are said to be the **coordinates of $P$** *relative to the chosen coordinate axes*; $\boldsymbol{a}$ is called the **$x$-coordinate** and $\boldsymbol{b}$ the **$y$-coordinate of $P$** *relative to the chosen coordinate axes*. Observe that the coordinates of $O$ are $(0, 0)$.



Now, by construction, the three angles of the quadrilateral $PaOb$ at the vertices $O$, $a$, and $b$ are right angles. By an observation on page 17, $PaOb$ is a rectangle. It follows from another observation on page 17 that the opposite sides of $PaOb$ are equal (in length). Thus the length of the side $bP$ is equal to the length of the side $Oa$. Since by the definition of length on the number line, the length of $Oa$ is just the number $a$, we see that the $x$-coordinate $a$ of the point $P$ is nothing other than the length of the perpendicular segment $Pb$ from $P$ to the $y$-axis. Likewise, the length of the perpendicular segment $Pa$ from $P$ to the $x$-axis is just the $y$-coordinate $b$ of $P$. The length of the perpendicular segment from a point to a given line is called the **distance of the point from the line**. We have therefore obtained a different interpretation of the coordinates of $P$ when $P$ is a point in the first quadrant:

> *The $x$-coordinate of a point $P$ in the first quadrant is the distance of $P$ from the $y$-axis and the $y$-coordinate of $P$ is the distance of $P$ from the $x$-axis.*

Conversely, with a chosen pair of coordinate axes understood, then given an ordered pair of fractions $(a, b)$, there is one and only one point in the plane with coordinates $(a, b)$: this is the point of intersection of the vertical line passing through $(a, 0)$ and the horizontal line passing through $(0, b)$.

It is common to just ***identify a point with its corresponding ordered pair of numbers.*** In the plane, we define $\boldsymbol{(a, b) = (c, d)}$ to mean that the points represented by $(a, b)$ and $(c, d)$ are the same point. (This is the analog in the plane of the definition that two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are *equal* if the points on the number line represented by $\frac{a}{b}$ and $\frac{c}{d}$ are the same point.) Since every point corresponds to one and only one ordered pair of numbers, we see that

$$(a, b) = (c, d) \quad \text{is the same as saying} \quad a = c, \quad b = d.$$

A plane in which a pair of coordinate axes have been set up is called a **coordinate plane**.

**Caution:** We should make explicit an assumption about the $x$- and $y$-axes that is usually taken for granted. We will state the assumption using intuitive language, but the assumption will be put on a firm foundation in high school (see (A7) on page

133 ). We assume that if we rotate the plane 90 degrees counter-clockwise around $(0, 0)$, then the numbers on the $x$-axis coincide with those on the $y$-axis, i.e., the unit segments on the two number lines have "the same length". (To understand the last statement, recall that the choice of the location of 0 and 1 on a number line is arbitrary.) Also if we reflect across the angle bisector of the $90°$ angle between the positive $x$-axis and the positive $y$-axis, then the numbers on the $x$-axis again coincide with those on the $y$-axis, and vice versa. As will be seen in high school, these assumptions are of critical importance for understanding the geometry of the plane.

The last major topic of grade 5 has to do with basic definitions of various triangles and quadrilaterals. In TSM (see page 2), there is some confusion about whether a square is a rectangle and whether an equilateral triangle is isosceles. However, the content of two of the standards in Standard 5.G is to remove this confusion:

> 3. *Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.*
>
> 4. *Classify two-dimensional figures in a hierarchy based on properties.*

Thus start with triangles. Recall that we defined an *isosceles triangle* as a triangle with two equal sides (see page 18 ); this means that so long as it has two equal sides, it has to be called "isosceles". In particular, even if the third side is also equal to the other two (in which case we call it an *equilateral triangle*; see page 18 ), it is still an isosceles triangle. This is the standard mathematical usage of the terms *isosceles* and *equilateral* and is consistent with the quoted standards above.

For quadrilaterals, there are other notable ones in addition to parallelograms and rectangles. For the sake of completeness, we list their definitions together. With a quadrilateral understood, we have:

> **Trapezoid:** One pair of parallel opposite sides, i.e., any quadrilateral with the property that it has one pair of parallel opposite sides qualifies to be called a trapezoid.
>
> **Parallelogram:** Two pairs of parallel opposite sides.

**Rectangle:** Four right angles.

**Square:** Four right angles and four equal sides.

**Kite:** Two pairs of equal adjacent sides. [8]

Then: every square is a rectangle, every rectangle is a parallelogram, and every parallelogram is a trapezoid. A square is always a kite. Moreover, there are kites that are not squares, there are trapezoids that are not parallelograms, there are parallelograms that are not rectangles, and finally, there are rectangles that are not squares.

Many exercises can be given on quadrilaterals and triangles in a coordinate plane so that the coordinates of their vertices are already given. In simple situations, these coordinates already allow us to determine if the triangle or quadrilateral has certain properties, such as rotational symmetry or symmetry with respect to a line.

[8] Two sides of a quadrilateral are said to be **adjacent** if they have a vertex in common.

# GRADE 6

## **The Number System** 6.NS

**Apply and extend previous understandings of numbers to the system of rational numbers.**

6. Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

## **Geometry** 6.G

**Solve real-world and mathematical problems involving area, surface area, and volume.**

1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

2. Find the volume of a right rectangular prism with fractional edge lengths by packing it with [rectangular prisms] of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = \ell wh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

3. Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

4. Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the

context of solving real-world and mathematical problems.

## Comments on teaching grade 6 geometry

The geometry of grade 6 is, in the main, about areas and volumes. Here will be found the common area formulas for triangles and quadrilaterals. The importance of the area formula for triangles is that it allows us, at least in principle, to compute the area of any polygon by "triangulation" (see page 39 ). In addition, along with the area formula for rectangles with side lengths that are fractions, we will also give the corresponding volume formula for rectangular prisms whose edge lengths are fractions.

We will also mention the four quadrants of the coordinate plane, the computation of lengths of horizontal and vertical segments in a coordinate plane, nets, and the definitions of tetrahedra and pyramids.

We begin by recalling the basic assumptions about what we call *area*. They were already mentioned in grade 5 and there are four of them:

(a) The area of a planar region is always a number $\geq 0$.

(b) The area of a unit square (a square whose sides have length 1) is by definition the number 1.

(c) If two regions are *congruent*, then their areas are equal.

(d) (*Additivity*) If two regions have at most (part of) their boundaries in common, then the area of the region obtained by combining the two is the sum of their individual areas.

The whole discussion in this sub-section hinges on the simple statement that

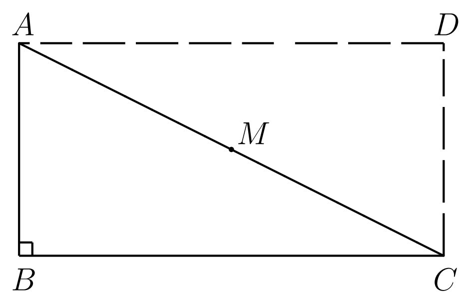
$$\text{Area of rectangle} \;=\; \text{product of the (lengths of the) sides} \qquad (4)$$

The validity of this formula when the lengths of both sides are fractions is exactly the content of equation ( 2 ) on page 24 . Students should be formally told at this point that there are numbers which are not fractions (they have probably heard of $\pi$

and it would make sense to let them know that $\pi$ is not a fraction). So the message of equation (4) is that the area of a rectangle is always the product of its sides no matter what the lengths of the sides may be.[9]

It is astonishing how much useful information can be extracted from the simple formula (4) alone. We will show how to exploit this area formula to compute the areas of triangles, parallelograms, trapezoids, and in fact any polygon (at least in principle).

We begin with triangles. Consider a right triangle $\triangle ABC$ with $AB \perp BC$. We compute its area by expanding it to a rectangle, as follows. Let $M$ be the midpoint of $AC$.




We observe that if we do a rotation of 180° around $M$, then the rotation interchanges $A$ and $C$ and moves $B$ to the position $D$ as shown. We will take for granted that such a rotation moves a triangle to a congruent one ("with the same size and same shape") and therefore one with the same area (by (c)).[10] Now use the fourth bullet on page 16 to explain why all the angles of the quadrilateral $ABCD$ are right angles and therefore $ABCD$ is in fact a rectangle. Therefore, by the additivity of area (i.e., (d) above),

$$\begin{aligned}
\text{area}(ABCD) &= \text{area}(\triangle ABC) + \text{area}(\triangle CDA) \\
&= \text{area}(\triangle ABC) + \text{area}(\triangle ABC) \\
&= 2 \cdot \text{area}(\triangle ABC).
\end{aligned}$$

[9]Strictly speaking, we are invoking the formal extension of the formula *area of rectangle = product of the side lengths* from fractional side lengths to side lengths that are arbitrary numbers by using the *Fundamental Assumption of School Mathematics.* See page 88 of H. Wu, Pre-Algebra.

[10]This is a wonderful opportunity to prepare students for geometry in grade 8.

It follows that

$$\text{area}(\triangle ABC) \;=\; \frac{1}{2}\,\text{area}(ABCD).$$

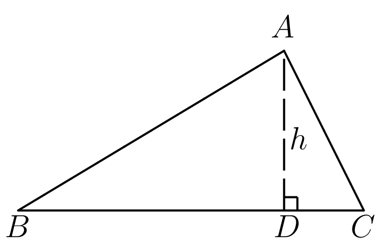
At this point, it becomes necessary to use a symbol, $|\boldsymbol{AB}|$, to denote the length of a segment $AB$. By equation (4), we get,

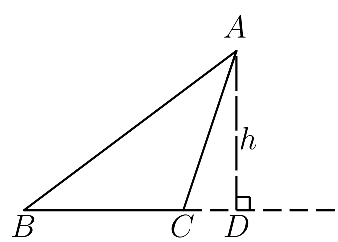
$$\text{area}(\triangle ABC) \;=\; \frac{1}{2}\,|AB|\cdot|BC|.$$

The sides $AB$ and $BC$ flanking the right angle in a right triangle are called the **legs** of $\triangle ABC$. We therefore have:

$$\text{Area of right triangle} \;=\; \frac{1}{2}\,(\text{product of (the lengths of) its legs}) \tag{5}$$

Now suppose $\triangle ABC$ is not a right triangle. Let $AD$ be the **altitude** from the vertex $A$ to line $BC$, i.e., $AD$ is the segment which joins $A$ to line $BC$ and is perpendicular to line $BC$). If $D = B$ or $D = C$, then $\triangle ABC$ would be a right triangle, contradicting the hypothesis. So we may assume that $D \neq B$ and $D \neq C$. Then we obtain two *right* triangles, $\triangle ABD$ and $\triangle ACD$, so that equation (5) becomes applicable to each of them. There are two cases to consider: the case where $D$, the **foot of the altitude**, is inside the segment $BC$, and the case where $D$ is outside segment $BC$. See the figures:




In either case, $AD$ is called the **height** with respect to the **base** $BC$. By the usual abuse of language, **height** and **base** are also used to signify the *lengths* of $AD$ and $BC$, respectively. With this understood, we shall prove in general that

$$\text{Area of triangle} \;=\; \frac{1}{2}\,(\text{base} \times \text{height}) \tag{6}$$

For convenience, we shall use $h$ to denote $|AD|$. Then this is the same as proving

$$\text{area}(\triangle ABC) \;=\; \frac{1}{2}\,|BC|\cdot h$$

In case $D$ is inside $BC$, we use the additivity of area ((d) on page 32) and refer to the figure above to derive:

$$\begin{aligned}
\text{area}(\triangle ABC) &= \text{area}(\triangle ABD) + \text{area}(\triangle ADC) \\
&= \frac{1}{2}\,|BD|\cdot h + \frac{1}{2}\,|DC|\cdot h \\
&= \frac{1}{2}\,\big(|BD| + |DC|\big)\,h \qquad \text{(by the dist. law)} \\
&= \frac{1}{2}\,|BC|\cdot h
\end{aligned}$$

Incidentally, here as well as in the computation for the second case ($D$ is outside $BC$), we see how important it is to know the distributive law. One cannot overstate the need for students from grade 5 and up to be fluent in the use of this law.

In case $D$ is outside $BC$, we again use the additivity of area and refer to the figure above to obtain:

$$\text{area}(\triangle ABD) \;=\; \text{area}(\triangle ACD) + \text{area}(\triangle ABC)$$

This is the same as

$$\frac{1}{2}\,|BD|\cdot h \;=\; \frac{1}{2}\,|CD|\cdot h + \text{area}(\triangle ABC).$$

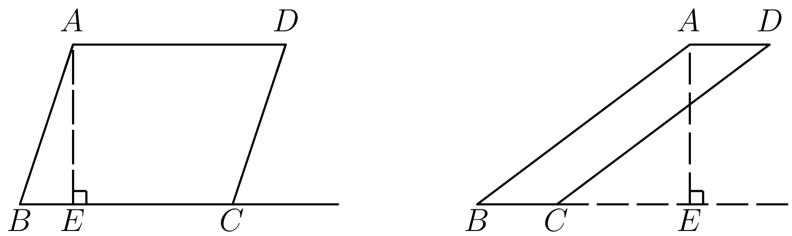
Therefore,

$$\begin{aligned}
\text{area}(\triangle ABC) &= \frac{1}{2}\,|BD|\cdot h - \frac{1}{2}\,|CD|\cdot h \\
&= \frac{1}{2}\,\big(|BD| - |CD|\big)\,h \qquad \text{(by the dist. law)} \\
&= \frac{1}{2}\,|BC|\cdot h
\end{aligned}$$

Thus the area formula for triangles has been completely proved.

Almost all school math textbooks mention the first case but not the second in deriving equation (6). Looking forward to the proofs of the area formulas for parallelograms and trapezoids below, one realizes that this omission creates a crucial gap in students' understanding of these formulas.

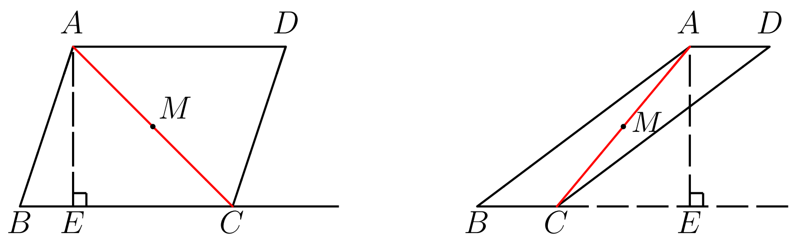
Next the area of a parallelogram $ABCD$. Drop a perpendicular from $A$ to the opposite side $BC$. Call it $AE$. In view of equations (4) and (5), we may assume $E \neq B$ or $C$. Then two cases are possible, as shown below.




$AE$ is called the **height of the parallelogram** with respect to the **base** $BC$. From the second bullet on page 16 , we know that $|AE|$ does not change if another point on $AD$ replaces $A$. As before, **height** and **base** are also used to designate the *lengths* of these segments. The formula to be proved is then:

$$\text{Area of parallelogram} \quad = \quad \text{base} \times \text{height} \tag{7}$$

The following proof of equation ( 7 ) works for both cases, and it goes as follows. Draw the diagonal $AC$ (see page 16 for a definition) and let $M$ be the midpoint of $AC$.

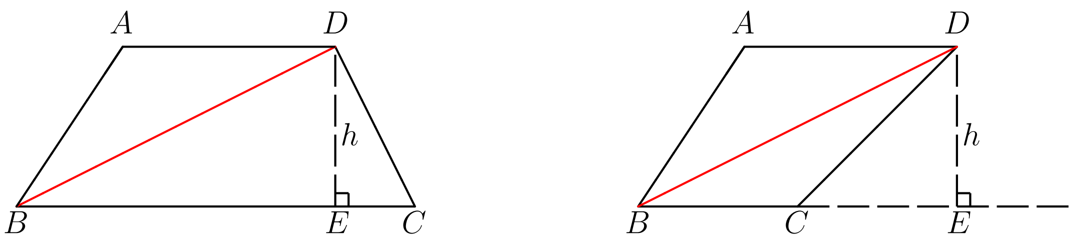



One verifies by hands-on experiments (e.g., cardboard cut-outs) that the rotation of $180°$ around $M$ moves $\triangle ABC$ to $\triangle CDA$. So as usual, we conclude that $\triangle ABC$ is congruent to $\triangle CDA$ and therefore, (c) of page 32 implies that area$(\triangle ABC)$ = area$(\triangle CDA)$. By (d) of page 32 , we therefore have:

$$\begin{aligned} \text{area}(ABCD) \quad &= \quad \text{area}(\triangle ABC) + \text{area}(\triangle CDA) \\ &= \quad 2 \cdot \text{area}(\triangle ABC) \\ &= \quad 2 \cdot \frac{1}{2}\big(|BC| \cdot |AE|\big) \\ &= \quad |BC| \cdot |AE| \end{aligned}$$

as desired. Note how, in the second case, we need equation ( 6 ) to be still valid when the foot of the altitude falls outside the base.

We also get the formula for the area of a trapezoid $ABCD$ with $AD \parallel BC$. Let $DE \perp BC$. Again, in view of equations (4) and (5), we may assume $E \neq B$ or $C$.




Then note that $|DE|$, being the distance between the parallel lines $L_{AD}$ and $L_{BC}$, is the height of both $\triangle ABD$ with respect to the base $AD$ and $\triangle BCD$ with respect to base $BC$, and is called the **height of the trapezoid**. Again we denote this height by $h$. The segment $AD$ and $BC$ are called the **bases of the trapezoid**. We are going to prove that the area of a trapezoid is $\frac{1}{2}$ the height times the sum of bases. Precisely,

$$\text{Area}(ABCD) \;=\; \frac{1}{2}\,h\,\big(|AD| + |BC|\big)$$

It is of some interest to observe that when the trapezoid $ABCD$ is a parallelogram (i.e., when $AB$ is also parallel to $CD$), this area formula reduces to equation (7). Indeed, in this case, we saw on page 36 that $\triangle ABC$ is congruent to $\triangle CDA$ so that in particular, $BC$ and $AD$ have the same length and the preceding formula becomes

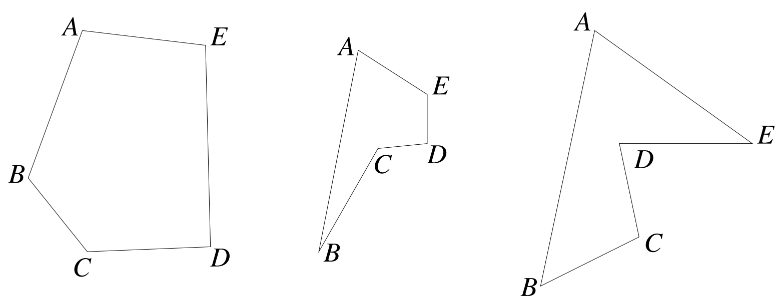
$$\text{Area}(ABCD) \;=\; \frac{1}{2}\,h\,\big(|BC| + |BC|\big) \;=\; \frac{1}{2}h\,(2|BC|) \;=\; h\,|BC|.$$

This is exactly equation (7). Incidentally, this is one reason why we want a parallelogram to be a trapezoid (see the discussion on page 29) because mathematical theorems about quadrilaterals such as these area formulas then make sense. As to the proof of the trapezoid area formula, we have

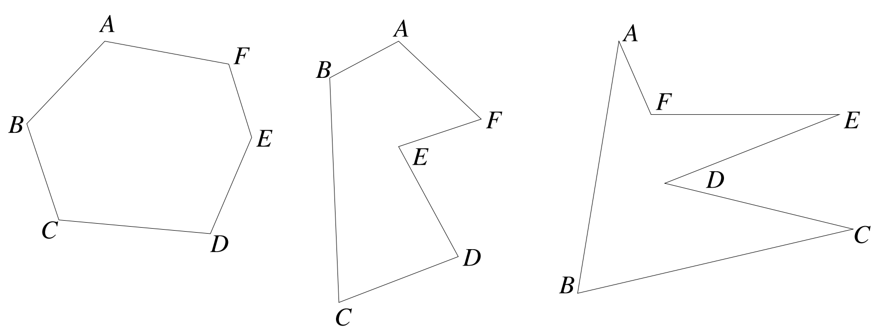
$$\begin{aligned}
\text{Area}(ABCD) \;&=\; \text{area}(\triangle BAD) + \text{area}(\triangle BDC) \\
&=\; \frac{1}{2}\,h \cdot |AD| + \frac{1}{2}\,h \cdot |BC| \\
&=\; \frac{1}{2}\,h\,\big(|AD| + |BC|\big) \qquad \text{(by the dist. law)},
\end{aligned}$$

as claimed. Note once again that in this proof, we need the area formula of a triangle when the foot of the altitude falls outside the given base. This is why one must know the proof of the area formula of a triangle for this case too.

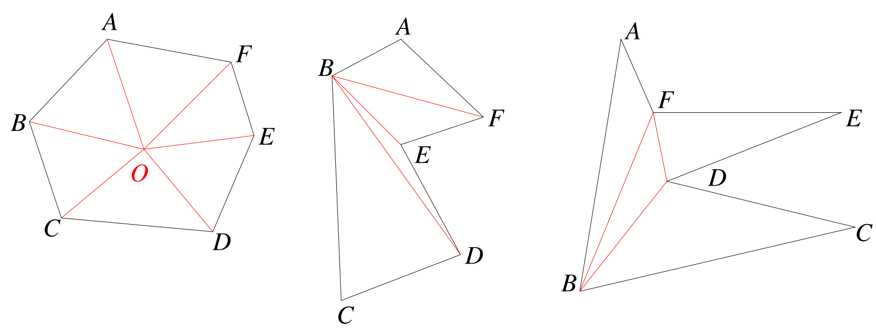
At this point, students need to be exposed to some general facts about *n-gons*, or *polygons*, including a correct definition. The general definition of a polygon requires the use of subscripts to denote the vertices. So for sixth grade, it is enough to define a pentagon and a hexagon and wave hands a bit. A **3-gon** is a triangle, and a **4-gon** is a quadrilateral. A **5-gon**, called a **pentagon**, is defined to be a geometric figure with 5 distinct points $A$, $B$, $C$, $D$, $E$ *in the plane*, together with the 5 segments $AB$, $BC$, $CD$, $DE$, and $EA$, so that *none of these segments intersects the others except at the endpoints as indicated*, i.e., $AB$ intersects $BC$ at $B$, $BC$ intersects $CD$ at $C$, etc. In symbols: the pentagon will be denoted by $ABCDE$. Here are some examples of pentagons.




In the same way, a **6-gon**, called a **hexagon**, is a geometric figure with 6 distinct points $A$, $B$, $C$, $D$, $E$, $F$ *in the plane*, together with the 6 segments $AB$, $BC$, $CD$, $DE$, $EF$, and $FA$ so that *none of these segments intersects the others except at the endpoints as indicated*, i.e., $AB$ intersects $BC$ at $B$, $BC$ intersects $CD$ at $C$, etc. In symbols: the hexagon will be denoted by $ABCDEF$. Here are some examples of hexagons:



In general, for any positive integer $n$, an ***n*-gon**, more commonly called a **polygon**, can be similarly defined. What we wish to observe is that, once we know how to compute areas of triangles, then the area of any polygon can be computed—at least in principle—through the process of **triangulation**. It is not necessary to define precisely, in sixth grade, what a triangulation is, because we just want to give students a general idea and, for this purpose, some picture-drawing is quite sufficient. As the name suggests, what we do is to partition any polygon into a collection of triangles which intersect each other at most on their boundaries. Since the areas of these triangles can be computed, we can apply repeatedly the additivity of area (see (d) on page 32) to get the area of the polygon itself. Let us illustrate with the above hexagons. Here are some of the possible triangulations.




Thus for the hexagon $ABCDEF$ on the left, its area is the sum of the areas of

$$\triangle OAB,\ \triangle OBC,\ \triangle OCD,\ \triangle ODE,\ \triangle OEF,\ \triangle OFA.$$

For the hexagon $ABCDEF$ in the middle, its area is the sum of the areas of

$$\triangle ABF,\ \triangle BCD,\ \triangle BDE,\ \triangle BEF.$$

As to the hexagon $ABCDEF$ on the right, its area is the sum of the areas of

$$\triangle ABF,\ \triangle BFD,\ \triangle BCD,\ \triangle DEF.$$

Be sure to take note of the fact that there are other possible triangulations in each case. For example, the hexagon on the left can also be triangulated by joining the vertex $A$ to $C$, $A$ to $D$, and $A$ to $E$. On the other hand, the triangulations of the other two hexagons illustrate the complications in obtaining a triangulation: while it can be done, there is no *simple algorithm* to always get it done.

Next, we revisit the volume of a rectangular prism. Recall that the assignment of a number to a (3-dimensional) solid, called its **volume** is conceptually *identical* to the case of area. We have the analogous assumptions on how volumes are assigned to solids (see page 25 ):

(a) The volume of a solid is always a number $\geq 0$.

(b) The volume of a *unit cube* (a rectangular prism whose *edges* all have length 1) is by definition 1 cubic unit.

(c) If two solids are *congruent*, then their volumes are equal.

(d) (*additivity*) If two solids have (at most part of) their boundaries in common, then the volume of the solid obtained by combining the two is the sum of their individual volumes.

Again, we will not define "congruent solids" precisely in sixth grade except to appeal to the intuitive idea that congruent solids are those with the "same size and same shape".

We want to show that, if a rectangular prism $\mathcal{R}$ has edge lengths equal to $\ell$, $w$ and $h$, *and $\ell$, $w$, $h$ are all fractions*, then the volume of $\mathcal{R}$ is the product of these three numbers, i.e.,

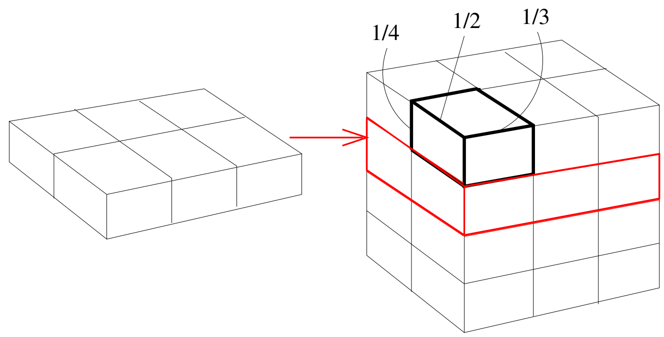
$$\text{volume } \mathcal{R} = \ell \times w \times h \tag{8}$$

In the interest of clarity, we will prove the special case $\ell = \frac{3}{2}$, $w = \frac{2}{3}$, and $h = \frac{5}{4}$. The reason for this expository decision is that the reasoning in this special case is in fact completely general. As in the case of area, we break up the reasoning into two steps.

($i$) The volume of a rectangular prism with edge lengths $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ is $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}$.

($ii$) The volume of a rectangular prism with edge lengths $\frac{3}{2}$, $\frac{2}{3}$, and $\frac{5}{4}$ is $\frac{3}{2} \times \frac{2}{3} \times \frac{5}{4}$.

For step ($i$), divide each of the three groups of parallel edges of a fixed unit cube into 2 equal parts, 3 equal parts, and 4 equal parts, respectively. For example, in the following picture, each of the group of four *vertical* edges is divided into 4 equal parts. Pass a plane through the corresponding division points, as shown. (Recall in the following picture that the cube is a unit cube and therefore each edge has length 1.)

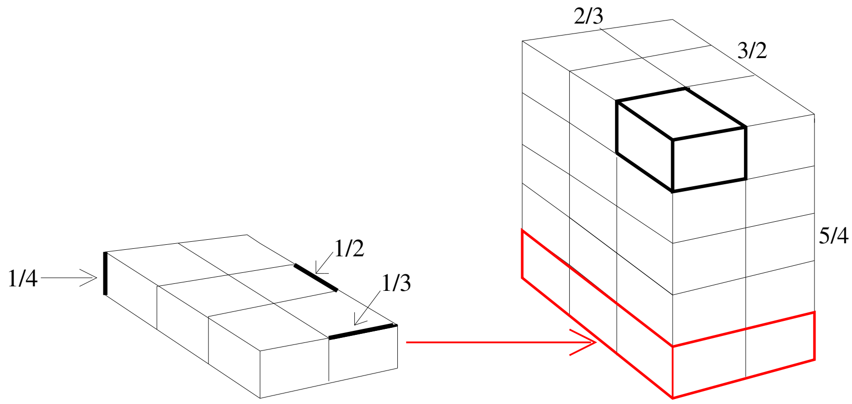



These planes partition the unit cube into twenty-four ($= 2 \times 3 \times 4$) congruent small rectangular prisms, and thus $2 \times 3 \times 4$ prisms of equal volume (see (c) above). By (d) above, the sum of the volumes of these $2 \times 3 \times 4$ prisms is equal to the volume of the unit cube, which is 1 (see (b)). Thus by the definition of a fraction, relative to a unit 1 that is the volume of a unit cube, the volume of one of these prisms, such as the thickened prism in the preceding picture, is the fraction

$$\frac{1}{24} = \frac{1}{2 \times 3 \times 4}\,.$$

Because by construction, the edges of the thickened rectangular prism in the preceding picture have lengths $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, we have finished explaining why ($i$) is correct.

Now part ($ii$). As with the case of area, we now ignore the unit cube and, instead, build a rectangular prism $P$ whose edge lengths are $\frac{3}{2}$, $\frac{2}{3}$, and $\frac{5}{4}$ by starting with the above thickened small rectangular prism with edge lengths $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. To this end, take three rows of these small prisms with two columns in each row; place them on a plane with the edges of length $\frac{1}{4}$ pointing up vertically, as shown in the picture below, to obtain a rectangular prism whose edge lengths are $\frac{3}{2}$, $\frac{2}{3}$, and $\frac{1}{4}$ . Now place 5 such layers on top of each other to get a rectangular prism whose edge lengths are now $\frac{3}{2}$, $\frac{2}{3}$, and $\frac{5}{4}$; this is the prism $P$ we are after, as shown below:




The volume of the rectangular prism with side lengths $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ is $\frac{1}{2\times3\times4}$, by part $(i)$. Since there are 30 $(= 3 \times 2 \times 5)$ of these prisms in $P$, the volume of $P$ is thus

$$\underbrace{\frac{1}{2\times3\times4} + \cdots\cdots + \frac{1}{2\times3\times4}}_{3\times2\times5} = \frac{(3\times2\times5)\times1}{2\times3\times4} = \frac{3}{2}\times\frac{2}{3}\times\frac{5}{4}$$

Note the critical fact about fraction multiplication that we have just used: the *product formula*, to the effect that $\frac{ac}{bd} = \frac{a}{b} \times \frac{c}{d}$ (see pp. 47–48 of H. Wu, Teaching Fractions According to the Common Core Standards for a proof).

The general explanation of equation (8) is entirely the same, and this can be seen from the fact that the preceding reasoning for the special case of $\ell = \frac{3}{2}$, $w = \frac{2}{3}$, and $h = \frac{5}{4}$ never makes use of the explicit values of $\frac{3}{2}$, $\frac{2}{3}$, or $\frac{5}{4}$.

In discussing area or volume, the role of property (c) (congruent figures give rise to the same area or volume) and property (d) (additivity) should be emphasized through the use of exercises that ask for the computations of areas of planar regions and volumes of solids formed by assembling congruent rectangles and rectangular solids of varying sizes. In particular, this will make students aware, at an early age, of *the fundamental importance of the concept of congruence.*

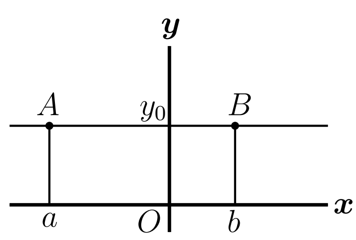
Next, we expand on the concept of coordinatizing the plane using rational numbers. Recall from grade 5 that our work with coordinates thus far has been limited to the first quadrant (page 27) because when a line passing through a point $P$ parallel to a coordinate axis intersects the other axis, we had to make sure that the point of intersection was a fraction. In other words, we had to make sure that the coordinates of $P$ are bigger than or equal to 0. In grade 6, however, we have rational numbers and we are no longer concerned with the location of the point of intersection of a line with a coordinate axis, so everything we said in grade 5 about coordinates, when suitably modified, will continue to hold. In particular, the analog of the interpretation of coordinates on page 28 is the following:

> *If the coordinates of a point $P$ are $(p, p')$, then $|p|$ (respectively, $|p'|$) is the distance of $P$ from the $y$ (resp. $x$) axis.*

The computation of the distance between two points in terms of their coordinates has to wait for the Pythagorean Theorem. There is a special situation, however, that allows for such a computation without the Pythagorean Theorem, and this is our next concern. Given a coordinate system in the plane, let $AB$ be a **horizontal segment** in the sense that it lies on a horizontal line. We want to compute the distance between $A$ and $B$, i.e., the length $|AB|$ of $AB$, in terms of the coordinates of $A$ and $B$. Since $AB$ is horizontal, the line $L$ containing $AB$ is parallel to the $x$-axis by definition. We know that the distances of $A$ and $B$ to the $x$-axis are equal; thus in the pictures below, $|Aa| = |Bb|$. Since opposite sides of a rectangle are equal (page 17), we know that the $y$-coordinates of $A$ and $B$ are equal, say $y_0$. Thus the coordinates of $A$ and $B$ must be $(a, y_0)$ and $(b, y_0)$, respectively. If $a$ or $b$ is equal to 0, for example if $a = 0$, then the distance between $A$ and $B$ is just $|b|$, by the definition of absolute value and there would be no need to compute. So we may assume $a$, $b$ are nonzero. Now there are two cases.

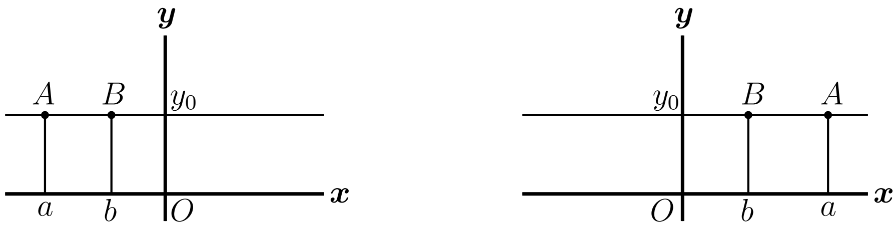
Case 1: $a < 0 < b$. Then

$$|AB| = (\text{distance from } a \text{ to } 0) + (\text{distance from } 0 \text{ to } b) = |a| + |b|.$$




Case 2: $a < b < 0$ or $0 < b < a$. Then

$$|AB| = (\text{distance from } a \text{ to } 0) - (\text{distance from } 0 \text{ to } b) = |a| - |b|.$$




What is worth pointing out is that, when students learn about the subtraction of rational numbers in grade seven, then both formulas above will be subsumed under one simple formula:

*The distance between two points* $(a, y_0)$ *and* $(b, y_0)$ *on a horizontal line*
$$= \ |a - b|$$

Such a formula gives a partial explanation of why it is desirable to learn about *absolute value* and *subtraction of rational numbers.*

The case of points on a vertical line can be treated in the same way. Thus:

The distance between two points $(x_0, a)$ and $(x_0, b)$ on a vertical line

$$= \begin{cases} |a| + |b| & \text{if } a < 0 < b \\ |a| - |b| & \text{if } \ a < b < 0 \ \text{ or } \ 0 < b < a. \end{cases}$$

In the terminology of 6th grade mathematics, we have:

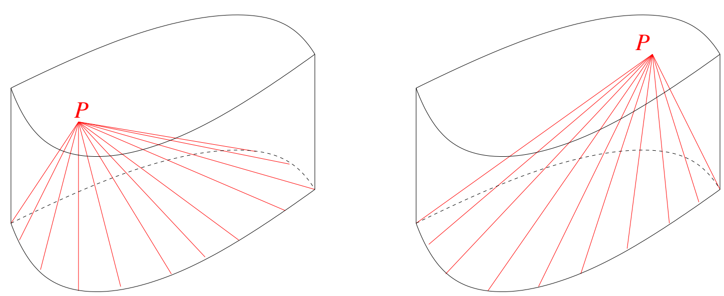
*The distance between two points* $(x_0, a)$ *and* $(x_0, b)$ *on a vertical line*
$$= \ |a - b|$$

Finally, a word about surface areas of solids. For rectangular prisms, the boundary of the prism consists of six rectangles. By the additivity property of area (see (d) of page 32 )[11], the total area of the boundary is the sum of the areas of these rectangles; since the equation ( 4 ) on page 32 gives the area of any one of these rectangles, one can add up these areas to get the total area. This total area is what is known as the **surface area** of the prism. *Nets* are sometimes used to help students visualize this surface to ease the computation of surface area. We will not go into any discussion of nets here for the simple reason that this is one example where interactive materials are much more instructive than verbal explanations. Click on the following link, for example, to see the nets of the most common solids:

[11] Strictly speaking, we need a generalized version of the additivity property as stated on page 32 because we are now talking about regions that are no longer lying in a plane.

In everyday life, there are also solids whose boundaries consist of geometric figures that are not rectangles. The simplest of these are *tetrahedra* and *pyramids*, and we define them as follows.

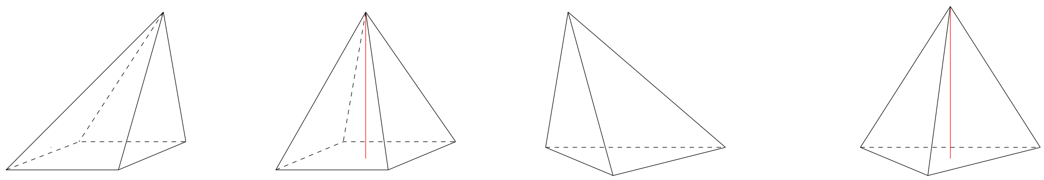
Let two *parallel planes*[12] be given. We may think of these planes as "horizontal" ones, with one called the *top plane* and the other the *bottom plane*. Let $P$ be a point in the top plane, then the collection of *all* the segments joining $P$ to points of a planar region $\mathcal{R}$ in the bottom plane is a solid $\mathcal{S}$. The point $P$ is called the **top vertex** of $\mathcal{S}$ and $\mathcal{R}$ the **base** of $\mathcal{S}$. Here are two examples of such a solid, where the base is fixed in the bottom plane but the vertex assumes two different positions in the top plane.



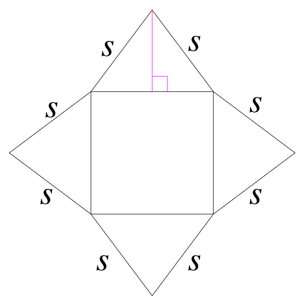

If the base $\mathcal{R}$ is a square, then the solid $\mathcal{S}$ is called a **pyramid** (see the two figures on the left below). If the top vertex of a pyramid lies on the line perpendicular to the base at the **center** of the square (the intersection of the diagonals), the pyramid is called a **right pyramid** (see figure second from left below). If the base is a triangle, the solid $\mathcal{S}$ is called a **tetrahedron** (see the two figures on the right below). If the base is an equilateral triangle and all the other three triangles on the boundary are congruent to the base, the tetrahedron is called a **regular tetrahedron** (see figure on the extreme right). It is known that the top vertex of a regular tetrahedron lies on the line perpendicular to the base passing through the **center**, which in this case is the point of intersection of the three *medians*[13] of the base.

[12] These are, by definition, planes that do not intersect, such as the planes containing the top and bottom of a rectangular prism.

[13] A median of a triangle is, by definition, a segment joining a vertex to the midpoint of the opposite side.



To compute the surface area of a right pyramid, for example, we can look at the *net* of the pyramid in this case. It is intuitively acceptable that the four edges above the base are all of the same length. Assuming this, if we cut the boundary of the right pyramid along all the edges coming out of the top vertex and then "flatten out" the boundary and lay it on the bottom plane of the base, we get the following plane figure whose eight boundary edges are all of the same length, say $s$:




If we denote the length of the side of the base square by $a$, then it is possible to compute the surface area of the right pyramid in terms of $s$ and $a$. But right now we don't have the necessary tools (the Pythagorean Theorem, which will be proved on page 102, or better yet, Heron's formula) so the only way the surface area can be computed is if the height of any one of the triangles (indicated by the magenta segment in the picture above) is also given. Let us say it is $h$. Then using equations (4) and (5) on pages 32 and 34, we get that the surface of this particular right pyramid is

$$a^2 + 4(\frac{1}{2}ha) \;=\; a(a+2h).$$

# GRADE 7

## **Geometry** 7.G

**Draw, construct, and describe geometrical figures and describe the relationships between them.**

1. Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

2. Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

3. Describe the two-dimensional figures that result from slicing three- dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

**Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.**

4. Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

5. Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.

6. Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

## Comments on teaching grade 7 geometry

We will concentrate on the three key ideas in this grade, namely:

the meaning of "scale drawing",

the need to develop students' geometric intuition, and

the area and circumference formulas of a circle.

The kind of problems mentioned in standards 5 and 6 of Geometry 7.G are by comparison more routine and therefore will get short shrift for now. (But keep in mind the distance formula in terms of rational number subtraction on page 44).

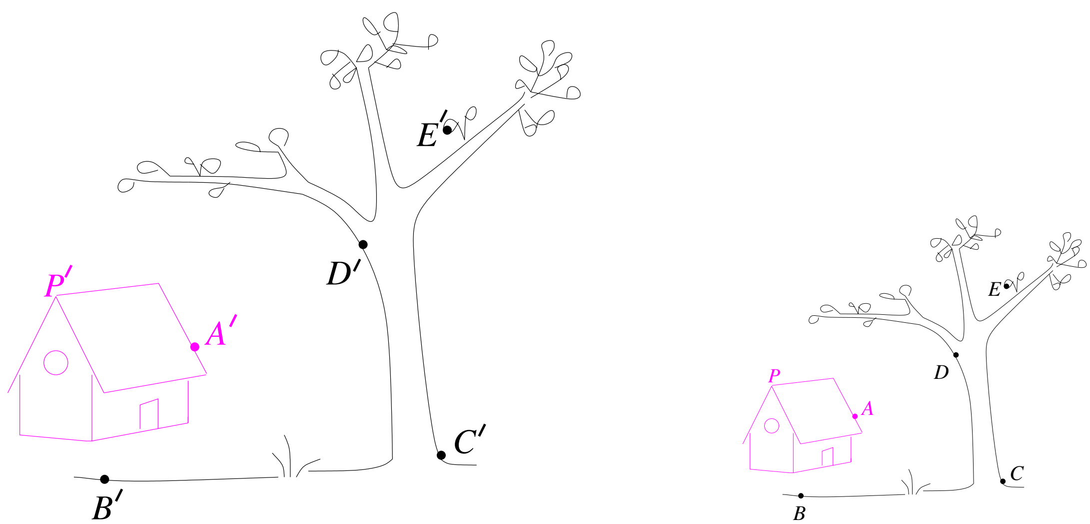
**Scale drawings**

Like many concepts in school mathematics, the concept of "scale drawing" is used indiscriminately and students routinely get tested about "scale drawings" on standardized tests without ever being told what a "scale drawing" is. The reason for this monstrous state of affairs is that, without the concept of similarity (which will be taken up in grade 8 and high school) it is not easy to give a simple definition of this concept. What we will do is to give an *intuitive* definition of "scale drawing" that is appropriate for grade 7, and leave the precise definition in terms of the concept of *similarity* to grade 8 (see page 95). Here is the easy part: if you are given a *scale drawing of a real-life object* (usually a car, a staircase, etc.), what it means is that you have to imagine that the three-dimensional object has been captured, in exactly the same size, as a *two*-dimensional image in a picture.[14] Then the so-called **scale drawing** refers to a reduced-size picture or an enlarged-size picture of that *two*-dimensional picture (but NOT of the three-dimensional object). We repeat: a scale drawing is either the reduction or the magnification of a *two*-dimensional picture, never the three-dimensional object itself.

Now the hard part: how to explain "reduction" or "magnification" in two dimensions. To this end, we adopt a procedure that is standard in mathematics: imagine

[14] Think of the gigantic pictures on some bill boards, for example.

that we have enlarged a picture (by using the "magnification" feature of a xerox machine or the same feature of geometry software) and then we compare it with the original picture to isolate some observable characteristics of the enlargement. We are not saying we already know what "enlargement" means; rather, we try to find out what it *could* mean by looking at something we all agree is already enlarged. Then we turn around and use these observed characteristics to *define* "magnification". So suppose we already have a picture of a house and a tree, to be called $\mathcal{S}$; see the picture on the right below. Let the left picture (to be denoted by $\mathcal{S}'$) be an "enlargement", in the intuitive sense, of $\mathcal{S}$. Then it is not difficult to see that, in the process of enlargement, a point in the right picture $\mathcal{S}$ goes to one and only one point in $\mathcal{S}'$ on the left. For example, the point $A$ in $\mathcal{S}$ goes to the point $A'$ in $\mathcal{S}'$, the point $B$ in $\mathcal{S}$ goes to the point $B'$ in $\mathcal{S}'$, etc.




Put differently, there is a precise *pairing* of each point of $\mathcal{S}$ on the right to one and only one point of $\mathcal{S}'$ on the left, and also of each point of $\mathcal{S}'$ on the left to one and only one point of $\mathcal{S}$ on the right. Thus we may symbolically represent this pairing as $A \leftrightarrow A'$, $B \leftrightarrow B'$, ..., $E \leftrightarrow E'$, and so on, for every point in either picture. Such pairing is an example of what is known as a *one-to-one correspondence*. But we can go further, it is intuitively clear that the paired points enjoy the property that the "enlarged distances are proportional to the original distances" in the following sense: if we use $|AB|$ to denote the distance between $A$ and $B$ as usual, then:

$$\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} = \ldots = \frac{|A'E'|}{|AE|} = \frac{|B'C'|}{BC|} = \cdots = \frac{|D'E'|}{|DE|},$$

and this proportional relationship is supposed to hold no matter what points $A$, $B$, ... in the original picture $\mathcal{S}$ are chosen. The number common to these ratios is called the **scale factor**, usually denoted by $\boldsymbol{r}$. Thus

$$\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} = \cdots = r$$

Such a proportional relationship between points in the two pictures will turn out to be the key feature that makes one picture an enlargement of the other. *This is a high-level mathematical insight* and you shouldn't worry about it if you didn't discover this yourself. All of us must learn from the wisdom of the past.

Obviously it is not important that these two pictures lie in the same plane; if they lie in different planes, we can also carry out the same discussion.

If the scale factor $r$ is bigger than 1, $\mathcal{S}'$ is called a **magnification** of $\mathcal{S}$; if $r < 1$, then $\mathcal{S}'$ is called a **reduction** of $\mathcal{S}$. What we have above is (visibly) a magnification. Of course if $r = 1$, then by definition, distances between points do not change; in this case, such a scale drawing turns out to be what we call a *congruence*. See the discussions on page 66 and page 81 .

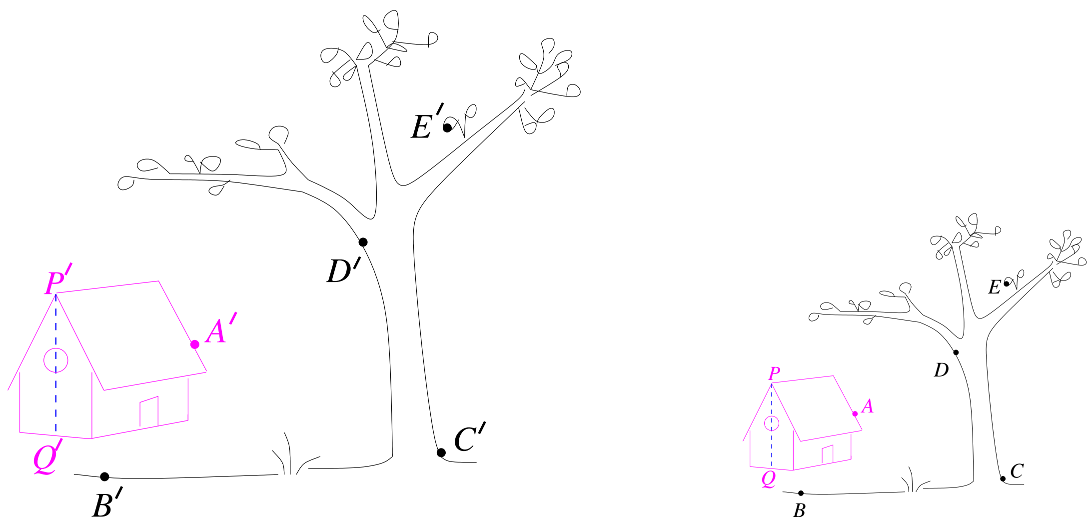
Now back to the intuitive definition of *scale drawing*. Two geometric figures, $\mathcal{S}$ and $\mathcal{S}'$, each lying in a plane, are said to be in **one-to-one correspondence** if there is a pairing between the points in $\mathcal{S}$ and the points $\mathcal{S}'$ so that, each point $P$ of $\mathcal{S}$ is paired with one and only one point $P'$ in $\mathcal{S}'$ and, likewise, each point $Q'$ in $\mathcal{S}'$ is paired with one and only one point $Q$ in $\mathcal{S}$.

Now suppose two geometric figures $\mathcal{S}$ and $\mathcal{S}'$ are given, each lying in a plane. We say $\mathcal{S}'$ is a **scale drawing of $\boldsymbol{\mathcal{S}}$ with scale factor $\boldsymbol{r}$** if there is a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{S}'$ so that, under the pairing of this one-to-one correspondence, the distance $|PQ|$ between *any* two points $P$ and $Q$ of $\mathcal{S}$ is related to the distance $|P'Q'|$ between the two corresponding points $P'$ and $Q'$ of $\mathcal{S}'$ by the equality

$$\frac{|P'Q'|}{|PQ|} = r\,.$$

Note that the essential feature of a scale drawing is that this equality holds true no matter what the points $P$ and $Q$ may be. One can check, by direct measurements, that for the two house-and-tree pictures above, the left picture is a scale drawing of the right picture with scale factor $\frac{5}{3}$.

A typical problem in this context is the following. Referring to the two pictures above, suppose we only know that the left picture is a scale drawing of the right, and $|B'D'| = 60$ mm, $|BD| = 36$ mm. If the *height of the house* on the right (defined to be the distance from the point $P$ to the point $Q$ which is the intersection of the vertical line from $P$ with the bottom edge of the house below $P$, as shown) is 15 mm, what is the height of the house on the left?




Knowing that the house on the left is a scale drawing of the house on the right, we know

$$\frac{\text{height of left house}}{\text{height of right house}} = r$$

where $r$ is the scale factor of the scale drawing. The reason is that if the vertical line from $P'$ meets the bottom edge below it at a point $Q'$, then $P$ corresponds to $P'$ and $Q$ corresponds to $Q'$ in the one-to-one correspondence, and therefore by the definition of a scale drawing, we have

$$\frac{|P'Q'|}{|PQ|} = r\,.$$

For the same reason,

$$\frac{|B'D'|}{|BD|} = r\,.$$

We therefore have the proportion

$$\frac{|P'Q'|}{|PQ|} = \frac{|B'D'|}{|BD|}$$

because both sides are equal to $r$. Since $|PQ| = 15$ mm, $|BD| = 36$ mm, and $|B'D'| = 60$ mm, we have

$$\frac{|P'Q'|}{15} = \frac{60}{36},$$

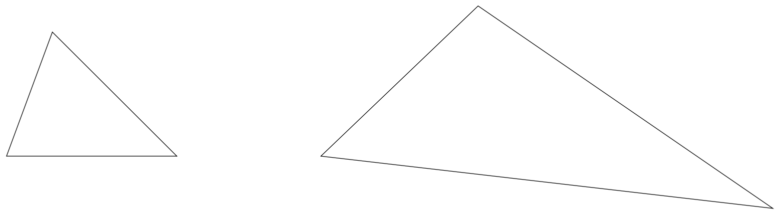
so that $|P'Q'| = 25$ mm, i.e., the height of the left house is 25 mm.

It should be remarked that the proportion $\frac{|P'Q'|}{|PQ|} = \frac{|B'D'|}{|BD|}$ is a *logical consequence* of the given hypothesis that we have a scale drawing and the fact that we have a precise definition of "scale drawing" in terms of the concept of "pairing". What we have done is to demonstrate how the teaching of mathematics can be done by (*i*) first giving students sufficient *precise* information (i.e., an explicit hypothesis), and (*ii*) asking them to use reasoning to make logical deductions from the given hypothesis.

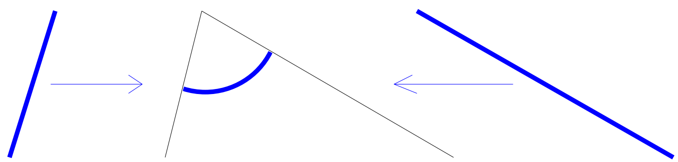
**Pedagogical Comments:** If we seem to be belaboring the point, it is because the way problems related to scale drawings are posed and solved in textbooks does not make any mathematical sense: there is no explanation of what a "scale drawing" means, yet students are asked to do scaling problems. This is simply defective mathematics education; in mathematics, one cannot reason on the basis of something not yet clearly defined. When students fail to preform in this environment, such inevitable failures then become topics in education research about students' weak basic skills and lack of problem-solving ability. (One such example is Case 3 in K. K. Merseth, *Windows on Teaching Math*, Teachers College Press, 2003.) At least in this instance, education research is treating the symptom and not the disease.

### Developing geometric intuition

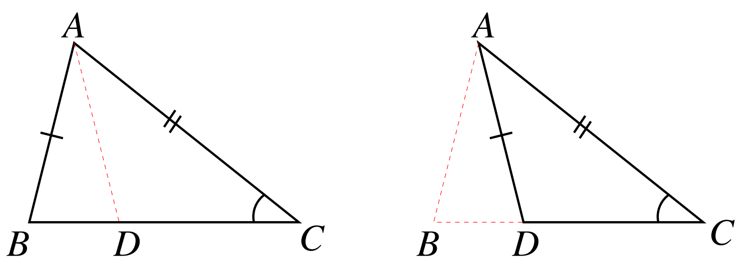
At this stage of students' learning trajectory, teachers should help students develop geometric intuition by drawing many pictures. Now "geometric intuition" is not a measurable quantity, so it is difficult to say whether one has "enough" geometric intuition or not. Nevertheless, one can say that, if asked to sketch a scale drawing with scale factor 2 of the triangle on the left, a student draws the triangle on the right, then her geometric intuition would need further development.



Students should also acquire the intuition, through extensive drawings, of why the classic criteria for triangle congruence—SAS, ASA, SSS—are correct, and why ASS, for example, is not. Consider SAS, once the angle and the lengths of its two adjacent sides are fixed, clearly the three vertices of the triangle are completely determined so that the congruence of two triangles both satisfying the conditions of SAS should not come as a surprise.



On the other hand, the following two triangles, $\triangle ABC$ and $\triangle ADC$, have two pairs of equal sides and a pair of equal angles, as indicated, but they are clearly "different", no matter how "different" is defined. So ASS cannot be true.




It remains to make a passing comment on Standard 7.G 3:

> Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

The general idea behind studying the slicing of a right rectangular prism by a plane is the standard method of approaching a geometric figure of a certain dimension by first studying its "sections" of one dimension lower. For example, we will get to know something about a triangle by looking at various line segments in it, e.g., angle bisectors, medians, etc., and we will also discover some properties of a circle by inspecting its various chords and radii. Therefore, to study a sphere or a cone, we look at their intersections with a plane, thereby obtaining circles in the former case, and circles, ellipses, parabolas and hyperbolas in the latter case. For this reason, looking at the plane sections of a rectangular prism is just an illustration of the attempt to develop students' three-dimensional geometric intuition but is not intended to be a subject to be pursued in depth. In any case, there are many websites with excellent graphics to illustrate what one can get by slicing a solid. See, for example, the solution to Problem C1 on Learning Math. [15]

**Circumference and area of a circle**

Although we have made use of circles since grade 4, this is the opportunity to formalize the terminology of the **center**, **radius** and **diameter** of a circle before embarking on a rather extensive discussion about the circle. Notice that, because of tradition, there is a tremendous abuse of language here: "circle" is of course the round curve, but "area of a circle" actually refers to the "area of the region inside the circle". Because of the very real danger of confusion, we hereby adopt a more reasonable terminology in this document: given a circle of radius $r$, we call the region inside the circle **the disk of radius $r$** and denote it by $\boldsymbol{D(r)}$. The circle itself we denote by $\boldsymbol{C(r)}$.

**Caution:** We will be talking about the "area of $D(r)$" and the "length of $C(r)$" presently, and there are some things we must be clear about at the outset. The concepts of the length and area of rectilinear objects were taken up in grade 6 (page 32). Because the concepts of "area" and "length" are part of everyday language, the familiarity with these words can easily delude us into believing that we *know* the meanings of the "area of a disk" and "length of a circle". A little thought would reveal that the *precise* meanings of the area and length of *curvy* objects such as the

[15] Reminder: this is an active link. Just click on it.

disk or the circle are anything but simple, and so long as we need precise meanings of the concepts we use in mathematics, we are forced to admit that we really don't know what these concepts mean. A main goal of the CCSSM in grade 7 is to take a first step in getting to know these sophisticated *mathematical* concepts. Anticipating the discussion on page 57, we will freely avail ourselves of these terms for the time being.

The length of $C(r)$ is called its **circumference**. The fundamental relation between the area of the disk and the circumference is this:

$$\text{area of } D(r) \;=\; \frac{1}{2}\,(\text{circumference of } C(r)) \cdot r \tag{9}$$

We will give an informal derivation of equation (9) presently. However, we will assume equation (9) for the moment and proceed to show why this equation is fundamental. The following sequence of derivations may not be well-known in school mathematics, but it is simple as well as mathematically enlightening; it deserves to gain a firm foothold in the school curriculum.

First, we introduce the number $\pi$ as the area of the **unit disk**, i.e., the disk of radius 1. In other words,

$$\pi \;\stackrel{\text{def}}{=}\; \text{area of } D(1)$$

**Caution:** Here, we are giving the precise meaning of $\pi$ for the first time as the area of the unit disk. There is an inherent danger that, because you have heard the number $\pi$ mentioned so often in daily conversations or popular writings that you begin to *imagine* you already know what it is. In that case, you would look at the equality, $\pi = \text{area of } D(1)$, as something that needs an explanation. But no explanation can be given in this case because, up to this point, we *do know what* $\pi$ *is* but are introducing $\pi$ for the first time as the symbol that stands for the area of the unit disk $D(1)$. However, the usual relationship of $\pi$ with the circumference (see equation (12) below) will then become something whose truth we must explain.

Our job now is to go on from here and get to know $\pi$ just a little bit. First, we will show informally that for all $r > 0$,

$$\text{area of } D(r) \;=\; r^2 \text{ (area of } D(1)). \tag{10}$$

Since the area of $D(1)$ is now denoted by $\pi$, we get from equation ( 10 ) the well-known area formula of a disk:

$$\text{area of } D(r) \;=\; \pi r^2 \tag{11}$$

But by equation ( 9 ), $\pi r^2 = \frac{1}{2}(\text{circumference of } C(r))\cdot r$. The following circumference formula now follows immediately:

$$\text{circumference of } C(r) \;=\; 2\pi r \tag{12}$$

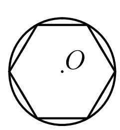
Recall that the number $2r$ (twice the radius) is called the *diameter* of the circle of radius $r$. Therefore equation ( 12 ) implies that

$$\pi \;=\; \frac{\text{circumference of } C(r)}{\text{diameter}}$$

This is the usual interpretation of the number $\pi$, but in terms of pedagogy, there is an excellent reason why $\pi$ should be defined as the area of the unit disk as we have done. The reason will be explained in the high school course (page 151 ).

We now give the informal arguments for equations ( 9 ) and ( 10 ). To this end, the first order of business is to give meaning to the concepts "area of a disk" and "length of a circle". They may be intuitive, but they turn out to be mathematically very subtle. We have to be content with oversimplified versions of the correct definitions of the area of a disk and the length of a circle.

A polygon (see page 39 ) is said to be **regular** if its vertices lie on a circle and all its sides are equal;[16] we then also say that the regular polygon is **inscribed** in that circle. Let $P_n$ be a regular *n-gon* (see page 39 ) inscribed in $C(r)$. The following shows the case of $n = 6$ for a disk with center $O$:

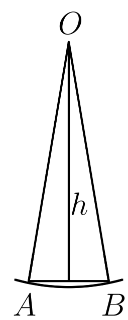



[16]This is not the standard definition of a regular polygon, but is equivalent to it. Moreover, this may be the simplest definition for use in the school classroom.

It is intuitively clear that as $n$ gets larger and larger without bounds, $P_n$ gets closer and closer to $C(r)$ and the region inside $P_n$ becomes virtually indistinguishable from the disk $D(r)$. Because the area of $P_n$ is something we know how to compute through triangulation (see page 39), it makes sense then to *define* the **area of $\boldsymbol{D(r)}$** to be the *limit* of the area of (the region inside) $P_n$ as $n$ increases without bounds. Here we have to be vague about the precise meaning of "limit", except to point out that it is one of the central concepts of advanced mathematics.

Next we explain the meaning of "length of $C(r)$". The **perimeter** of a polygon is by definition the sum of the lengths of its sides. Then using the notation of the preceding paragraph, we define the **length of $\boldsymbol{C(r)}$** to be the "limit" of the perimeter of $P_n$ as $n$ gets arbitrarily large, in the sense that as $n$ gets larger and larger, the perimeter of $P_n$ will be observed to get closer and closer to a certain number and this number is what we mean by the "length of $C(r)$".

We can now compute the length of $C(r)$. Letting as usual the center of $D(r)$ be $O$, let us join $O$ to two consecutive vertices $A$ and $B$ of $P_n$ to form a triangle $OAB$:




From $O$ we drop a perpendicular to side $AB$ of $\triangle OAB$ and we denote the length of this perpendicular by $h$. If we denote the length of the segment $AB$ by $s_n$ (the subscript $n$ here indicates that it is the length of one side of $P_n$), then the area of $\triangle OAB$ is $\frac{1}{2}s_n h$, by equation (6) on page 34. Now remember that $\triangle OAB$ is one of $n$ congruent triangles which "pave" $P_n$, so the area of $P_n$ is $n(\frac{1}{2}s_n h)$, by virtue of the additivity of area (see (c) and (d) of page 32). We rewrite it as

$$\text{area of } P_n \;=\; \frac{1}{2}(ns_n)h$$

But $ns_n$ is the perimeter of $P_n$ because the boundary of $P_n$ consists of $n$ equal sides ($AB$ being one of them). So as $n$ increases to infinity, $ns_n$ becomes in the limit the length of $C(r)$, by the definition just given. Moreover, as $n$ increases to infinity, $AB$ gets smaller and smaller and therefore closer and closer to the circle $C(r)$. Consequently the perpendicular from $O$ to $AB$ becomes $OA$ in the limit and therefore $h$ becomes the radius of $C(r)$, which is $r$. Needless to say, the area of $P_n$ becomes the area of $D(r)$, by the above definition. Letting $n$ increase to infinity, the preceding equation then becomes

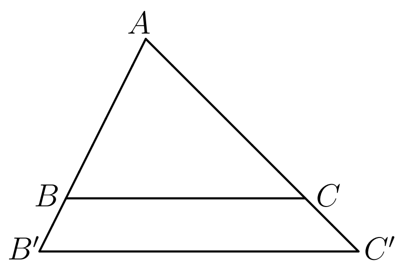
$$\text{area of } D(r) \;=\; \frac{1}{2}\,(\text{length of } C(r)) \cdot r,$$

which is the same as equation (9).

Next, we tackle equation (10).

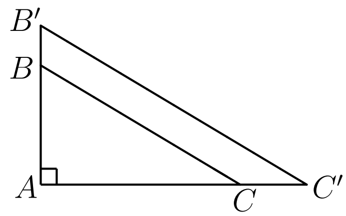
The heart of the matter in this case is the following fact about triangles: If $B'$ and $C'$ are points on the rays $AB$ and $AC$ of $\triangle ABC$ so that, for some positive number $r$, we have $|AB'| = r|AB|$ and $|AC'| = r|AC|$, then:

$$\text{area of } \triangle AB'C' \;=\; r^2 \cdot \text{area of } \triangle ABC \tag{13}$$




Note this feature of equation (13): nothing is said about the third sides $BC$ and $B'C'$. (It will turn out that also $|BC'| = r|BC|$, but that would belong in a high school course in geometry.)

Without some facts from similar triangles, we cannot give a complete explanation of equation (13). However, we will provide at least the proof of a special case, and will try to make this equation appear entirely reasonable by suggesting two hands-on experiments. The special case in question is that of a right triangle $\triangle ABC$ so that $\angle A$ is a right angle. For the sake of definiteness, we assume $r > 1$. Let $B'$ and $C'$ be points on the rays $AB$ and $AC$, respectively, of $\triangle ABC$ so that $|AB'| = r|AB|$ and $|AC'| = r|AC|$ (shown here for the case $r > 1$):

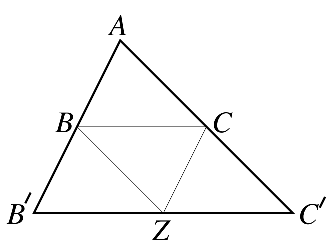


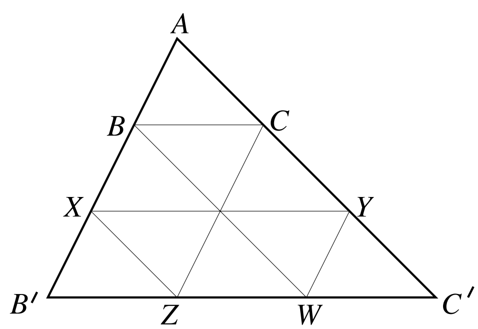

We now prove equation (13) for this case as follows. By equation (5), we have

$$\begin{aligned}
\text{area of } \triangle AB'C' &= \frac{1}{2}\,|AB'|\cdot|AC'| \\
&= \frac{1}{2}\,r\,|AB|\cdot r\,|AC| \\
&= \frac{1}{2}\,r^2\,|AB|\cdot|AC| \\
&= r^2\cdot\text{area of } \triangle ABC,
\end{aligned}$$

as desired.

Next we give an intuitive argument for the case of $r = 2$ and $r = 3$ in equation (13) when $\angle A$ is not a right angle. For the case of $r = 2$, we refer to the left picture below.




If we let $Z$ be the midpoint of $B'C'$ and join $BZ$, $CZ$, and $BC$, then a well-drawn picture would show quite convincingly that $\triangle AB'C'$ is now "paved" by four triangles each of which is congruent to $\triangle ABC$. By (c) and (d) of page 32, we have that the area of $\triangle AB'C'$ is 4 times the area of $\triangle ABC$, i.e.,

$$\text{area of } \triangle AB'C' = 2^2\cdot\text{area of } \triangle ABC.$$

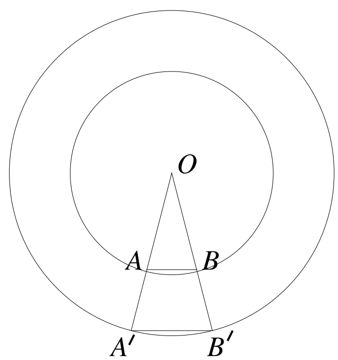
This then gives a visual verification of equation (13) for $r = 2$. In the case of $r = 3$, we refer to the above picture on the right. If we let $X$ be the midpoint of $BB'$, $Y$ be the midpoint of $CC'$ and let $Z$ and $W$ trisect $B'C'$, then again a well-drawn figure will show that the segments $BW$, $CZ$, and $XY$ meet at a point as shown, and that $\triangle AB'C'$ is paved by nine triangles each of which is congruent to $\triangle ABC$. Again, by (c) and (d) of page 32, we have that the area of $\triangle AB'C'$ is 9 times the area of $\triangle ABC$, i.e.,

$$\text{area of } \triangle AB'C' \;=\; 3^2 \cdot \text{area of } \triangle ABC.$$

This gives a visual verification of equation (13) for $r = 3$.

The general reasoning for equation (13) will have to come from the study of similarity in a high school course on geometry.

We now use equation (13) to verify equation (10).




Given a circle of radius $r$; for convenience of drawing, let $r > 1$. Then with the same center $O$, we draw the circles $C(r)$ and $C(1)$, which are the larger circle and the smaller circle, respectively, in the picture above.

Let $AB$ be one side of a regular $n$-gon $P_n$ inscribed in $C(1)$, and let the rays $OA$ and $OB$ intersect $C(r)$ at $A'$ and $B'$, respectively. It is then easy to believe that $A'B'$ is also one side of a regular $n$-gon $P'_n$ inscribed in $C(r)$. The reasoning on page 57 shows that

$$\text{area of } P_n \;=\; n \cdot \text{area of } \triangle OAB.$$

Similarly, we have

$$\text{area of } P'_n \;=\; n \cdot \text{area of } \triangle OA'B'.$$

But because of equation (13) and the fact that $|OA'| = r = r \cdot 1 = r \cdot |OA|$, we have

$$\text{area of } \triangle OA'B' \;=\; r^2 \cdot \text{area of } \triangle OAB,$$

so that we now have:

$$\begin{aligned}
\text{area of } P_n' &= n \cdot \text{area of } \triangle OA'B' \\
&= n \cdot r^2 \cdot \text{area of } \triangle OAB \\
&= r^2 \cdot (n \cdot \text{area of } \triangle OAB) \\
&= r^2 \cdot \text{area of } P_n
\end{aligned}$$

As $n$ increases without bound, the area of $P_n'$ becomes the area of $D(r)$ by the definition of the latter, while the area of $P_n$ becomes the area of $D(1)$, which is $\pi$. Therefore, when $n$ increases without bound, the preceding equation becomes

$$\text{area of } D(r) \;=\; r^2 \cdot \text{area of } D(1) \;=\; \pi r^2,$$

which then shows equation ( 10 ) is correct.

# GRADE 8

## **Geometry** 8.G

**Understand congruence and similarity using physical models, transparencies, or geometry software.**

1. Verify experimentally the properties of rotations, reflections, and translations:
a. Lines are taken to lines, and line segments to line segments of the same length.
b. Angles are taken to angles of the same measure.
c. Parallel lines are taken to parallel lines.

2. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

3. Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

4. Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

5. Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.

**Understand and apply the Pythagorean Theorem.**

6. Explain a proof of the Pythagorean Theorem and its converse.

7. Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

8. Apply the Pythagorean Theorem to find the distance between two points in a coordinate system. Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

9. Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

### **Expressions and Equations** 8.EE

6. Use similar triangles to explain why the slope $m$ is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at $b$.

## **Goals of eighth grade geometry**

There are six in all, and they are intended to be achieved with an emphasis on the intuitive *geometric* content through the ample use of hands-on activities:

1. An *intuitive* introduction of the concept of *congruence* using rotations, translations, and reflections, and their compositions (**page 65**)
2. An *intuitive* introduction of the concepts of *dilation* and *similarity* (**page 84**)
3. An informal argument for the angle-angle criterion (AA) of similar triangles (**page 95**)
4. Use of AA for similarity to give a correct definition of the slope of a line and prove the Pythagorean Theorem (**page 99**)

5. An informal argument that the angle sum of a triangle is 180 degrees (**page 104**)

6. Introduction of some basic volume formulas (**page 105**)

Grade 8 is a pivotal grade in the CCSSM. Unlike other standards of the past two decades, they provide proper preparations for eighth graders to learn beginning algebra and they also furnish them with a firm foundation for the more formal development of high school geometry. The key fact regarding the former is that the CCSSM equip students with a working knowledge of similar triangles so that a correct *definition* of the slope of a line can be given. Contrast this with the definition-by-rote of slope given in TSM (see page 2) as "rise over run". Because the concept of slope underlies the discussion of linear equations and functions—which are the mainstay of the curriculum of introductory algebra—the importance of this curricular decision cannot be overstated. By emphasizing the intuitive content of geometry, the eighth grade standards of the CCSSM also manage to maximize students' chances of learning *geometry* in a high school course rather than learning how to memorize two-column proofs. At this point, the reader may find it profitable to read the pedagogical comments on page 150 and page 154 to get a coherent overview of the geometry curriculum in the CCSSM.

It is worth pointing out, explicitly, that the discussion of geometry in this grade includes also Standard 6 from *Expressions and Equations* because a principal application of similar triangles—the definition of the slope of a line—is hidden in that standard.

Since the release of the CCSSM in June of 2010, very little has been written on such an approach to geometry that is mathematically sound. For this reason, the following commentary on the standards of Grade 8 will be more detailed than those in the earlier grades. If even more details are needed, one may consult the following (recall: these are active links):

H. Wu, Pre-Algebra (Chapters 4–7)

H. Wu, Teaching Geometry in Grade 8 and High School According to the Common Core Standards

## 1. Basic rigid motions and congruence



### Overview

The main new ideas in the eighth grade are the concepts of translations, reflections, rotations, and dilations in the plane. The first three—translations, reflections, rotations—are collectively referred to as the **basic rigid motions**, and they will be discussed in this section. Dilation will be explained in the next.

Before proceeding further, we note that the basic rigid motions are quite subtle concepts whose precise definitions should be preceded by other advanced concepts, but for grade eight,

> *it is the intuitive geometric content of the basic rigid motions that needs to be emphasized.*

Fortunately, the availability of abundant teaching tools makes it easy to convey this intuitive content. In this document, we will make exclusive use of overhead projector transparencies as an aid to the explanation of basic rigid motions. This expository decision should be complemented by two remarks, however.

The first is to caution against the premature use of computer software for learning about basic rigid motions. While computer software will eventually be employed for the purpose of geometric explorations, it is strongly recommended, on the basis of professional judgment and experience, that students begin the study of basic rigid motions with transparencies rather than with computer software. Primitive objects such as transparencies afford to beginners the advantage of direct control so that unforeseen software-related subtleties do not interfere with the learning process. Let students first be given an extended opportunity to gain the requisite geometric intuition through direct, tactile experiences *before* they approach the computer. A second remark is that if you believe some other manipulatives are more suitable for your own classroom needs and *you are certain that these manipulatives manage to convey the same message*, then you should feel free to use them.

In the following, a **basic rigid motion** will mean a translation, a reflection, or a rotation in the plane; a preliminary definition of each will be given below. In high school, these concepts will be precisely defined. In general terms, a basic rigid motion is a rule $F$ that assigns to each point of the plane $P$ another point of the plane,[17] to be denoted by $\boldsymbol{F(P)}$, and this rule will be described separately below for each of translation, reflection, and rotation. Before doing that, we are going to introduce a piece of terminology for the sake of clarity:

> instead of saying *"a basic rigid motion $F$ assigning $F(P)$ to $P$"*,
>
> we will sometimes say: *"a basic rigid motion $F$ **moves** a point $P$ to another point $F(P)$."*

This terminology expresses the intuitive content of "motion" better than the original language of a "rule of assignment". It may be instructive at this point to give the historical origin of the term "rigid motion": it literally means moving the points from one part of the plane to another "rigidly" so that the relative positions of the points are unchanged, i.e., the distances between points remain the same in the motion. The modern terminology for "rigid motion" is **isometry**. Still speaking intuitively, the way an arbitrary "rigid motion" moves the points of the plane around may be hard to describe precisely, but it is known that every such rigid motion can be achieved also by a sequence of translations, rotations, and reflections.[18] Because the behavior of translations, rotations, and reflections can be easily understood through the manipulation of transparencies, the behavior of an arbitrary "rigid motion" becomes understandable. This is why translations, rotations, and reflections are called the *basic* rigid motions. We now proceed to show that the motion of each of these three basic rigid motions is very easy to visualize.

---

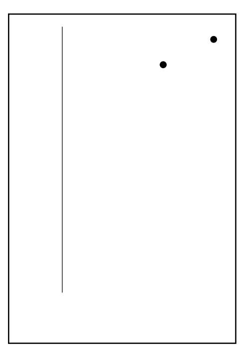
[17] This should remind you of the definition of a function of one variable.

[18] For further details, see the forthcoming volume, H. Wu, *Mathematics of the Secondary School Curriculum, II.*

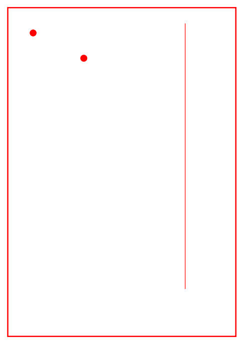
In the following, we will describe how to move a transparency over a piece of paper in order to demonstrate the effect of a basic rigid motions on the points in the plane. In reading the description, please bear in mind that, ***in the classroom, a face-to-face demonstration with transparency and paper is far easier to understand than the clumsy verbal description given below.*** In order to compensate for this clumsiness, we are going to provide animations on pp. 68, 70, 73, 77, 82, 102, and 144 to give an idea of such a face-to-face demonstration. Moreover, there are digital *document cameras* at the time of writing (October, 2013) that are capable of making a teacher's demonstrations with transparencies easily visible to the whole class or even to a large audience. Thus the potential of hands-on activities in the learning of geometry is far from being exhausted.

**Definitions of the basic rigid motions**

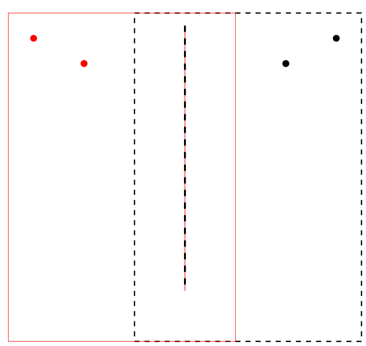
We begin with the basic rigid motion called a *reflection*. Let a line $L$ be given. Then the **reflection $\boldsymbol{R}$ across $\boldsymbol{L}$** moves all the points in the plane by "reflecting" them across $L$ as if $L$ were a mirror. The line $L$ is called the **line of reflection** of $R$. For definiteness, let us say $L$ is a vertical line and let us say two arbitrary points in the plane are given. We now describe how $R$ moves these points. Let the line $L$ and the dots be drawn on a piece of paper in black, as in the picture below. The black rectangle indicates the border of the paper.



Note that *we are using a finite rectangle to represent the plane which is infinite in all directions, and a finite segment to represent a line which is infinite in both directions.* With this understood, trace the line and the points exactly on a transparency (of exactly the same size as the paper, of course) using a different color, say red. Keeping the paper fixed, now flip the transparency across the vertical line (interchanging left and right) while keeping every point on the red vertical line on top of the same point on the black vertical line. The position of the red figure of two dots and the red line on the transparency now represents how the original figure has been reflected. (The red rectangle indicates the border of the transparency.)



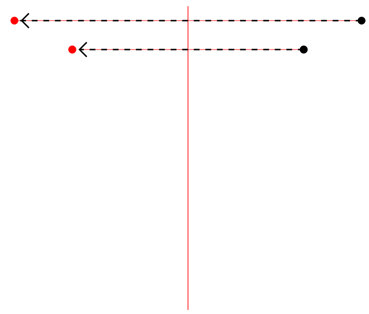
One should be aware of how the reflected figure compares with the original, so we draw them together below. Since the black figure represents where the red figure *used to be*, and is therefore just background information, we have drawn the black rectangular border and the black vertical line in dashed lines to emphasize this fact.



The following animation by Sunil Koswatta realizes this definition of a reflection and will be helpful to beginners:

http://www.harpercollege.edu/˜skoswatt/RigidMotions/reflection.html

We now look at the rule of assignment of the reflection $R$ which moves the points in the plane represented by the black dots to the corresponding points in the plane represented by the red dots.



Of course every point on the vertical line remains unmoved. In partial symbolic notation, we have:

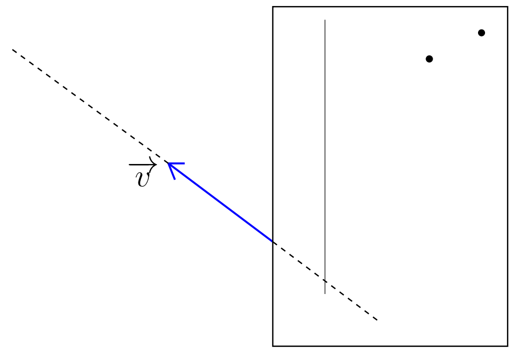
$$R(\text{upper black dot}) = \text{upper red dot}, \qquad R(\text{lower black dot}) = \text{lower red dot}$$

It goes without saying that $R$ moves *every* point in the plane not lying on $L$ to the "opposite side" of $L$, and the two points above are meant to merely suggest what happens in general. A key observation is that, if we take two points $A$ and $B$ in the plane and

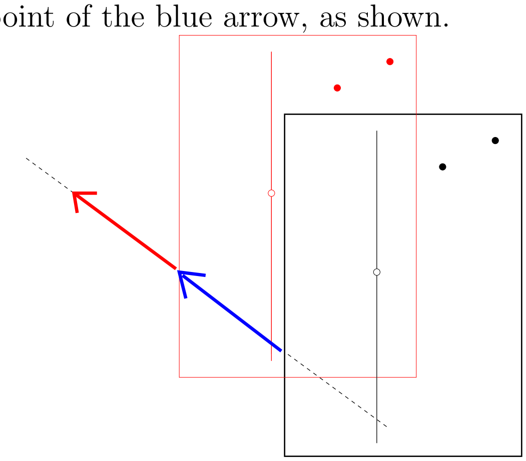
> *if $R$ moves the points $A$ and $B$ to $R(A)$ and $R(B)$, respectively, then the distance between $A$ and $B$ is equal to the distance between $R(A)$ and $R(B)$.*

This is because the red dots are the *exact* replicas (on the transparency) of the black dots on the paper, and so the distance between the red dots is exactly the same as that between the black dots. We refer to this property of $R$ as the **distance-preserving property.**

Next, we define the basic rigid motion of a **translation along a given vector** $\overrightarrow{v}$. A **vector** is a segment in the plane together with the designation of one of its two endpoints as a **starting point**; the other endpoint will be referred to simply as **the endpoint**. Pictorially, the arrowhead on the endpoint of a vector will distinguish it from the starting point. Let us continue with the same picture of a vertical line with two dots on a piece of paper, and we keep the paper fixed, as before. We are going to define the translation $T$ along a given blue vector, to be called $\overrightarrow{v}$, as shown.



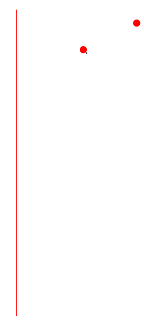

We copy the vertical line, the two dots, and the vector on a transparency in red; in particular, the copy of $\overrightarrow{v}$ on the transparency will be referred to as the *red* vector. Let the line containing the blue vector be denoted by $\ell$ (this is the slant dashed line in the picture above). We now **slide the transparency along $\overrightarrow{v}$**, in the sense that we move the transparency in the direction of $\overrightarrow{v}$ so that the red vector on the transparency remains in the line $\ell$, and so that the starting point of the red vector rests on the endpoint of the blue arrow, as shown.



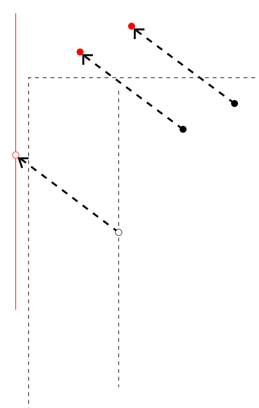
The whole red figure is seen to move "in the direction of $\overrightarrow{v}$ and by the same distance". Then by definition, $T$ moves the black dots to the red dots. Precisely, the rule of assignment of $T$ moves the point *in the plane* represented by the upper (respectively, lower) black dot to the point in the plane represented by the upper (respectively, lower) red dot. The following animation of essentially this translation by Sunil Koswatta would be helpful to a beginner:

http://www.harpercollege.edu/˜skoswatt/RigidMotions/translation.html

If we draw the translated figure by itself without reference to the original, it is visually indistinguishable from the original:



So we put in the black figure as background information to show where the red figure *used to be*. Then $T$ moves the points represented by the black dots to the corresponding points represented by the red dots, and moves each point in the black vertical line to a point in the red vertical line. The dashed arrows are meant to suggest the assignment.

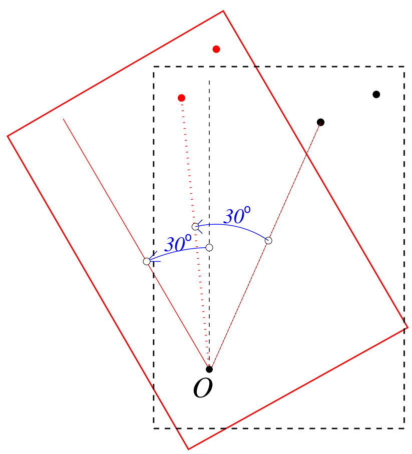


For exactly the same reason as in the case of a reflection, *a translation is distance-preserving*: if $A$ and $B$ are any two points in the plane and if $T$ assigns the points $T(A)$ and $T(B)$ to $A$ and $B$, respectively, then the distance between $A$ and $B$ is equal to the distance between $T(A)$ and $T(B)$.

We make an observation about translations: If $\overrightarrow{0}$ is the **zero vector**, i.e., the vector with 0 length, which is a point, the translation along $\overrightarrow{0}$ then leaves every point unchanged. This is the **identity** basic rigid motion, usually denoted by $I$.

Thus $I(P) = P$ for every point $P$. (This is the transformation analog of the identity function of one variable: $f(x) = x$ for every number $x$.)

Finally, we define the basic rigid motion called the **rotation *Ro* around a given point *O* of a fixed degree**. The point $O$ is called the **center of rotation** of $Ro$. The center $O$ could be any point, but for definiteness, let it be the lower endpoint of the vertical line segment we have been using, and let the rotation be 30 degrees *counterclockwise* around this point (one could also do a clockwise rotation). Again, we trace the vertical line segment and the two dots on a transparency in red. Then we pin the transparency down at the center of our chosen rotation, which is the lower endpoint of the segment and (keeping the paper fixed of course) rotate the transparency counterclockwise 30 degrees, i.e., so that the angle between the black segment and the red segment is 30 degrees. In the picture below, the rotated figure is superimposed on the original figure and, as usual, the red rectangle represents the border of the transparency. By definition, the rotation moves the upper black dot to the upper red dot, and the lower black dot to the lower red dot.




Observe that the angle formed by the ray from the center of rotation to a black dot and the ray from the center of rotation to the corresponding red dot is also 30 degrees.

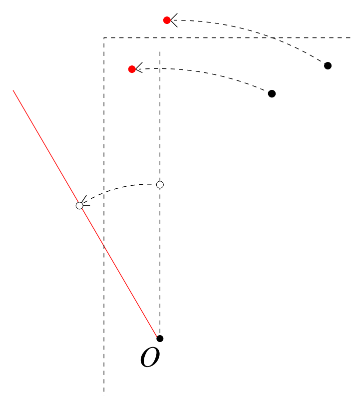
So far we have discussed rotations of positive degrees, and they are, by definition, the counter-clockwise rotations. A rotation of negative degree is defined exactly as above, except that the transparency is now rotated *clockwise*. Therefore, a rotation of $-35$ degrees around a point $B$ means we rotate the transparency *clcokwise* 35 degrees around the point $B$.

The following two animations by Sunil Koswatta show how a rotation of 35 degrees (respectively, $-35$ degree) rotates a geometric figure consisting of three points and an angle whose vertex is the center of the rotation:

http://www.harpercollege.edu/˜skoswatt/RigidMotions/rotateccw.html

http://www.harpercollege.edu/˜skoswatt/RigidMotions/rotatecw.html

We now draw the rotated figure as a geometric figure in the plane. Again we emphasize that the dotted black figure is provided only as background information; it serves as a reminder of where the red figure *used to be*. The dotted arcs indicate the rule of assignment by $Ro$.

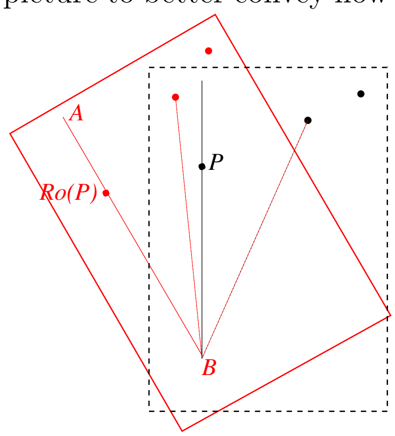



*A rotation is distance-preserving,* for the usual reason that what is on the transparency is nothing but an exact replica of what is on the paper. Note that a rotation of 0 degrees is also the identity basic rigid motion $I$.

### Motions of entire geometric figures

We now introduce some terminology to facilitate the ensuing discussion. Given a basic rigid motion $F$, it assigns a point $F(P)$ to a point $P$ in the plane. We say $F(P)$ is the **image** of $P$ under $F$, or that $F$ **maps** $P$ to $F(P)$.

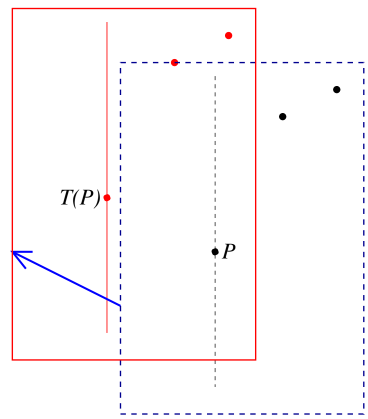
We have given a description of how a reflection, a translation, or a rotation moves each point. What we do next is to examine a bit how a basic rigid motion moves, not just a point, but *a whole geometric figure,* in the following sense. Given a geometric figure $\mathcal{S}$ in the plane, then each point $P$ in $\mathcal{S}$ is mapped by $F$ to another point $F(P)$. Now focus entirely on $\mathcal{S}$ and observe what the total collection of *all* the points $F(P)$ looks like *when $P$ ranges over all the points of $\mathcal{S}$, and only the points in $\mathcal{S}$.* For understandable reasons, we denote such a collection by the symbol $\boldsymbol{F(\mathcal{S})}$ and call it the **image of $\boldsymbol{\mathcal{S}}$ by $\boldsymbol{F}$**. (We also say **$\boldsymbol{F}$ maps $\boldsymbol{\mathcal{S}}$ to $\boldsymbol{F(\mathcal{S})}$.**) For example, in the preceding picture of the 30 degree counterclockwise rotation $Ro$ around the lower endpoint of the vertical segment, let the lower endpoint be denoted by $B$ and let $\mathcal{S}$ denote this vertical segment.[19] Then $Ro(\mathcal{S})$ is the red segment $AB$, as shown. It makes a 30 degree angle with $\mathcal{S}$. (Again, the red rectangle represents the border of the transparency, and it is left in the picture to better convey how the plane has been rotated.)




Now, let $T$ be the translation along the blue vector that we encountered earlier on page 70 , and if $\mathcal{S}$ continues to denote the same vertical segment, then $T(\mathcal{S})$ becomes the red segment which is now parallel to $\mathcal{S}$ rather than making a 30 degree angle with $\mathcal{S}$ at its lower endpoint. (In the picture below, the fact that the upper horizontal

[19]Notice that in this instance, we are taking the picture literally and regard the segment for what it really is: a segment. By contrast, we have, up to this point, used this segment to represent the whole vertical line.

dashed line passes through the lower red dot is entirely accidental.)




We see that by looking at the image of a *segment*, we obtain at a glance a fairly comprehensive understanding of the basic difference between a rotation and a translation, something that is not possible if we just look at the cut-and-dried descriptions of how these basic rigid motions move the points, *one point at a time*. Students should be exposed to many such hands-on activities to get a better feel for the "image of a figure".

We observe that *rotation maps lines, rays, and segments to lines, rays, and segments, and are distance- and degree-preserving* for the reason that each such image is nothing but an exact replica (on the transparency) of the corresponding geometric figure on the paper.

### Assumptions on basic rigid motions

Let us summarize our findings thus far. Hands-on experiences, such as those above, predispose us to accept as true that the basic rigid motions (reflections, translations, and rotations) share three common "rigidity" properties:

1. They map lines to lines, rays to rays, and segments to segments.

2. They are **distance-preserving**, meaning that the distance between the images of two points is always equal to the distance between the original two points.

3. They are **degree-preserving**, meaning that the degree of the image of an angle is

always equal to the degree of the original angle.

Notice that property 1 implies that a basic rigid motion maps rays to rays, and since angles are two rays with a common vertex (see page 8 ), a basic rigid motion also maps angles to angles. This is why in property 3 we can speak about "the degree of the image of an angle" because this image *is* an angle.

The three properties above are believable, so why not just declare that these are the starting points of our reasoning? In other words, on the basis of these properties, what else can we say about geometric figures using logical reasoning? This will in fact be our official position. In the ordinary parlance of mathematics, properties 1, 2, 3 above will be our **assumptions** about basic rigid motions, i.e., we will henceforth agree that every basic rigid motion has these properties, and will use these properties to **make logical deductions** about geometric figures.

For now, will also accept as part of our **assumptions** that there are "plenty" of basic rigid motions, in the following sense:

**R** Given any line, there is a reflection across that line.

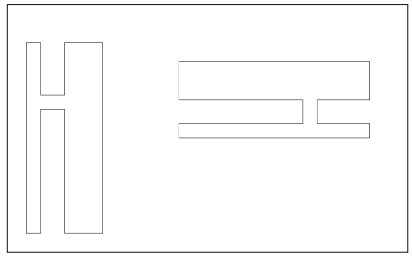
**T** Given any vector, there is a translation along that vector.

**Ro** Given a point and a degree, there is a rotation (clockwise or counterclockwise) of that degree around the point.

In the high school course where we will have to make explicit other assumptions, we will see that, in fact, **R, T**, and **Ro** can be *proved* and there will be no need to accept them as an article of faith (see Lemmas 2–4 on page 138 ).

### Compositions of rigid motions

Having explained the meaning of the basic rigid motions, it is time to go to the next level and explain the concept of the **composition of basic rigid motions**, which means moving the points of the plane by use of two or more basic rigid motions *in succession*, one after another. The need for composition is easily seen by considering an example. Suppose the following two identical H's are paced in the plane as shown. Is there a single basic rigid motion so that it maps one of these H's to the other?

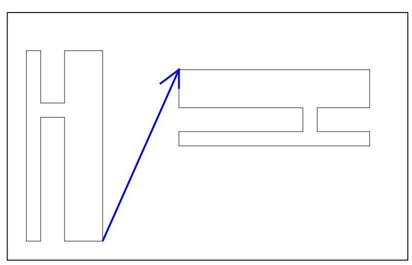


Because there is no obvious such rigid motion to get the job done,[20] we do the next best thing by devising a *sequence* of basic rigid motions to accomplish the same thing. Before describing this sequence of basic rigid motions, I want to point out that Larry Francis has created an animation for this purpose:

Composition of Rigid Motions (translation, rotation, and reflection)

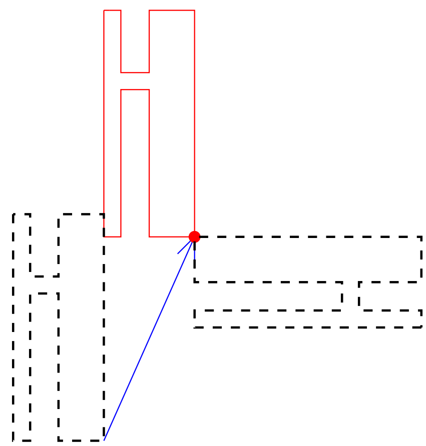
I think it would be profitable to watch this animation and also read the following description; they complement each other.

We begin by mapping the lower right corner of the left vertical H to the upper left corner of the right horizontal H because those two points are "well-matched" intuitively. Once we have done that, then we can better think of what to do next. So we translate the plane along the blue vector, as shown:



Now we trace the left vertical H onto a transparency in red, and then we slide the transparency along the blue vector (see page 70 for the definition of *slide*). Then the vertical (black) H on the left is moved to the position indicated by the red figure of the transparency as indicated below:

[20] In fact it is impossible to use one basic rigid motion to map either H onto the other.



Note that the translation along the blue vector also moves the horizontal black H to a higher position (not shown here), but for our purpose, there is no need to keep track of where this horizontal black H goes because we are only interested at this point in the new position of the vertical black H. However, we put the black *dashed* H's in the picture above in order to remind ourselves where the original figure (of the two H's) used to be. In particular, they remind us that our goal is to further move the vertical red H to where the dashed horizontal H is.

We have to ***move the plane*** (i.e., the transparency) again by using basic rigid motions until the red vertical H on the transparency[21] coincides with the black dashed horizontal H. If we rotate the plane (i.e., the transparency) around the red dot (i.e., the endpoint of the blue vector) 90 degrees clockwise, the red figure will assume the following position:

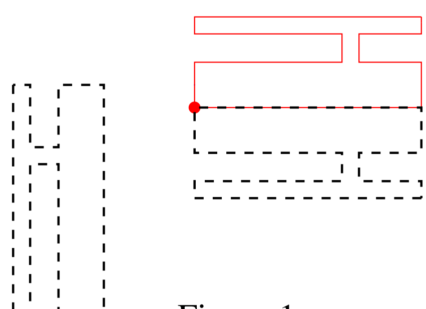
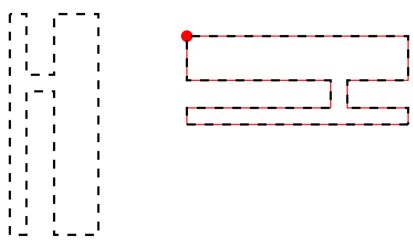


Figure 1

Let us be clear about what the red figure means: this is the position of the (original) black vertical H moved first by the translation along the blue vector, and

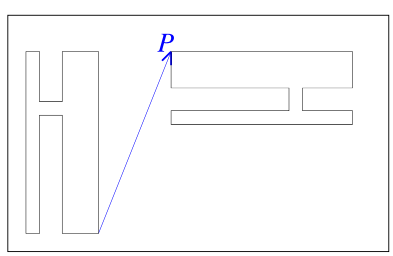
[21]More correctly, all the *points of the plane in the area at present occupied by the red vertical H.*

then *followed by* the clockwise rotation of 90 degrees around the red dot. Specifically, under the consecutive actions of these two rigid motions, the vertical black H has been moved to the position of the horizontal red H in Figure 1 above (the horizontal black H has also been moved, but we are not concerned with that). At this point, it is clear that our goal will be achieved if can move the horizontal red H in Figure 1 to the dashed horizontal black H of Figure 1). This can be done by reflecting the plane across the horizontal line that passes through the red dot (this line is not drawn in Figure 1). We proceed to realize this reflection by flipping the transparency across the horizontal line containing the red dot, interchanging the part of the plane (i.e., the transparency) above the line and the part below it (while keeping every point of the horizontal line fixed). When we do that, the new position of the red figure is the following:



By applying three basic rigid motions in succession, we finally achieve our goal.

To summarize: In order to move the left vertical H of the following picture to coincide with the right horizontal H, we apply three basic rigid motions in succession: (1) We first translate along the blue vector (whose endpoint we call $P$), then (2) follow the translation by a 90 degree clockwise rotation around $P$, and then (3) follow the rotation by a reflection across the horizontal line that contains $P$.



We now give a formal definition of composition. Let $F$ and $G$ be two basic rigid motions. Then the **composition $\boldsymbol{F \circ G}$**, or **$\boldsymbol{G}$ composed with $\boldsymbol{F}$** [22] is defined to be the rule that assigns to a given point $P$ the point $F(G(P))$. In symbols:

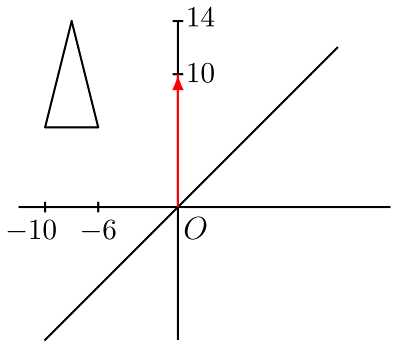
$$\boldsymbol{(F \circ G)(P) \stackrel{\text{def}}{=} F(G(P))}$$

Let us make sure that this definition makes sense. First of all, $G$ moves $P$ to the point $G(P)$ of the plane, so it makes sense for the rigid motion $F$ to move the given point $G(P)$ to the point $F(G(P))$. Thus the rule that assigns the point $F(G(P))$ to the point $P$ does make sense. Moreover, to find out which point $F \circ G$ assigns to a given point $P$, first we obtain $G(P)$ and then we focus on what $F$ does to $G(P)$. In terms of transparencies, this observation corresponds to our insistence that, once we have moved the transparency according to the first basic rigid motion $G$, we are no longer concerned with the effect that the second rigid motion has on the points in their original positions in the plane (i.e., on the paper) itself, but *only on the points in their displaced position as a result of* $G$ (i.e., the points on the transparency).

Notice that $F \circ G$ so defined also satisfies properties 1–3 on page 75 shared by the basic rigid motions. Indeed, consider property 1. If we think back on our use of transparencies to define basic rigid motions, then it is clear that the image of a figure under $F \circ G$ is just a relocation of the same figure on the transparency to a different part of the plane, and therefore if the figure is a line, or a ray, or a segment, the image remains a line, or a ray, or a segment. For the same reason, distances and degrees are preserved by $F \circ G$. The conclusion: the composition $F \circ G$ enjoys the same properties 1–3 on page 75 shared by the basic rigid motions.

One way to familiarize students with the concept of composition of basic rigid motions is to use rigid motions to map geometric figures drawn in a coordinate system. For example, consider the following triangle in the coordinate plane;

[22] $G$ comes before $F$, as the following definition makes clear. It is unfortunate that the writing of $F \circ G$ gives the opposite impression when read from left to right.




Suppose $R$ is the reflection across the slant line (the "diagonal") that makes a 45° angle with the positive $x$-axis and $T$ is the translation along the vector shown in red, what is the image of the triangle by the composition $T \circ R$?

### The concept of congruence

A main reason for introducing the concept of the composition of basic rigid motions is that we need it to define *congruence*. In general, we say two geometric figures are **congruent** if a composition of a finite number of basic rigid motions maps one figure onto the other. We also call the composition of a finite number of basic rigid motions a **congruence**. From the definition, we see that a composition of congruences is also a congruence.

Because the concept of congruence has been treated in a cavalier manner in K–12 for so long ("same size, same shape"), we call attention to the fact that once we have this precise definition, we are duty-bound to use *only* this meaning of *congruence* in all subsequent mathematical discussion. For example, we can no longer claim that two geometric figures are congruent just because they seem to have the same size and same shape, but must now produce a composition of basic rigid motions that moves one figure to the other. As another example, this precise definition of congruence sheds new light on the all-too-familiar criterion of (for example) ASA for triangle congruence. In order to claim that ASA is true, we must now exhibit a composition of basic rigid motions that moves one triangle to another when they satisfy the requisite conditions (see below). This is what we are going to do. We first recall the three classical criteria for congruence.

**SAS criterion for congruence.** If two triangles have a pair of equal angles (i.e., same degree) and corresponding sides of these angles in the triangles are pairwise equal, then the two triangles are congruent.

**ASA criterion for congruence.** If two triangles have two pairs of equal angles and the common side of the angles in one triangle is equal to the corresponding side in the other triangle, then the triangles are congruent.

**SSS criterion for congruence.** If two triangles have three pairs of equal sides, then they are congruent.

At this stage, it suffices for students to verify these theorems experimentally by drawing pictures or by use of a geometric software. However, if more details are needed, we now offer an *informal* proof of ASA ("informal" means that, while the overall idea is correct, some details are glossed over) together with a link to an animation of the proof created by Larry Francis:

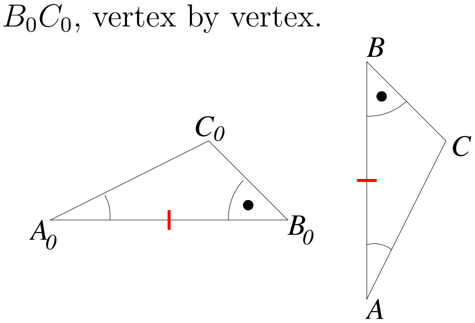
Angle-Side-Angle Congruence by Basic Rigid Motions

Please also keep in mind that *when the following proof is given in class by moving (plastic, cardboard, or wooden) models of triangles on the blackboard, it is much more understandable than the purely verbal explanation given here.*

Let us begin the informal proof. Thus we have two triangles $ABC$ and $A_0B_0C_0$ so that $|\angle A| = |\angle A_0|$, $|AB| = |A_0B_0|$, and $|\angle B| = |\angle B_0|$. We have to produce a congruence (see page 81 ) $F$ so that $\boldsymbol{F(\triangle ABC) = \triangle A_0B_0C_0}$, where the notation means:

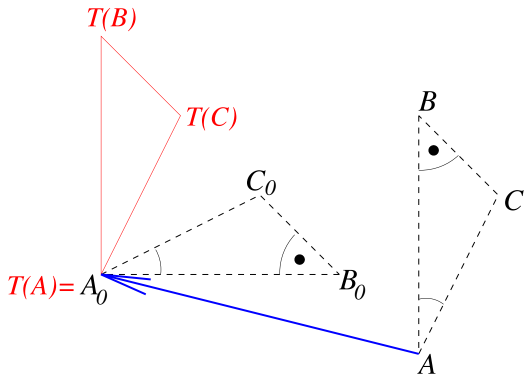
$$F(A) = A_0, \quad F(B) = B_0, \quad F(C) = C_0.$$

In greater detail, we have to produce a composition of basic rigid motions that maps $\triangle ABC$ exactly on $\triangle A_0B_0C_0$, vertex by vertex.

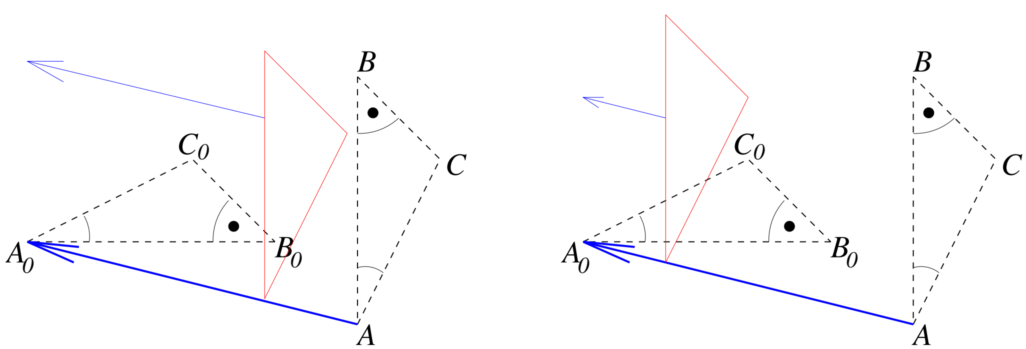



*Step 1: Bring vertices $A$ and $A_0$ together.* If $A = A_0$ already, do nothing. If not, let $T$ be the translation along the vector $\overrightarrow{\boldsymbol{AA_0}}$ (the vector consisting of the segment

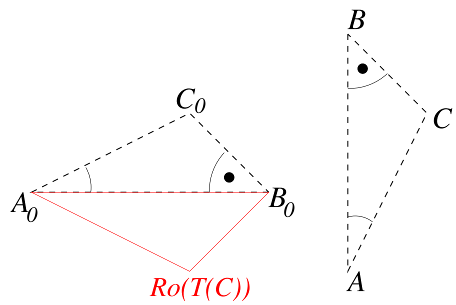
joining $A$ to $A_0$, with $A$ as the starting point and with $A_0$ as the endpoint) . Then $T(\triangle ABC)$ is a triangle with one vertex in common with $\triangle A_0B_0C_0$.



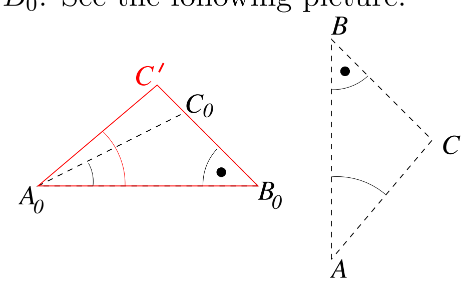

To realize this by use of a transparency, we trace out $\triangle ABC$ in red and then slide $\triangle ABC$ along the vector $\overrightarrow{AA_0}$. We can show two of the positions of $\triangle ABC$ (in red) in its transition from $A$ to $A_0$; the upper blue arrow in each picture indicates how much further the red triangle has yet to go. In the background, we draw the original positions of $\triangle ABC$ and $\triangle A_0B_0C_0$ in *dashed* lines as a reminder of where things used to be.




*Step 2: Bring the sides $AB$ and $A_0B_0$ together.* If the translated segment $T(AB)$ of $AB$ already coincides with $A_0B_0$, do nothing. Otherwise, since $A = A_0$, a rotation $Ro$ of a suitable degree around $A_0$ would bring the ray from $A_0$ to $T(B)$ to coincide with the ray from $A_0$ to $B_0$. Then because of the assumption that $|AB| = |A_0B_0|$, the same rotation would bring $T(B)$ to $B_0$.




*Step 3: Bring vertices $C$ and $C_0$ together.* If the point $Ro(T(C))$ and the point $C_0$ are on opposite sides of the line joining $A_0$ to $B_0$, then the reflection $R$ across this line would bring the point $Ro(T(C))$ to the same side of $C_0$. We may therefore assume that, after (possibly) a translation and a rotation and a reflection, the point $C$ is brought to a point $C'$ which lies on the same side as the point $C_0$ with respect to the line joining $A_0$ to $B_0$. See the following picture.

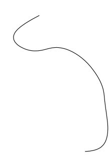



Now, we claim that, *appearance to the contrary* (as in the above picture), the ray from $A_0$ to $C'$ must coincide with the ray from $A_0$ to $C_0$. This is because the basic rigid motions preserve degrees of angles (see page 75 ) and therefore $\angle C'A_0B_0$ is equal to $\angle A$, which is assumed to be equal to $\angle C_0A_0B_0$. Thus $|\angle C'A_0B_0| = |\angle C_0A_0B_0|$, and since $C'$ and $C_0$ are on the same side of the line joining $A_0$ to $B_0$, the two sides $A_0C_0$ and $A_0C'$ coincide *as rays*. Similarly, the ray from $B_0$ to $C_0$ coincides with the ray from $B_0$ to $C'$. But $C'$ is the intersection of the ray from $A_0$ to $C'$ and the ray from $B_0$ to $C'$, while $C_0$ is the intersection of the ray from $A_0$ to $C_0$ and the ray from $A_0$ to $C_0$. Thus $C' = C_0$, which means after (possibly) a translation and a rotation and a reflection, $A$, $B$, and $C$ are brought respectively to $A_0$, $B_0$, and $C_0$. We have proved the ASA criterion.

## 2. Dilation and similarity

**Dilations and the Fundamental Theorem of Similarity**

So far we have dealt with rules of assignment in the plane that move points in a distance-preserving manner (see page 75 for the definition). Now we will confront an important class of such rules that definitely are not distance-preserving. Consider this question: Given a wiggly curve such as the following, how can we "double its size"? (Note: Please do not be misled by the imperfection of the graphics; this wiggly curve has no width.)

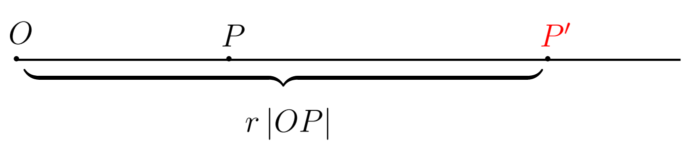


One of the purposes of this section is to show how this can be done and, in the process, clarify what it means to "double the size" of a geometric figure. The basic idea of getting this done turns out to be the astonishingly simple one of fixing an arbitrary point $O$ in the plane and then pushing every point in the plane away from $O$ by doubling its distance from $O$. Once this idea takes root, it becomes clear that not only "doubling the size", but also "halving the size" can be achieved by a variation on this theme. This is the concept of a *dilation*, and we formulate it as follows.

**Definition.** *A* **dilation $D$ with center $O$ and scale factor $r$ ($r > 0$)** *is a rule that assigns to each point* $\boldsymbol{P}$ *of the plane a point* $\boldsymbol{D(P)}$ *so that*

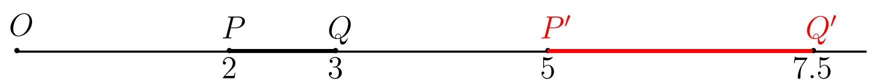
(1) *$D(O) = O$.*

(2) *If $P \neq O$, the point $D(P)$, to be denoted more simply by $P'$, is the point on the ray $R_{OP}$ so that $|OP'| = r|OP|$.*



Intuitively, if $r > 1$, the dilation pushes every point of the plane away from the center $O$ by "proportionally the same amount", and if $r < 1$, then it pulls every point toward the center $O$ also by "proportionally the same amount".

There are two things of note about this definition. One is that unless the scale factor $r$ is equal to 1, a dilation is not a congruence. The easiest way to see this is to consider a simple situation where the scale factor $r$ of the dilation $D$ is (let us say) 2.5 and the center $O$ is the origin of a coordinate system. Let the ray from $O$ be the positive $x$-axis and let two points $P$ and $Q$ be the numbers 2 and 3, respectively.




Then $|PQ| = 3 - 2 = 1$. Then if $P' = D(P)$ and $Q' = D(Q)$, we have $|OP'| = r\,|OP| = 2.5 \times 2 = 5$ and $|OQ'| = r\,|OQ| = 2.5 \times 3 = 7.5$. It follows that $P' = 5$ and $Q' = 7.5$ and $|P'Q'| = 7.5 - 5 = 2.5$. In particular, $|P'Q'| = 2.5 \cdot |PQ| > |PQ|$ and $D$ is not distance-preserving (see page 75 ).

The same reasoning shows that, for any dilation $D$ with center $O$ and scale factor $r$, the equality $|D(P)\,D(Q)| = r\,|PQ|$ holds for any two points $P$ and $Q$ on a ray issuing from $O$. Therefore unless $r = 1$, $D$ cannot be a congruence.

A second thing to note about the preceding definition of dilation is that the point $P'$ in condition (2) is chosen from the *ray* $R_{OP}$ and not the *line* $L_{OP}$. Thus the point $P$ and its image $P'$ under $D$ are always *on the same side* of the center $O$ on the line $L_{OP}$.

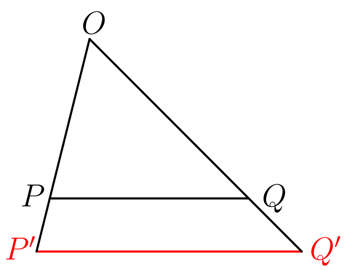
Now suppose points $P$ and $Q$ are on a line passing through the center $O$ but do not lie on the same ray. Does the equality $|P'Q'| = r\,|PQ|$ continue to hold? The answer is affirmative and the computation is quite similar.


However, if line $L_{PQ}$ does not contain $O$, then the fact that $|P'Q'| = r\,|PQ|$ continues to hold is much more subtle and this is the content of the following **Fundamental Theorem of Similarity**, usually abbreviated to **FTS**. In the statement of the theorem, we adopt a common abuse of notation: Let $\boldsymbol{L_{PQ}}$ (respectively, $L_{P'Q'}$) denote the line joining $P$ and $Q$ (respectively, $P'$ and $Q'$). Then instead of saying $L_{P'Q'} \parallel L_{PQ}$, we usually write:

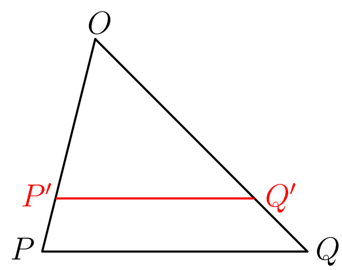
$$P'Q' \parallel PQ$$

**Theorem (FTS).** *Let $D$ be a dilation with center $O$ and scale factor $r > 0$ and $r \neq 1$. Let $P$ and $Q$ be two points so that $L_{PQ}$ does not contain $O$. If $D(P) = P'$ and $D(Q) = Q'$, then*

$$P'Q' \parallel PQ \quad \text{and} \quad |P'Q'| \;=\; r\,|PQ|$$




The case $r > 1$




The case $r < 1$

Eighth grade is not a suitable place to prove this theorem; a high school course will be better able to handle such a proof, or at least a part of it. What one can do in an eighth grade class is to verify simple cases of FTS by direct measurements to gain confidence in its validity. For example, one can start with $r = 2, 3, 4$, and then verify that (within the bounds of measurement errors), indeed, $|P'Q'| = 2|PQ|$, $|P'Q'| = 3|PQ|$, $|P'Q'| = 4|PQ|$, respectively, and $PQ \parallel P'Q'$ each time. Then do the same with $r = \frac{1}{2}$, $r = \frac{2}{3}$, $r = \frac{3}{4}$, etc.

However, the importance of FTS to eighth grade geometry is not so much to learn to prove it but learn to *use* it. We often come across such a situation in school mathematics, namely, using a powerful tool without proof, because the proofs of such tools can be given without *circular reasoning*.[23] We will use FTS to deduce the most basic properties of a dilation.

[23] Briefly, this means that the reasoning used in affirming the validity of these tools is logically independent of the application we have in mind for these tools.

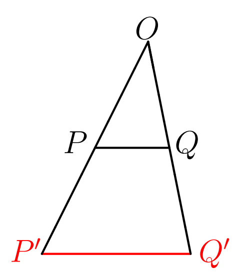
**Basic properties of dilations**

There are four of them. Their proofs can be found in a high school course, but in line with the curricular goal of eighth grade geometry, we will only give an informal proof for the last one.

First of all, notice that insofar as a dilation is a rule of assignment in the plane, we will simply take over the terminology associated with basic rigid notions such as *maps*, *image*, *composition*, etc. We will also fix the notation as in FTS, i.e., we have a dilation $D$ with center at $O$ and scale factor $r$. Then the discussion leading up to FTS about $|P'Q'|$ may now be summarized as follows:

> ***(i)*** *Let $D$ be a dilation with scale factor $r$. Then the distance between the images $P' = D(P)$, $Q' = D(Q)$ of any two points $P$ and $Q$ is always $r$ times the distance between $P$ and $Q$, i.e.,*
>
> $$|P'Q'| \ = \ r\,|PQ|.$$

We usually paraphrase ($i$) by saying that *a dilation with scale factor $r$ changes distance by a scale factor of $r$.* Here is an example where $r = 2$:




A second property is this:

> ***(ii)*** *A dilation maps lines to lines, rays to rays, and segments to segments.*

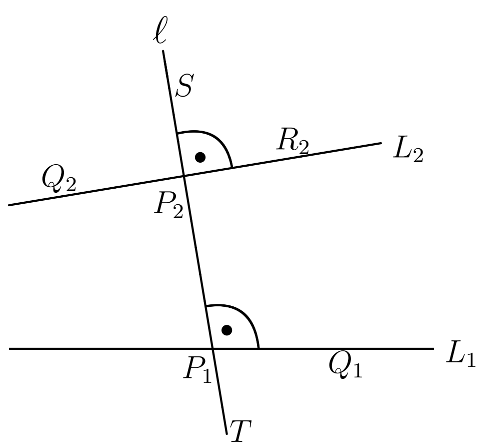
The main point here is the assertion about lines. Again, for grade eight, it suffices to verify this assertion in special cases by picture drawing. Once that we know a dilation $D$ maps a line $PQ$ to the line $P'Q'$, where $P'$, $Q'$ are the images of $P$, $Q$ under $D$, FTS now implies:

> ***(iii)*** *A dilation maps a line not containing the center of dilation to a parallel line.*

A fourth basic property of dilation is the following.

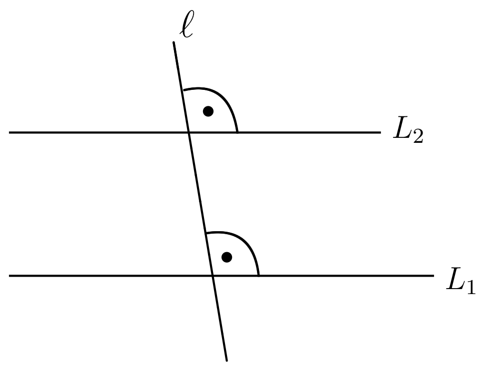
**(*iv*)** *A dilation preserves degrees of angles.*

We note first of all that $(iv)$ makes sense because by $(ii)$ above, a dilation maps rays to rays and therefore angles to angles. So it makes sense to ask for the degree of the image angle by a dilation. For eighth grade students, the following informal argument for $(iv)$ will be enough and a rigorous proof can be postponed to a high school course. We begin with a fact about parallel lines. Let $L_1$ and $L_2$ be two distinct lines and let $\ell$ be a **transversal** of $L_1$ and $L_2$ in the sense that $\ell$ intersects both. Suppose $\ell$ meets $L_1$ and $L_2$ at $P_1$ and $P_2$, respectively. Then the angles $\angle SP_2R_2$ and $\angle P_2P_1Q_1$ in the picture below, with vertices at $P_1$ and $P_2$ and lying on the same side of the line $\ell$, are called a pair of **corresponding angles** of the transversal $\ell$ with respect to $L_1$ and $L_2$. Replacing one of them by its **opposite angle** (or **vertical angle**) such as $\angle P_1P_2Q_2$, then $\angle P_1P_2Q_2$ and $\angle P_2P_1Q_1$ are called **alternate interior angles** of $\ell$ with respect to $L_1$ and $L_2$. Similarly, $\angle R_2P_2P_1$ and $\angle Q_1P_1T$ are also corresponding angles of $\ell$ with respect to $L_1$ and $L_2$.




Then we have the following theorem.

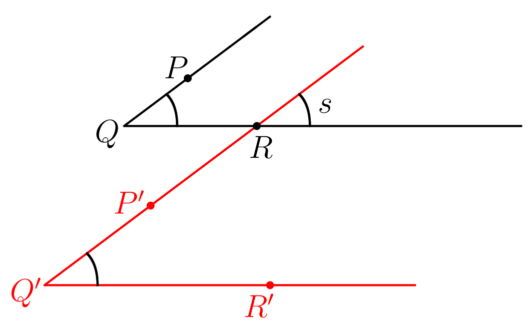
**Theorem 1.** $(a)$ *Corresponding angles of a transversal with respect to parallel lines are equal, and conversely, if a pair of corresponding angles of a transversal with respect to two lines are equal, then the two lines are parallel.* $(b)$ *Alternate interior angles of a transversal with respect to parallel lines are equal, and conversely, if a pair of* *alternate interior angles of a transversal with respect to two lines are equal, then the two lines are parallel.*




Since opposite angles are (easily seen to be) equal, it suffices to prove part ($a$). We suggest that eighth graders simply verify special cases of this theorem by direct measurements. In fact, it would be very instructive to teach them *how to use the converse statement of part* ($a$) *to draw a line parallel to a given line and passing through a given point with the help of a ruler and a plastic triangle.*

We can now return to the original problem which inspired this detour into parallelism: How to prove that a dilation preserves degrees of angles (page 89 ). Let $D$ be a dilation (with some center $O$) and let $\angle PQR$ be given. Let $D(QP) = Q'P'$ and without loss of generality, we may assume $R$ is the intersection of $L_{QR}$ and $L_{Q'P'}$. Let $D(QR) = Q'R'$, so that $D(\angle PQR) = \angle P'Q'R'$.

We have to prove that
$|\angle PQR| = |\angle P'Q'R'|$.




Let the angle formed by $L_{Q'P'}$ and $L_{QR}$ at $R$, as indicated in the picture, be denoted by $\angle s$. Since $D(QR) = Q'R'$, ($iii$) implies that $QR \parallel Q'R'$ (page 88 ), so that, by Theorem 1($a$),

$$|\angle P'Q'R'| \;=\; |\angle s|$$

Since also $D(QP) = Q'P'$ by assumption, Theorem $1(a)$ implies that

$$|\angle s| \;=\; |\angle PQR|$$

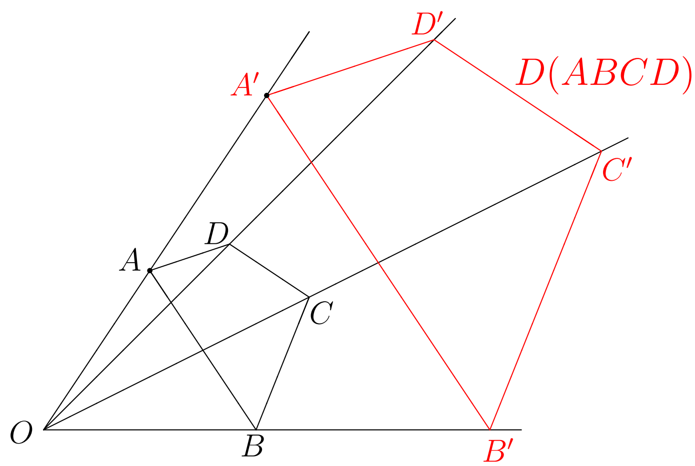
Hence $|\angle PQR| = |\angle P'Q'R'|$, as desired.

We have just given the essential idea of why a dilation preserves degrees of angles.

**The dilated image of a figure**

Property $(iii)$ of a dilation makes it very easy to draw the dilated image of a **rectilinear** geometric figure, i.e., one that is the **union** of segments (in the sense of the collection of all the segments). Consider a segment $PQ$ and a dilation $D$, then the image $D(PQ)$ by $D$ is simply the segment $P'Q'$, where $P'$, $Q'$ are the images of $P$ and $Q$ by $D$, respectively. This is because $(iii)$ says the image $D(PQ)$ is a segment joining $D(P) = P'$ and $D(Q) = Q'$, and since $P'Q'$ is also a segment joining $P'$ and $Q'$, we must have $D(PQ) = P'Q'$ (there is only one line joining two distinct points).

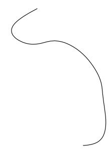
For example, if we have to get the dilated image of a given quadrilateral $ABCD$ with a scale factor of 2.1, we take a point $O$ as the center of dilation, draw rays from $O$ to the vertices. On each of these rays, say the ray from $O$ to $A$, mark down $A'$ so that $|OA'| = (2.1) \cdot |OA|$. Do likewise to the other rays. We thus obtain a quadrilateral $A'B'C'D'$. By assertion $(iii)$, $D(ABCD) = A'B'C'D'$.



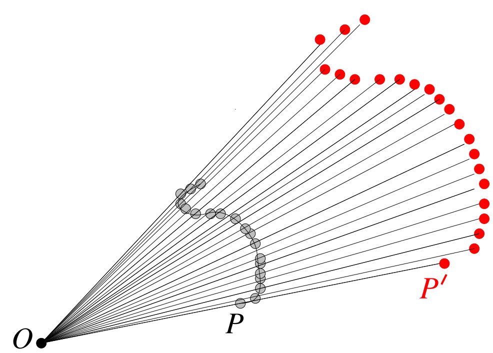

This $D(ABCD)$ is by definition the **magnification of $ABCD$ to 2.1 times its size**. So for rectilinear figures, how to magnify them is straightforward.

Notice that, in an intuitive sense, $ABCD$ and $D(ABCD)$ do "look alike", i.e., they have "the same shape".

We can now return to the curve at the beginning of this section:



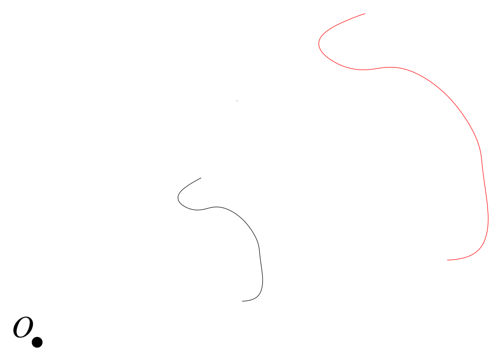
How to "double its size"? We choose an arbitrary point $O$ outside the curve as the center of a dilation and dilate the curve with a scale factor of 2. Now by definition, dilating the curve means dilating it point by point, and since the curve contains an infinite number of points, we have to compromise for the sake of drawing by dilating only a *finite* number of points on the curve. We start simply: take a point $P$ on the curve and on the ray $OP$, we mark off a point $P'$ so that $|OP'| = 2|OP|$. Now repeat this for a small number of such $P$'s and get something like the following. The contour of a curve that is bigger than, but "looks like" the original is unmistakable.




Now if we choose, let us say 1500 points on the original curve,[24] and dilate them one-by-one, we get the usual curve that appears on the computer screen. We have omitted the radial lines but retained the center of the dilation; on a normal computer screen, of course even the center is omitted.

Given a curve $\mathcal{S}$, if there is a dilation $D$ with scale factor $r$ (and some center) so that $D(\mathcal{S}) = \mathcal{S}'$, then we say the curve $\mathcal{S}'$ is **a magnification with a scale factor $\boldsymbol{r}$ of the curve $\boldsymbol{\mathcal{S}}$**. (For lack of a better term, we let the word "magnification" be used even when $r < 1$.) In this sense, the above dilated image of the original curve is "double the size" of the original, and our problem is solved.

[24]This is an estimate of how many points the graphing software uses.

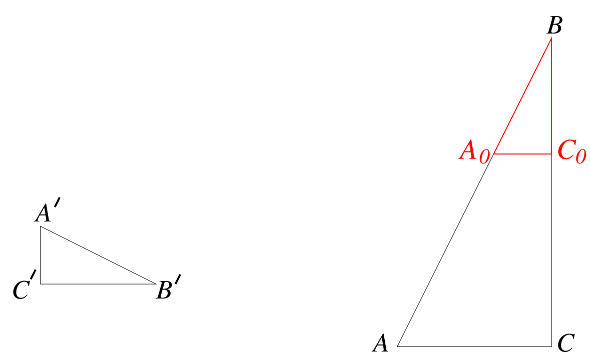



Of course, you may be puzzled by this new meaning of "magnification". After all, don't we *recognize* whether something is a "magnification" of a given figure or not, and whether or not they have the same shape? Perhaps, but in mathematics, what is needed ultimately is not some vague feelings that any particular individual happens to possess, but a *precise articulation of the properties or qualities that we try to express in a way that leaves no room for misunderstanding.* From this point of view, the preceding definition of "double the size" (i.e., magnification with scale factor 2) is sufficiently precise to be usable. Furthermore, this definition of "double the size" would not be acceptable if the image curve under the dilation fails to have "the same shape" as the original curve or fails to look "twice as big", but the image curve passes this common-sense test with flying colors as well. (In a school classroom, be sure to let students draw many magnifications with various scale factors of different curved figures so as to give them confidence in this definition of "magnification".) So this is the definition we shall abide by in our mathematical work related to "magnification". Incidentally, what we have described here is the underlying principle of magnification in digital photography; this is how an image is enlarged or reduced in size in the digital world.

In a classroom, getting students to do the magnification of a curvy figure by dilation (with a reasonable number of points chosen and strategically placed on the original figure), and with different scale factors, would be a very worthwhile learning experience. It would reinforce their confidence in, and their understanding of the definition of dilation, and it is also a "fun" activity because it is not at all obvious how a figure can be enlarged or reduced in size.

### Similarity

We have seen that the dilated image of a figure "has the same shape" as the original, so a geometric figure would most certainly "have the same shape" as its image under any dilation. But does this suggest that we can simply define two geometric figures to be "similar" if one is the dilation of the other? The answer is unfortunately *no*. Consider, for example, the dilated image of a triangle $ABC$ to a triangle $A_0BC_0$ by a dilation $D$ centered at $B$ as shown, with a scale factor $r < 1$. Of course these two triangles "have the same shape". Now let a congruence $F$ move $\triangle A_0BC_0$ to $\triangle A'B'C'$, as shown (more precisely, $F$ is the composition of a 90 degree clockwise rotation around $B_0$ followed by a translation).




Because $\triangle A_0BC_0$ and $\triangle A'B'C'$ "have the same size and the same shape", we have to agree that $\triangle ABC$ and $\triangle A'B'C'$ also have the same shape. Yet there is no dilation $D'$ that maps $\triangle ABC$ to $\triangle A'B'C'$ because if there were, we'd have

$$D(A) = A' \quad \text{and} \quad D(C) = C'$$

so that by property $(iii)$ of a dilation (page 88), we would have $AC \parallel A'C'$, which is not the case. Therefore similarity between geometric figures cannot be limited to those so that one is obtained from the other by a dilation. At the same time, the preceding example also suggests how to define similarity correctly: we should include the composition with a congruence in the definition.

We therefore define a figure $\mathcal{S}$ in the plane to be **similar** to another figure $\mathcal{S}'$ if there is a dilation $D$ and a congruence $F$ so that $(F \circ D)(\mathcal{S}) = \mathcal{S}'$. According to this definition, we now see that $\triangle ABC$ is similar to $\triangle A'B'C'$ because if $D$ is the dilation that maps $\triangle ABC$ to $\triangle A_0BC_0$ and $F$ is the congruence that maps $\triangle A_0BC_0$ to $\triangle A'B'C'$, then

$$\begin{aligned}(F \circ D)(\triangle ABC) &= F(D(\triangle ABC)) \quad \text{(definition of composition, page 80)} \\ &= F(\triangle A_0BC_0) \quad (D \text{ maps } \triangle ABC \text{ exactly to } \triangle A_0BC_0) \\ &= \triangle A'B'C' \quad (F \text{ maps } \triangle A_0BC_0 \text{ exactly to } \triangle A'B'C')\end{aligned}$$

Thus the dilation $D$ followed by the congruence $F$ bring $\triangle ABC$ to $\triangle A'B'C'$.

According to this definition, it is also the case that if a dilation $D$ maps a figure $\mathcal{S}$ to another figure $\mathcal{S}'$, then $\mathcal{S}$ is similar to $\mathcal{S}'$ because we can let the congruence $F$ be the identity basic rigid motion $I$ (page 71) so that $(I \circ D)(\mathcal{S}) = \mathcal{S}'$. Likewise, if a figure $\mathcal{S}$ is congruent to another figure $\mathcal{S}'$, then $\mathcal{S}$ is similar to $\mathcal{S}'$ because we can let the dilation be the identity dilation (scale factor 1).

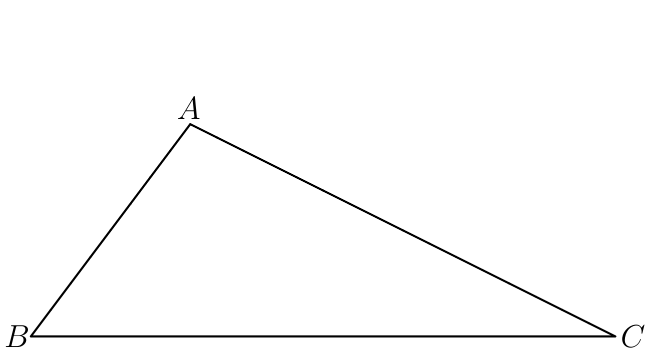
The composition $F \circ D$ of a dilation followed by a congruence is called a **similarity**. The scale factor of the dilation $D$ is then called the **scale factor of the similarity $\boldsymbol{F \circ D}$**.

We conclude this section by bringing closure to the discussion of scale drawings started in grade 7 (page 50). If there are two geometric figures in the plane, $\mathcal{S}$ and $\mathcal{S}'$, then by definition, $\mathcal{S}'$ is said to be a **scale drawing** of $\mathcal{S}$ with **scale factor $\boldsymbol{r}$** if there is a similarity $F \circ D$ with scale factor $r$ so that $(F \circ D)(\mathcal{S}) = \mathcal{S}'$.

To see how this definition of scale drawing is related to the discussion on page 50, notice first of all that a one-to-one correspondence is a transformation and that a composition of one-to-one correspondences remains a one-to-one correspondence (this is a straightforward exercise). Now a dilation is necessarily a one-to-one correspondence of the plane with itself, as is any congruence. Thus a similarity, being a composition of a dilation and a congruence, is also a one-to-one correspondence. Furthermore, property $(i)$ on page 88 shows that a similarity satisfies the proportionality relationship given at the bottom of page 51. Thus the concept of similarity is nothing but a precise formulation of the concept of scale drawing.

## 3. The angle-angle criterion (AA) for similarity

Let $\triangle ABC$ be similar to $\triangle A'B'C'$.




Thus there is a dilation $D$ and a congruence $F$ so that $(F \circ D)(\triangle ABC) = \triangle A'B'C'$. It is convenient to denote the similarity $F \circ D$ by a single letter, say $H = F \circ D$. (We use $H$ in this document, but it may be better in a school classroom to use only $F \circ D$ to remind students that a similarity is a composition of a dilation and a congruence.) Recalling the convention regarding the notation of congruent triangles, we explicitly point out that the notation $H(\triangle ABC) = \triangle A'B'C'$ carries the convention that

$$H(A) = A', \quad H(B) = B', \quad \text{and} \quad H(C) = C'$$

Let the scale factor of the dilation $D$ be $r$, and let

$$D(A) = A^*, \quad D(B) = B^*, \quad \text{and} \quad D(C) = C^*$$

By properties $(i)$ and $(iv)$ of dilations (pages 88 ), we get

$$|\angle A| = |\angle A^*|, \quad |\angle B| = |\angle B^*|, \quad |\angle C| = |\angle C^*|$$

$$\text{and} \qquad \frac{|AB|}{|A^*B^*|} = \frac{|BC|}{|B^*C^*|} = \frac{|AC|}{|A^*C^*|} \quad (= \ r)$$

Now $F$ is a congruence which preserves lengths and degrees. Therefore all this information about $\triangle A^*B^*C^*$ will be transferred to $\triangle A'B'C'$ and we arrive at the following theorem.

**Theorem 2.** *Let $\triangle ABC$ be similar to $\triangle A'B'C'$. Then*

$$|\angle A| = |\angle A'|, \quad |\angle B| = |\angle B'|, \quad |\angle C| = |\angle C'|$$

and

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|AC|}{|A'C'|}$$

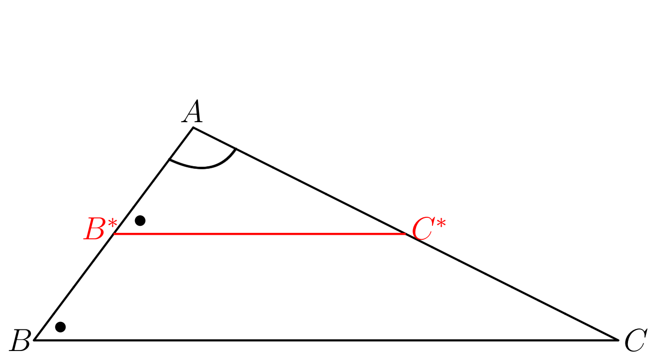
It is worth remarking that whereas the content of this theorem is usually taken to be the *definition* of similar triangles, for us this theorem is a confirmation of the fact that two triangles with "the same shape" (equal angles and proportional sides) are indeed similar in a precise sense.

The converse of Theorem 2 is also true. However, as in the case of congruence (page 81 ), much more is true. The following are the counterparts in similarity of the SAS, ASA and SSS criteria for congruence, respectively.

> **SAS criterion for similarity.** Given two triangles $ABC$ and $A'B'C'$, suppose
> $$|\angle A| = |\angle A'| \quad \text{and} \quad \frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|}.$$
> Then the triangles are similar.
>
> **AA criterion for similarity.** Given two triangles $ABC$ and $A'B'C'$, suppose two pairs of corresponding angles are equal. Then the triangles are similar.
>
> **SSS criterion for similarity.** Given two triangles $ABC$ and $A'B'C'$, suppose the ratios of the (lengths) of three pairs of corresponding sides are equal. Then the triangles are similar.

The proofs of these theorems are more suitable for a high school course than an eighth grade class, but we are going to give a proof of the AA criterion because it is so central to the discussion of the slope of a line. The proof is based on FTS.

**Proof of the AA criterion for similarity.** Suppose we are given triangles $ABC$ and $A'B'C'$ such that $|\angle A| = |\angle A'|$ and $|\angle B| = |\angle B'|$. We have to show that the triangles are similar.




If $|AB| = |A'B'|$, then triangles $ABC$ and $A'B'C'$ are congruent because of the ASA criterion for congruence (pages 82 and 82); there would be nothing to prove. Thus we may assume that the sides $AB$ and $A'B'$ have different lengths: either $|A'B'| < |AB|$ or $|A'B'| > |AB|$. Since the proofs for these cases are entirely similar, we take up the former for definiteness. Thus we assume $|A'B'| < |AB|$. On the segment $AB$, let $B^*$ be the point so that $|AB^*| = |A'B'|$. Also let $r = |A'B'|/|AB|$ ($= |AB^*|/|AB|$). Denote the dilation with center $A$ and scale factor $r$ by $D$, and let $C^*$ be the point in the segment $AC$ so that $C^* = D(C)$. By FTS (page 87), $B^*C^* \parallel BC$ and therefore $|\angle AB^*C^*| = |\angle B|$, by Theorem 1 (page 89). But by hypothesis, $|\angle B| = |\angle B'|$, so

$$|\angle AB^*C^*| \ = \ |\angle B'|\,.$$

Since $|\angle A| = |\angle A'|$ by hypothesis, triangles $AB^*C^*$ and $A'B'C'$ now satisfy the conditions of ASA and are congruent. Hence there is a congruence $F$ so that $F(\triangle AB^*C^*) = \triangle A'B'C'$. But by the definition of $D$, we already have $D(\triangle ABC) = \triangle AB^*C^*$. Thus we now see that the similarity transformation, which consists of the dilation $D$ bringing $\triangle ABC$ to $\triangle AB^*C^*$ followed by the congruence $F$ bringing $\triangle AB^*C^*$ to $\triangle A'B'C'$, maps $\triangle ABC$ exactly to $\triangle A'B'C'$. In symbols, $(F \circ D)(\triangle ABC) = \triangle A'B'C'$ because

$$(F \circ D)(\triangle ABC) = F(D(\triangle ABC)) = F(\triangle AB^*C^*) = \triangle A'B'C'.$$

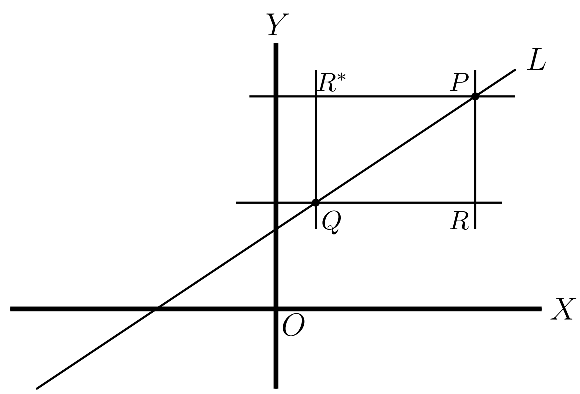
The two triangles $ABC$ and $A'B'C'$ are therefore similar.

We remark that this proof makes use of the ASA criterion for congruence, FTS, and Theorem 1. It is a remarkable example of making use of the available tools to solve a problem. If we want students to be proficient in problem-solving, we have to give them excellent examples of how it is done. For this reason, although one cannot mandate the teaching of this proof in all the eighth grade classrooms, it is nevertheless recommended that this proof be presented in class if at all possible.

## 4. Slope of a line and the Pythagorean Theorem

For eighth grade, the significance of the above three criteria for similarity lies not so much in getting students to know how to prove them as in *getting them to learn how to put them to use.* In this section, we give two examples of such applications, the second one being a proof of the Pythagorean theorem.

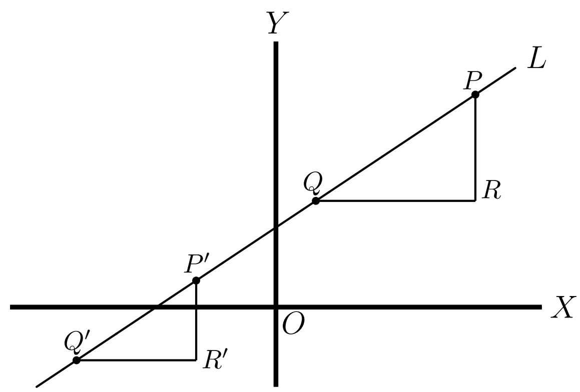
A typical example arising from algebra is the following. Given a line $L$ in the coordinate plane together with two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ on $L$. *For this discussion, let the line slant to the right as in the following picture.* Let lines parallel to the coordinate axes and passing through $P$ and $Q$ be drawn. In general, these pairs of parallel lines meet at two points $R$ and $R^*$, as shown:




It will be clear presently that it does not matter whether $R$ or $R^*$ is used for this discussion, but the prevailing convention is to use the lower point $R$. We will follow this convention. With this understood, the ratio $\frac{|PR|}{|QR|}$, comparing the *vertical* side of the right triangle $\triangle PQR$ to the *horizontal* side of the same triangle, is commonly proposed as a measure of the "degree of steepness" of the line $L$, and this ratio is called the *slope of the line* $L$.

*We hasten to observe that if* $R^*$ *is used instead of* $R$*, then the two corresponding ratios* $\frac{|PR|}{|QR|}$ *and* $\frac{|QR^*|}{|PR^*|}$*, are equal because* $|PR| = |QR^*|$ *and* $|QR| = |PR^*|$ (*opposite* *sides of a rectangle are equal, see page 17). This is the reason that it doesn't matter whether $R$ or $R^*$ is used.*

In school textbooks, $\frac{|PR|}{|QR|}$ is called the slope of the line $L$. This should be cause for some reflection: After all, if two other points $P'$ and $Q'$ are chosen on $L$ instead and we form the ratio $\frac{|P'R'|}{|Q'R'|}$ accordingly, as shown,




then we would have the quotient $\frac{|P'R'|}{|Q'R'|}$ instead of $\frac{|PR|}{|QR|}$. So which is the slope of $L$, $\frac{|P'R'|}{|Q'R'|}$ or $\frac{|PR|}{|QR|}$? Fortunately, it turns out that they are equal:

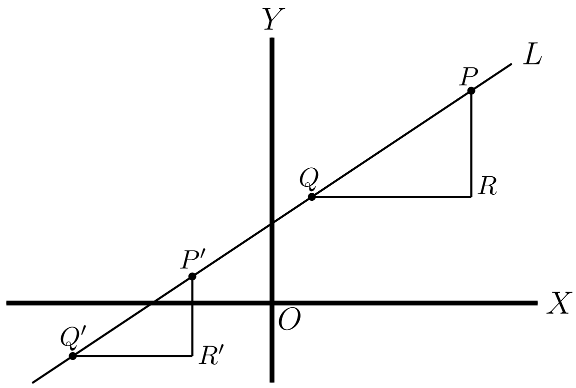
$$\frac{|PR|}{|QR|} = \frac{|P'R'|}{|Q'R'|}.$$

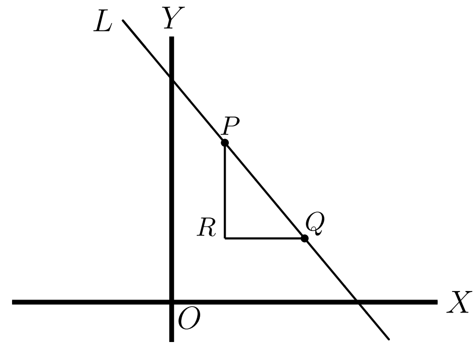
Therefore, the ratio $\frac{|PR|}{|QR|}$ *is independent of the choice of the two points $P$ and $Q$* and is well qualified to be called **the slope of $L$**.

The failure of school textbooks to address this issue accounts for much of the nonlearning of linear equations and straight lines in introductory algebra; it also implicitly encourages sloppy thinking. A main reason for the CCSSM to introduce similar triangles in the eighth grade is precisely to resolve this issue affirmatively, thereby showing that the concept of slope is **well-defined**, in the sense that the ratio $\frac{|PR|}{|QR|}$ remains the same no matter which two points $P$ and $Q$ on the line $L$ are chosen. For a fuller discussion, see Section 4 of H. Wu, Introduction to School Algebra.

Let us start from the beginning and do it properly. Given a line $L$ in the coordinate plane together with two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ on $L$. Let lines parallel to the coordinate axes and passing through $P$ and $Q$ be drawn and let them meet at

$R$ (recall the convention above about the point $R$). Now there are two possibilities for $L$: it can slant to the right (left picture below) or to the left (right picture below).




For the line on the left, the **slope of the line *L*** is defined as the ratio $\frac{|PR|}{|QR|}$, whereas for the line on the right, the slope is $-\frac{|PR|}{|QR|}$. Thus the minus sign in the slope differentiates between the two different slants of the lines. It suffices to consider the case of the line on the left because the reasoning is entirely similar.

We are going to show that the slope of the line $L$ is independent of the two chosen points $P$ and $Q$. Thus we choose two arbitrary points $P'$, $Q'$ on $L$ and we get a point of intersection $R'$ in like manner (see left picture above), and we are going to prove:

$$\frac{|PR|}{|QR|} = \frac{|P'R'|}{|Q'R'|},$$

Therefore this ratio is a property of the line $L$ after all and not of the pair of points chosen.

To prove the preceding equality, we are going to use the AA criterion for similarity (page 97) to prove that triangles $PQR$ and $P'Q'R'$ are similar. Indeed, the lines $QR$ and $Q'R'$, being both parallel to the $x$-axis, are parallel to each other. Theorem 1 (page 89) implies that $|\angle PQR| = |\angle P'Q'R'|$. Since $\angle PRQ$ and $\angle P'R'Q'$ are right angles, they are also equal. So the triangles $PQR$ and $P'Q'R'$ have two pairs of equal angles and are therefore similar. By Theorem 2 on page 96, we get

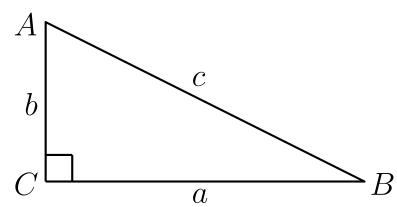
$$\frac{|PR|}{|P'R'|} = \frac{|QR|}{|Q'R'|}$$

This implies

$$\frac{|PR|}{|QR|} = \frac{|P'R'|}{|Q'R'|}$$

by the cross-multiplication algorithm.[25] We are done.

As our second application of the AA criterion for similarity, we prove the Pythagorean Theorem. Let us fix the terminology. Given a right triangle $ABC$ with $C$ being the vertex of the right angle, then the sides $AC$ and $BC$ are called the **legs** of $\triangle ABC$, and $AB$ is called the **hypotenuse** of $\triangle ABC$.

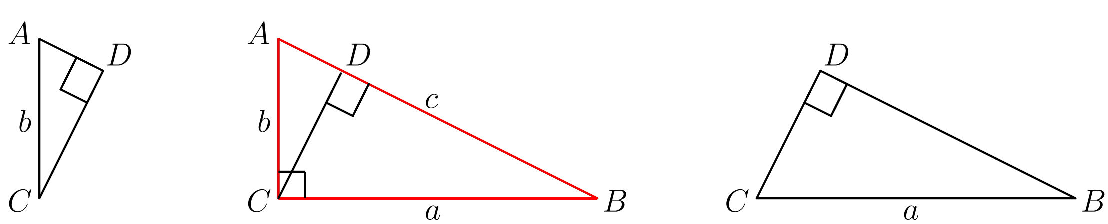



**Theorem 3 (Pythagorean Theorem).** *If the lengths of the legs of a right triangle are $a$ and $b$, and the length of the hypotenuse is $c$, then* $a^2 + b^2 = c^2$.

**Proof.** There is an animation of the following proof created by Larry Francis:

Pythagorean Theorem proof from similar right triangles

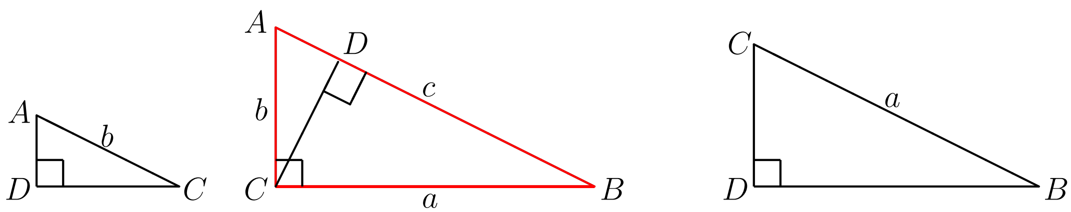
We will have more to say about this animation later. For the proof proper, we draw a perpendicular $CD$ from $C$ to side $AB$ of the given $\triangle ABC$, as shown:




We draw this perpendicular because it creates, from the point of view of the AA criterion for similarity, three similar triangles. For example, right triangles $CBD$ and $ABC$ are similar because they share $\angle B$ in addition to having equal right angles. Likewise, right triangles $ACD$ and $ABC$ are similar because they share $\angle A$. For

[25] Or rather, by the formal extension of the cross-multiplication algorithm from complex fractions to arbitrary numbers using the *Fundamental Assumption of School Mathematics.* See page 88 of H. Wu, Pre-Algebra (remember, this is an active link).

beginning students, it may help them to see the similarity of the three triangles better if we do the following: (1) To $\triangle ACD$, apply a suitable counterclockwise rotation around the vertex $C$, a reflection across the line $L_{BC}$, and then a translation to the left to obtain the $\triangle ACD$ on the left below. (2) To $\triangle CBD$, apply a suitable counterclockwise rotation around the vertex $B$, a reflection across the line $L_{BC}$, and then a translation to the right to obtain the $\triangle CBD$ on the right in the following picture:




Specifically, for the similar triangles $\triangle ABC$ and $\triangle ACD$, in order to set up the correct proportionality of sides, Theorem 2 (page 96) tells us that we need the correct correspondences of the vertices. The vertices of the two right angles obviously correspond, so $C$ of $\triangle ABC$ corresponds to $D$ of $\triangle CDB$. The two triangles share $\angle B$, so $B$ of $\triangle ABC$ corresponds to $B$ of $\triangle CDB$. Now there is no choice but that $A$ of $\triangle ABC$ corresponds to $C$ of $\triangle CDB$. Thus we have:

$$C \leftrightarrow D, \quad B \leftrightarrow B, \quad A \leftrightarrow C$$

Hence $\frac{|BA|}{|BC|} = \frac{|BC|}{|BD|}$, so that by the cross-multiplication algorithm,

$$|BC|^2 = |AB| \cdot |BD|$$

By considering the similar right triangles $ABC$ and $ACD$, we conclude likewise that $\frac{|AC|}{|AB|} = \frac{|AD|}{|AC|}$ and

$$|AC|^2 = |AB| \cdot |AD|$$

Adding, we obtain

$$|BC|^2 + |AC|^2 = |AB| \cdot |BD| + |AB| \cdot |AD| = |AB|\,(\,|BD| + |DA|\,) = |AB|^2$$

This is the same as $a^2 + b^2 = c^2$. The proof is complete.

Now the preceding *algebraic* computation that leads to $|BC|^2 + |AC|^2 = |AB|^2$ is very natural and one tends to accept it as is. However, in the animation by Larry Francis (cited at the beginning of the proof), he makes a very nice observation that, in fact, the algebra has a *geometric* interpretation in terms of area. If you have not watched the animation, I highly recommend that you do.

**To the teacher:** There are all kinds of "cute" proofs of the Pythagorean Theorem out there, but you are strongly encouraged to present the preceding proof in your eighth grade classroom. There are two reasons. One is that at this point of the geometry curriculum according to the CCSSM, students need all the exposure to the concept of similar triangles they can get, and this proof of the Pythagorean Theorem serves this purpose surpassingly well. A second reason is that most of the "cute" proofs of the Pythagorean Theorem involve the concept of area as well as subtle (and usually hidden) arguments that depend on the Parallel Postulate. The "cuteness" of these proofs is usually the result of covering up how sophisticated the concept of area really is and omitting any reference to the Parallel Postulate. These "cute" proofs deserve to be learned, but should be learned *without* any cover-ups. In the meantime, please present the preceding proof, as it is most attractive when viewed from the perspective of Larry Francis' animation.

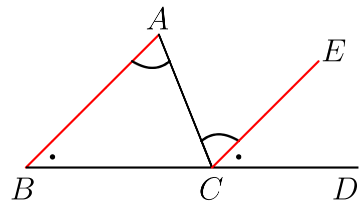
## 5. The angle sum of a triangle

We now bring closure to the discussion of the AA criterion for similarity. If you look at all six criteria for congruence and similarity (page 81 and 97 ), you will notice that the hypothesis of each of them consists of three equalities except for the AA criterion, which has *two* equalities for angles. It is time to point out that the apparent difference is an illusion because we will prove the following theorem.

**Theorem 4 (Angle Sum Theorem).** *The* **angle sum** *of a triangle* (*i.e., the sum of the degrees of the angles in a triangle*) *is always 180 degrees.*

Thus if two pairs of angles in the triangles are equal, then all *three* pairs of angles are equal.

To prove that the angle sum of a triangle is always equal to 180 degrees, let triangle $ABC$ be given. On the ray from $B$ to $C$, take a point $D$ so that the segment $BD$ contains $C$. Through the point $C$, draw a line $CE$ parallel to $AB$, as shown.



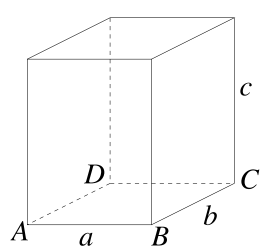

Now $|\angle A| = |\angle ACE|$ as they are alternate interior angles of $AC$ relative to the parallel lines $AB$ and $CE$ (Theorem 1(b) on page 89). In addition, $|\angle B| = |\angle ECD|$ because they are corresponding angles of $BD$ (Theorem 1(a) on page 89). Therefore the angle sum of triangle $ABC$ is equal to the sum of the angles that make up the straight angle $\angle BCD$, and we are done.

> *[Without going into details, we should mention that there are subtle issues inherent in this proof that we have chosen to neglect.]*

## 6. Volume formulas

In grade 6 (see page 40), we explained why if a (right) rectangular prism has dimensions $a$, $b$, $c$, its volume is $abc$ cubic units (i.e., if the linear unit is inches, the unit of the volume measure is inches$^3$, if the linear unit is cm., then the volume measure is in terms of cm.$^3$), etc. In grade eight, we expand the inventory of volume formulas to include those of a (generalized) right cylinder, a cone, and a sphere.

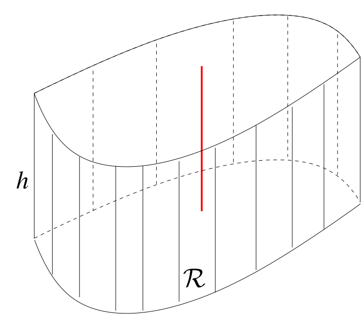
First we recall an interpretation of the volume formula for a rectangular prism. If we have such a prism, as shown,



and if we call the rectangle $ABCD$ the **base** of the prism and $c$ its **height**, then the area of the base is $ab$. Therefore the volume $abc$ of the prism can now be expressed as follows:

(A) volume of rectangular prism = (area of base) $\times$ height

In this form, this formula can be generalized in the following way. Let $\mathcal{R}$ be a region in the plane, then the **right cylinder over $\mathcal{R}$ of height $h$** is the solid which is the union of all the line segments of length $h$ lying above the plane, so that each segment is perpendicular to the plane and its lower endpoint lies in $\mathcal{R}$. When a right cylinder is understood, we usually say "cylinder" rather than "right cylinder". The region $\mathcal{R}$ is called the **base** of the cylinder. Notice that when $\mathcal{R}$ is a rectangle, the right cylinder over $\mathcal{R}$ is a rectangular prism, so that we are back to our starting point. Notice also that the **top** of a right cylinder (i.e., the points in the cylinder of maximum distance from the base) over $\mathcal{R}$ is also a planar region which is *congruent to* $\mathcal{R}$, but we will not spend time to explain what "congruent" means in three dimensions and will use the term in a naive sense.




Then we have:

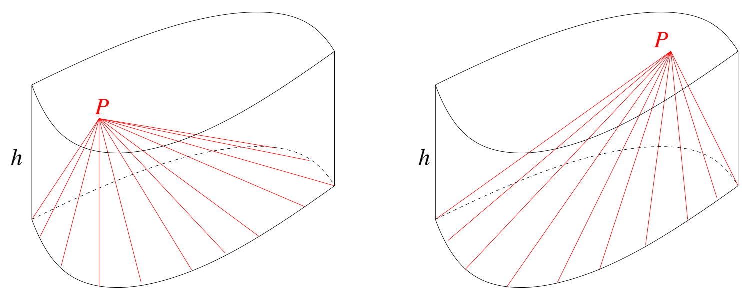
(B) volume of right cylinder over $\mathcal{R}$ of height $h$ = (area of $\mathcal{R}$) $\times$ $h$

So if $\mathcal{R}$ is a rectangle, this yields volume formula (A) for a rectangular prism, but if $\mathcal{R}$ is a circle of radius $r$, then the right cylinder over a circle of radius $r$ is called a **right circular cylinder**. The preceding formula then implies

(C) volume of right circular cylinder of radius $r$ and height $h = \pi r^2 h$

The case of a right circular cylinder is the most important example of a "cylinder" in school mathematics, but the reason we introduce the more general concept of a cylinder over an arbitrary planar region is that the explanations of the volume formulas (B) and (C) are the same. It is also important to recognize that there is only one general volume formula for cylinders, i.e., (B).

Let $P$ be a point in the plane that contains the top of a cylinder of height $h$. Then the union of all the segments joining $P$ to a point of the base $\mathcal{R}$ is a solid called a **cone with base $\boldsymbol{\mathcal{R}}$ and height $\boldsymbol{h}$** (see discussion in grade 6, page 45). The point $P$ is the **top vertex** of the cone. Here are two examples of such cones.

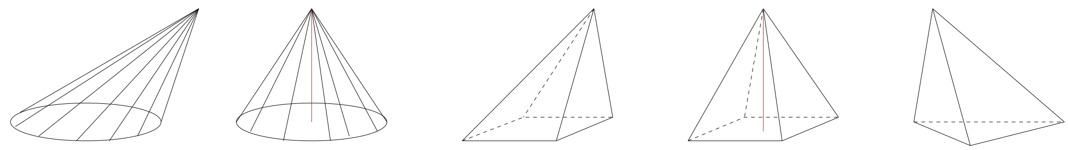



One has to be careful with the meaning of the word "cone" in the literature. If the base $\mathcal{R}$ is a circle, then this cone is called a **circular cone** (see left figure below). If the vertex of a circular cone happens to lie on the line perpendicular to the circular base at its center, then the cone is called a **right circular cone** (see second figure from left below). In everyday life, a "cone" is implicitly a right circular cone, and in many textbooks, this is how the word "cone" is used. The convention being used in this article regarding "cone" and "cylinder" is the one universally adopted in mathematics outside of K–12.

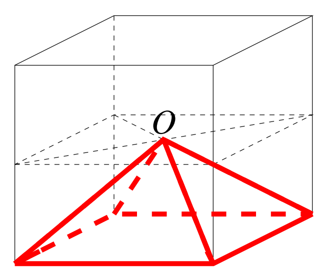
If the base of a cone is a square, then the cone is called a **pyramid** (see middle figure below) and if the base is a triangle, the cone is called a **tetrahedron** (see figure on extreme right below); see the discussion in grade 6 on page 45).

The fundamental formula here is

(D) volume of cone with base $\mathcal{R}$ and height $h$
$$= \frac{1}{3}\,\text{(volume of cylinder with same base and same height)}$$



Of great interest here is the factor $\frac{1}{3}$, which is independent of the shape of the base. How this factor comes about is most easily seen through the actual computations using calculus. However, even without the full arsenal of calculus, one can see the reason for the $\frac{1}{3}$ in an elementary way, as follows. Consider the **unit cube**, i.e., the rectangular prism whose sides all have length 1. The unit cube has a **center** $O$, and the simplest definition of $O$ may be through the use of the **mid-section**, which is the square that is halfway between the top and bottom faces (see the dotted square in the following picture), and let $O$ be the intersection of the diagonals of the mid-section. It is easy to convince oneself, at least at the intuitive level by looking at pictures, that $O$ is equidistant from all the vertices and also from all six faces.




Then the cone obtained by joining $O$ to all the points of one face is congruent[26] to the cone obtained by joining $O$ to all the points of any other face. There are six such cones, one for each face.

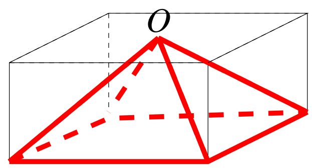
Let $\mathcal{C}$ be the cone joining $O$ to the base of the unit cube; it is the red cone above. Because congruent geometric figures have the same volume (see (c) on page 40 ), and because six cones congruent to $\mathcal{C}$ make up the unit cube, and the unit cube has volume 1 by definition, we obtain:

$$\text{volume of } \mathcal{C} \;=\; \frac{1}{6}$$

We have to interpret this formula the right way in order to bring out its significance.

[26]Again we leave undefined the meaning of "congruent" in this context and allow it to be understood in a naive sense.

Consider the rectangular prism which is the lower half of the unit cube, i.e., the part of the unit cube that is below the mid-section:




This particular rectangular prism has volume $\frac{1}{2}$, and since $\frac{1}{6}$ is equal to $\frac{1}{3} \times \frac{1}{2} = \frac{1}{3}$ $\times$ (volume of this short prism), we have

$$\text{volume of the cone } \mathcal{C} = \frac{1}{3} \text{ (volume of cylinder with same base and same height)}$$

Here we see the emergence of the factor of $\frac{1}{3}$, and this is no accident because, using ideas from calculus, one can show that if the preceding formula is true for *one* cone $\mathcal{C}$, then it is true for *all* cones.

Finally, we come to the volume formula of a sphere of radius $r$:

$$\text{(E)} \quad \text{volume of sphere of radius } r \ = \ \frac{4}{3}\,\pi\, r^3$$

The derivation of this formula is sophisticated and will have to be left to a high school course. The discovery of this formula by Archimedes (287–212 B.C.) was a major event in the mathematics of antiquity. Some seven centuries later, the same formula was independently discovered in China by Zu Chongzhi (A.D. 429–501) and his son Zu Geng (circa A.D. 450–520) by essentially the same method, which has come to be known as *Cavalieri's Principle* (Bonaventura Cavalieri, 1598-1647). Cavalieri was a foreunner of calculus and a main reason his name is associated with the method is that, by the time he enunciated it, his contemporaries were ready for calculus and therefore he had an audience. Archimedes and the Zus were centuries ahead of their times.

# HS GEOMETRY

## **Congruence** G-Co

### Experiment with transformations in the plane

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

### Understand congruence in terms of rigid motions

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from

the definition of congruence in terms of rigid motions.

### Prove geometric theorems

9. Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

10. Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

11. Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.

### Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.

13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

## **Similarity, right triangles, and trigonometry** G-Srt

### Understand similarity in terms of similarity transformations

1. Verify experimentally the properties of dilations given by a center and a scale factor:

a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar. Prove theorems involving similarity.

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

### Define trigonometric ratios and solve problems involving right triangles

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

7. Explain and use the relationship between the sine and cosine of complementary angles.

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

#### Apply trigonometry to general triangles

10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

### **Circles** G-C

#### Understand and apply theorems about circles

1. Prove that all circles are similar.

2. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

4. (+) Construct a tangent line from a point outside a given circle to the circle.

### **Geometric measurement and dimension** G-Gmd

#### Explain volume formulas and use them to solve problems

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's*

*principle, and informal limit arguments.*

2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

## Some key ideas in high school geometry

There are many geometry standards for high school; we have listed only those that make up the core of a typical high school geometry course. The following commentary on such a course will focus on a few key ideas:



One cannot understand the CCSSM approach to high school geometry without a knowledge of the pedagogical problems inherent in such a course in TSM (see page 2) and the reality of the high school geometry classroom in the past two decades. One such discussion is given in (recall: this is an active link):

H. Wu, Euclid and high school geometry

For further details of the logical development outlined below, together with a treatment of other topics not mentioned here (e.g., similarity, circles, etc.), see

H. Wu, Teaching Geometry in Grade 8 and High School
According to the Common Core Standards

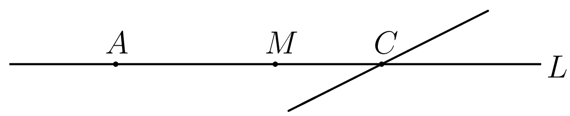
A main feature of the CCSSM is the seamless transition from eighth grade geometry to high school geometry. In broad terms, the key geometric ideas (basic rigid motions and dilations) are gently introduced in grade eight through hands-on activities; the high school course then begins the normal mathematical study of the plane by building on students' prior empirical experience with these transformations. In lieu of the usual axioms, the high school course begins with a precise summary this empirical experience in the form of eight assumptions (see in particular page 75 ). Insofar as these assumptions go straight to the essence of congruence—one of the mysteries of most students' learning experience—they are more intuitive and more accessible than the axioms of the usual abstract approach. Due to space limitations, *the following commentary is devoted mainly to the* transition *from the discussion of rigid motions to the proofs of the three congruence criteria for triangles* (*SAS, ASA and SSS*)*, but not to the subsequent mathematical development once these criteria are established.* For a detailed discussion of this development, one may consult the second of the documents listed above.

In order to lay a firm foundation for the geometric proofs, we need a clear enunciation of the basic definitions and assumptions. The two are intertwined. Sometimes a definition must be in place first before the assumption can be stated: the Parallel Postulate would make no sense if we do not know the meaning of "parallel lines". At other times, a definition becomes meaningful only after an assumption has been made: the definition of a "ray" has mathematical substance only because the Line Separation assumption (page 118 ) guarantees that every point on a line determines two rays with that point as vertex. The first section below will briefly summarize the main points of this foundational material. The second and third sections outline the path from these assumptions to the proofs of the basic congruence criteria for triangles (ASA, SAS, and SSS) and, in so doing, they realize the content of Standard G-Co 8:

> *Explain how the criteria for triangle congruence* (*ASA, SAS, and SSS*) *follow from the definition of congruence in terms of rigid motions.*

It will be seen that this can be done quite quickly, so that students will be spared the usual doldrums of axiomatic treatments that spend more than a hundred pages to lay the groundwork for the proofs of theorems of the following type:

- Any two right angles are congruent.
- Every angle has exactly one bisector.
- If $M$ is a point between points $A$ and $C$ on a line $L$, then $M$ and $A$ are on the same side of any other line that contains $C$.




There is something to be said about the value of being able to *prove* these geometrically obvious facts, but for most students, spending two to three months of the school year just to learn about such proofs does not constitute an inspiring learning experience. The goal of the CCSSM is to steer clear of such an approach to geometry by putting geometry on an equal footing with any other part of school mathematics. Geometry can be learned the same way fractions or algebra is learned: *there should be reasoning and there should be proofs in every part of the mathematics curriculum, but there should also be a minimum of formalism.*

In two passages on page 150 and page 154, we point out how this mathematical approach to plane geometry impacts the teaching of geometry.

## 1. Basic assumptions and definitions

We start from the beginning and go through all the known geometric concepts one by one in a systematic fashion, with precision but without any preconceptions. We will begin with a precise enunciation of ***what we assume to be known about the plane***.

We will not explain what a **point** is or what a **line** is other than to say that we understand them in the intuitive sense and that a "line" stands for a "straight line" that is *infinite in both directions.* The eight assumptions are listed as (A1)–(A8) and they can all be found in this section. Every single one of them is intuitively obvious, and the only reason we enunciate them is to make sure that we all have a clearly defined common starting point.

**(A1)** *Through two distinct points passes a unique line.*

Two lines are said to be **distinct** if there is at least one point that belongs to one but not the other; otherwise we say the lines are **the same**. Lines that have no point in common are said to be **parallel**. In symbols, $L_1$ parallel to $L_2$ is denoted by $\boldsymbol{L_1 \parallel L_2}$.

**(A2) (Parallel Postulate)** *Given a line $L$ and a point $P$ not on $L$. Then through $P$ passes at most one line that does not intersect $L$.*

In other words, we assume as obvious that for a point $P$ in the plane not lying on a line $L$, every line that contains $P$ intersects $L$ except possibly for one line. In school textbooks, the statement of the Parallel Postulate also includes the statement that there *is* a line passing through $P$ and parallel to $L$. However, (A2) above is the original version of the Parallel Postulate, and we shall see in the Corollary to Theorem 1 on page 135 that the existence of such a parallel line can in fact be proved once we know there are enough rotations in the plane. Thus, contrary to what is normally done in school textbooks, our formulation of the Parallel Postulate merely asserts that there is *no more than one parallel line.*

It will be seen that the Parallel Postulate dominates plane geometry by intervening in logical arguments at critical moments.

If $A$ and $B$ are two distinct points, then by (A1), there is a unique line containing $A$ and $B$. We may consider this as a number line. Denote it by $\boldsymbol{L_{AB}}$ and call it **the line joining $\boldsymbol{A}$ and $\boldsymbol{B}$**. On $\boldsymbol{L_{AB}}$, denote by $\boldsymbol{AB}$ the collection of all the points *between* $A$ and $B$ together with the points $A$ and $B$ themselves (recall that on a number line, we know what it means for a number to be between two other numbers). We call $AB$ the **line segment**, or more simply the **segment joining** $A$ **and** $B$, and the points $A$ and $B$ are called the **endpoints** of the segment $AB$. The term **segment** will be used in general to refer to the segment joining a pair of points.

With the concept of segment available, we are now in a position to define a polygon. (In the classroom, one would start with the definition of a triangle and a quadrilateral before tackling the general case, and care should be given to motivating the use of subscripts.) Let $n$ be any positive integer $\geq 3$. An **n-sided polygon** (or more simply an **n-gon**) is by definition a geometric figure consisting of $n$ distinct points $A_1, A_2, \ldots, A_n$ *in the plane*, together with the $n$ segments $A_1A_2$, $A_2A_3$, $\ldots$, $A_{n-1}A_n$, $A_nA_1$ so that *none of these segments intersects any other except at the endpoints as indicated*, i.e., $A_1A_2$ intersects $A_2A_3$ at $A_2$, $A_2A_3$ intersects $A_3A_4$ at $A_3$, etc. In symbols: the polygon will be denoted by $A_1A_2 \cdots A_n$. If $n = 3$, the polygon is called a **triangle**; $n = 4$, a **quadrilateral**; $n = 5$, a **pentagon**; and if $n = 6$, a **hexagon**. Given polygon $A_1A_2 \cdots A_n$, the $A_i$'s are called the **vertices** and the segments $A_1A_2$, $A_2A_3$, etc. the **edges** or sometimes the **sides**.

In order to define angles, we need to know a little bit more about lines. To this end, we first introduce a definition. A subset $\mathcal{R}$ in a plane is called **convex** if given any two points $A$, $B$ in $\mathcal{R}$, the segment $AB$ lies completely in $\mathcal{R}$.

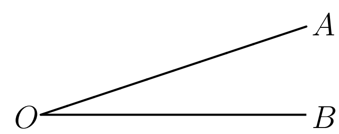
**(A3) (Line separation)** *A point $P$ on a line $L$ separates $L$ into two* non-empty *convex subsets $L^+$ and $L^-$, called* **half-lines***, so that:*

> $(i)$ *Every point of $L$ is in one* and only one *of the sets $L^+$, $L^-$, and the set $\{P\}$ consisting of the point $P$ alone.*
>
> $(ii)$ *If two points $A$ and $B$ belong to different half-lines, then the line segment $AB$ contains $P$.*

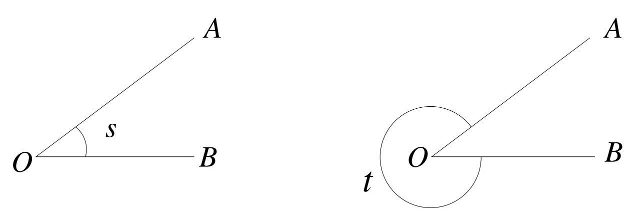

It follows from $(i)$ that any two of the sets $L^+$, $L^-$, and $\{P\}$ are **disjoint**, i.e., do not share a point in common. It also follows from the convexity of $L^+$ and $L^-$ that if two points $A$, $B$ belong to the same half-line, then the line segment $AB$ does not contain $P$:


The set consisting of the point $P$ and the points from a half-line, $L^+$ or $L^-$, is called a **ray**. We also say these are **rays issuing from $P$**. If we want to specifically refer to the ray containing $A$, we use the symbol $\boldsymbol{R_{PA}}$. We will also refer to $\boldsymbol{R_{PA}}$ as the **ray from $\boldsymbol{P}$ to $\boldsymbol{A}$**. Similarly, the ray containing $\boldsymbol{B}$ issuing from $\boldsymbol{P}$ is denoted by $\boldsymbol{R_{PB}}$. The point $P$ is the **vertex of $\boldsymbol{R_{PB}}$**. If $P$ is between $A$ and $B$, then the two rays $R_{PA}$ and $R_{PB}$ have only the vertex $P$ in common, and each ray is, intuitively, infinite in only one direction.

Two rays are **distinct** if there is a point in one that does not lie in the other. An **angle** is the union of two distinct rays with a common vertex. (This is the definition adopted by the CCSSM; see page 8.) The angle formed by the two rays $R_{OA}$ and $R_{OB}$ will be denoted by $\angle \boldsymbol{AOB}$.



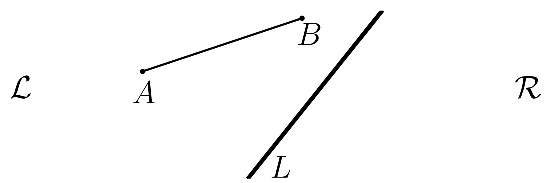

If $A$, $O$, $B$ are **collinear** (i.e., lie on a line so that $O$ is between $A$ and $B$), we say the angle is a **straight angle**. If $R_{OA}$ and $R_{OB}$ coincide, then we do not have an angle according to the definition above, but we make an exception and call it the **zero angle**. Now we have to face up to the fact that the intuitive concept of an angle is not just "two rays with a common vertex" but also "the space between these two rays". In other words, if $\angle AOB$ is neither the zero nor the straight angle, which of the following two subsets of the plane do we have in mind when we say $\angle AOB$, the space indicated by $s$ or the one indicated by $t$?



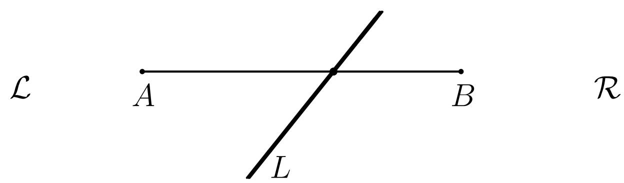

To resolve this difficulty, we need a precise way to differentiate between the two.

**(A4) (Plane separation)** *A line $L$ separates the plane into two* non-empty *convex subsets, $\mathcal{L}$ and $\mathcal{R}$, called* **half-planes***, so that*:

> $(i)$ *Every point in the plane is in one* and only one *of the sets $\mathcal{L}$, $\mathcal{R}$, and $L$.*




*(ii) If two points $A$ and $B$ in the plane belong to different half-planes, then the line segment $AB$ must intersect the line $L$.*



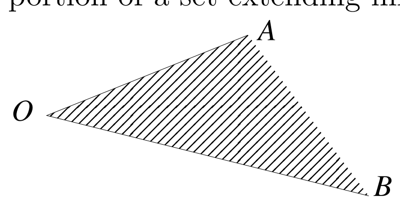

Two points that lie in the same half-plane of $L$ are said to be **on the same side of $L$**, and two points that lie in different half-planes are said to be **on opposite sides of $L$**. The union of either $\mathcal{L}$ or $\mathcal{R}$ with $L$ is called a **closed half-plane**.

Now we return to an angle $\angle AOB$ which is neither the zero angle nor a straight angle. The rays $R_{OA}$, $R_{OB}$ determine two subsets of the plane, one of them is the intersection of the following two closed half-planes:

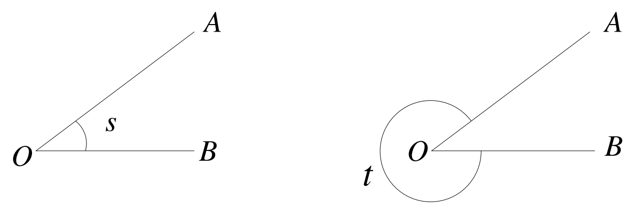
the closed half-plane of the line $L_{OA}$ containing $B$, and

the closed half-plane of the line $L_{OB}$ containing $A$.

It is straightforward to show that the intersection of a finite number of convex sets is convex, and that closed half-planes are also convex (we already know that the half-planes are convex). Therefore the intersection of these two closed half-planes is convex, and is suggested by the shaded set in the following figure (note that the shading only covers a finite portion of a set extending infinitely to the right).

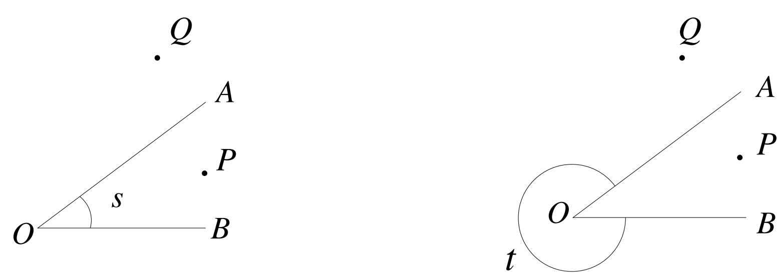



We will refer to this set as the **convex part** of $\angle AOB$, and this is the set that corresponds to our intuitive notion of what $\angle AOB$ is. When we refer to the convex part of an angle, we sometimes denote it by a single letter, e.g., $\angle \boldsymbol{O}$, if there is no danger of confusion. See the part indicated by $s$ in the following:




On the other hand, there will be occasions to use the other subset of the plane determined by the rays $R_{OA}$, $R_{OB}$. This would be the **nonconvex part** of $\angle AOB$ as indicated by $t$ above. Precisely, this is all the points that do not lie in the convex part of $\angle AOB$, together with all the points on both of the rays, $R_{OA}$ and $R_{OB}$.

Given an angle $\angle AOB$ that is neither the straight angle nor the zero angle, then we see that it is necessary to specify whether we mean the convex part or the nonconvex part of $\angle AOB$. Once that is done, it makes sense to say whether a point of the plane **belongs to the angle** or not. For example, in the pictures below, if $\angle AOB$ refers to the convex part, then $P$ belongs to $\angle AOB$ while $Q$ does not, but if the nonconvex part is meant, then $P$ does not belong to $\angle AOB$ but $Q$ does.




*Unless stated otherwise, a* (*nonzero and non-straight*) *angle will refer to the convex part of the angle.*

Our next goal is to formalize the concept of the *length* of a segment in the plane and introduce the measurement of angles in terms of *degree*. Length has to come first, so we start with that. We have thus far taken the concept of length lightly because it seems to come naturally to us. However, if we look at it critically, it is not as simple as it appears. Suppose we have a number line, then even the length of a segment $I$ *on the number line* has to be defined with the use of a *translation*. Indeed, let us recall how that is done: we translate the segment $I$ until its left endpoint is over 0, then the number on which its right endpoint rests is by definition the length of $I$. If the segment now lies on a line $L$ in the plane, then because there is no pre-ordained unit on $L$ (i.e., we don't know in advance how far apart 0 and 1 may be), the length of the segment becomes indeterminate unless there is a way to consistently "organize", so to speak, all the unit segments on all the lines which lie in the plane. This "organization" is made possible by the concept of *distance* in the plane. The next assumption spells out part of the basic properties we expect of *distance*; the remaining properties are included in assumption (A7) on page 133.

**(A5)** *To each pair of points $A$ and $B$ of the plane, we can assign a number* $\boldsymbol{\mathrm{dist}(A,B) \geq 0}$ *so that*

(*i*) $\mathrm{dist}(A,B) = \mathrm{dist}(B,A)$.

(*ii*) $\mathrm{dist}(A,B) \geq 0$, *and* $\mathrm{dist}(A,B) = 0 \iff A$ and $B$ *coincide.* (The symbol "$\iff$" stands for "is equivalent to".)

(*iii*) *If $A$, $B$, $C$ are collinear points, and $C$ is between $A$ and $B$, then*

$$\mathrm{dist}(A,B) = \mathrm{dist}(A,C) + \mathrm{dist}(C,B)$$

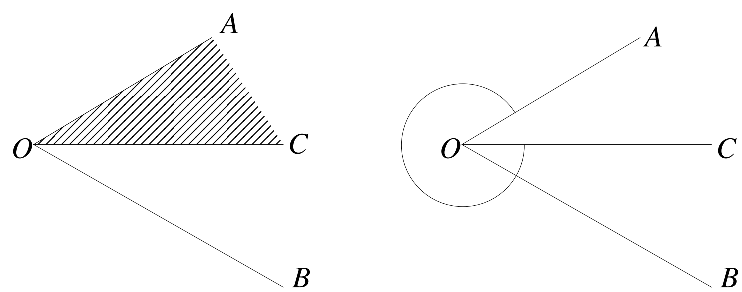
Of course, condition (*iii*) is one reason why the assignment of a nonnegative number $\mathrm{dist}(A,B)$ to each pair of points $A$ and $B$ cannot be random or arbitrary. (The other reason is encoded in assumption (A7) below.) Once we have the concept of distance between points in the plane, we can define, for any two points $A$ and $B$ in the plane, the **length of the segment $\boldsymbol{AB}$**, denoted by $|\boldsymbol{AB}|$, to be $\mathrm{dist}(A,B)$. In effect, what the concept of *distance* has done is to decree that, on each line, a unit segment has to be a segment whose endpoints are apart by a distance equal to 1. That done, one can see that the *length* between two points $A$ and $B$ in the plane will now be the same as the usual length of the segment $AB$ on the line $L_{AB}$, which is now a number line with a prescribed unit segment.

We say **two segments are equal** if they have the same length.

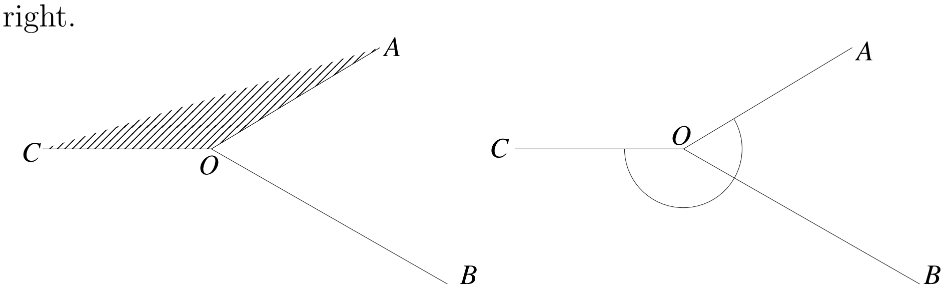
The concept of distance allows us to formally introduce the concept of a *circle*. Fix a point $O$. Then the set of all the points $A$ in the plane so that $\mathrm{dist}(O,A)$ is a fixed positive constant $r$ is called the **circle of radius $\boldsymbol{r}$ about $\boldsymbol{O}$.** The point $O$ is called the **center** of the circle. A line passing through the center $O$ will intersect the circle at two points, say $P$ and $Q$; the segment $PQ$ is then called a **diameter** of the circle and the segment $OP$ (or $OQ$) is called a **radius** of the circle.

A circle whose radius is of length 1 is called a **unit circle**.

We need one more definition before we can introduce the concept of the *degree* of an angle. Given an angle $\angle AOB$ with either the convex part or the nonconvex part specified, we say two angles $\angle AOC$ and $\angle COB$, with a common side $R_{OC}$, are **adjacent angles with respect to $\angle AOB$** if $C$ belongs to $\angle AOB$ (see definition on page 121) and $\angle AOC$ and $\angle COB$ are understood to be part of $\angle AOB$. For example, if $\angle AOB$ is understood to denote the convex part, then in the following picture, $\angle AOC$ has to mean the convex (shaded) part on the left rather than the nonconvex part indicated by the arc on the right.




On the other hand, if $\angle AOB$ is understood to denote the nonconvex part as in the picture below, then the point $C$ again belongs to $\angle AOB$, and $\angle AOC$ and $\angle COB$ are adjacent angles with respect to $\angle AOB$, provided $\angle AOC$ is understood to be the convex (shaded) part on the left rather than the nonconvex part indicated by the arc on the right.




Adjacent angles $\angle AOC$ and $\angle COB$ (with respect to $\angle AOB$) are the analogs, among angles, of segments $AC$, $CB$ so that $A$, $B$, $C$ are collinear and $C$ is between $A$ and $B$; they will allow us to formulate the analog of condition $(iii)$ in assumption (A5) above.

Now we can introduce the concept of the degree of an angle by way of an assumption. Intuitively, every angle has a degree, a straight angle should be 180 degrees, and the "full" angle should be 360 degrees. Our assumption now takes the following form:

**(A6)** *To each angle $\angle AOB$, we can assign a number $|\angle \boldsymbol{AOB}|$, called its* **degree**, *so that*

> $(i)$ $0 \le |\angle AOB| < 360°$, *where the small circle ° is the abbreviation of* degree. *Moreover, if a ray $R_{OB}$ and a number $x$ are given so that* $0 < x < 360$ *but* $x \neq 180$, *and if a half-plane of the line $L_{OB}$ is specified, then there is a unique angle $AOB$ so that $|\angle AOB| = x°$ and the ray $R_{OA}$ lies in the specified half-plane of $L_{OB}$.*
>
> $(ii)$ $|\angle AOB| = 0° \iff \angle AOB$ *is the zero angle, and* $|\angle AOB| = 180° \iff \angle AOB$ *is a straight angle.*
>
> $(iii)$ *If $\angle AOC$ and $\angle COB$ are adjacent angles with respect to $\angle AOB$, then*
>
> $$|\angle AOC| + |\angle COB| \;=\; |\angle AOB|$$

Notice that $(i)$ and $(ii)$ together cover all angles regardless of whether the degree is 180 or not.

One can give an intuitive discussion of how to reconcile this definition of degree with the one on page 10 , as follows.

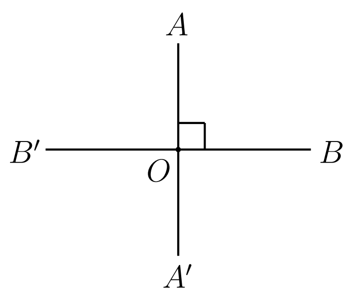
Suppose $\angle AOB$ is the zero angle (so that the rays $R_{OA}$ and $R_{OB}$ coincide). Then the angle corresponding to the nonconvex subset of the zero angle, which is then the whole plane, is sometimes called the **full angle**. We will agree to define the degree of a full angle to be 360° so that $(iii)$ in (A6) is now true without restrictions.

The existence of the concept of degree allows us to define that **two angles are equal** if they have the same degree.

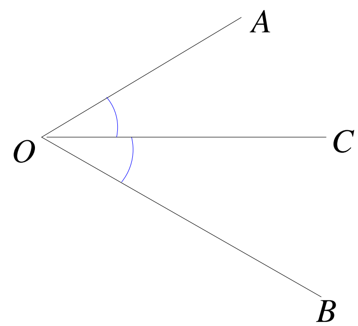
We note that by themselves, assumptions (A5) on distance and (A6) on degree do not have much substance. Their significance will be revealed only when we make the next assumption that the basic rigid motions are distance-preserving and degree-preserving and prove that there are "plenty of" basic rigid motions in the plane (for the latter, see Lemmas 2, 3, and 5 on page 138, page 138, and page 140, respectively).

With the measurements of angles available, we can introduce some standard terminology for angles and polygons. Two angles are defined to be **equal** if they have the same degree. An angle of $90^\circ$ is called a **right angle**. An angle is **acute** if it is less than $90^\circ$, and is **obtuse** if it is greater than $90^\circ$. There are analogs of these names for triangles, namely, a triangle is called a **right triangle** if one of its angles is a right angle, an **acute triangle** if all of its angles are acute, and an **obtuse triangle** if (at least) one of its angles is obtuse. (In view of the Angle Sum Theorem in grade 8, page 104, at most one angle of a triangle can be obtuse.)

Let two lines meet at $O$, and suppose one of the four angles, say $\angle AOB$ as shown, is a right angle.

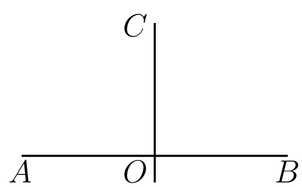



Then one sees easily that all the remaining angles are also right angles. It is therefore unambiguous to define the two lines to be **perpendicular** if an angle formed by the two lines at the point of intersection is a right angle. In symbols: $\boldsymbol{L_{AO} \perp L_{OB}}$ in the notation of the preceding figure, although it is equally common to write instead, $\boldsymbol{AO \perp OB}$. In general, if a point $C$ belongs to an angle $\angle AOB$ (see page 121), the ray $R_{OC}$ is called an **angle bisector** of $\angle AOB$ if the adjacent angles $\angle AOC$ and $\angle COB$ (with respect to $\angle AOB$) are equal.




Sometimes we also say less precisely that **the line $L_{OC}$ (rather than the ray $R_{OC}$) bisects the angle $AOB$.**

It is clear that an angle has one and only one angle bisector (by $(i)$ of assumption (A6)). Therefore if $CO \perp AB$ where $O$ is a point of $AB$, as shown below,



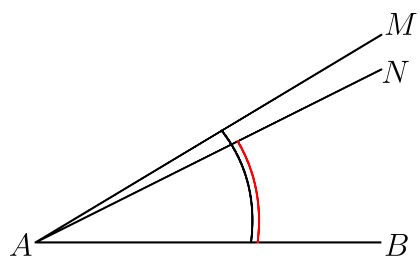

then $CO$ is the unique angle bisector of the straight angle $\angle AOB$. Hence,

> *Let $L$ be a line and $O$ a point on $L$. Then there is one and only one line passing through $O$ and perpendicular to $L$.*

We can now complete the list of standard definitions about lines and segments. If $AB$ is a segment, then the point $O$ in $AB$ so that $|AO| = |OB|$ is called the **midpoint** of $AB$. Analogous to the angle bisector, the **perpendicular bisector** of a segment $AB$ is the line perpendicular to $L_{AB}$ and passing through the midpoint of $AB$. It follows from the uniqueness of the line perpendicular to a line passing through a given point that *there is one and only one perpendicular bisector of a segment.*

We now introduce some common names for certain triangles and quadrilaterals. An **equilateral triangle** is a triangle with three sides of the same length, and an **isosceles triangle** is one with at least two sides of the same length. A quadrilateral all of whose angles are right angles is called a **rectangle**. A rectangle all of whose sides are of the same length is called a **square**. Be aware that at this point, we do not know whether there is a square or not, or worse, whether there is a rectangle or not. A quadrilateral with at least one pair of opposite sides that are parallel is called a **trapezoid**. A trapezoid with two pairs of parallel opposite sides is called a **parallelogram**.

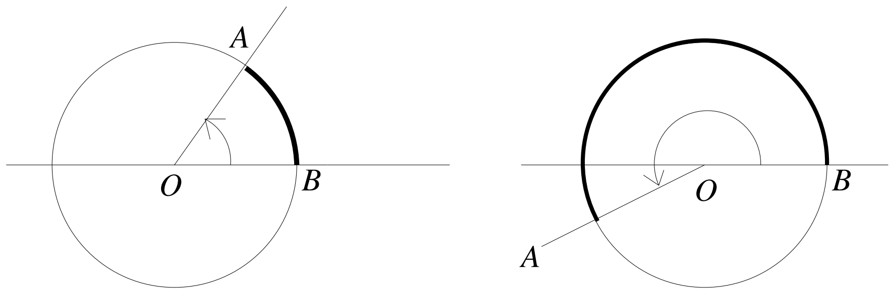
We conclude by making a general observation about angles that follows easily from assumption (A6)($i$):




**Lemma 1.** *Given two equal angles $\angle MAB$ and $\angle NAB$ with one side $R_{AB}$ in common and $M$ and $N$ are on the same side of the line $L_{AB}$, then the other sides $R_{AM}$ and $R_{AN}$ coincide.*

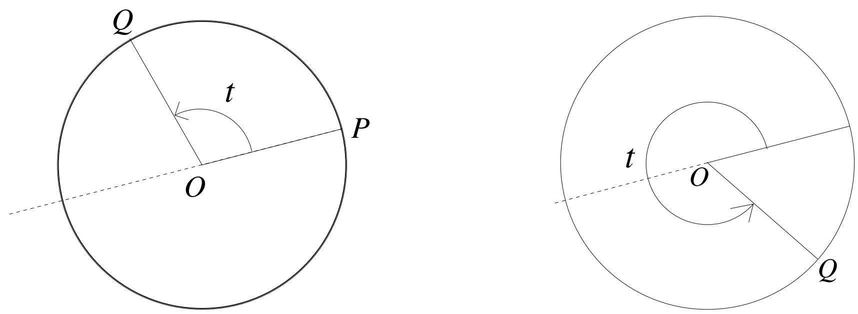
At this point, we begin the discussion of the most substantial part of definitions and assumptions by taking on the **basic rigid motions**. In eighth grade, we introduced them as rotations, reflections, and translations, and we did so mostly through the use of transparencies. Now we are going to define them precisely, and to this end, we introduce the concept of a **transformation** $F$ of the plane as a rule that assigns to each point $P$ of the plane a (single) point $F(P)$ of the plane. As in grade 8, $F(P)$ is called the **image of $P$ by $F$** and often we speak of **$F$ mapping $P$ to $F(P)$.** If $\mathcal{S}$ is a **geometric figure** in the plane (i.e., a subset of the plane), then the collection of all the points $F(Q)$ where $Q$ is a point of $\mathcal{S}$ is called the **image of $\mathcal{S}$ by $F$**, which is usually denoted by **$F(\mathcal{S})$**. We likewise say **$F$ maps $\mathcal{S}$ to $F(\mathcal{S})$.**

We now give in succession the definitions of the basic rigid motions: rotation, reflection, and translation. Before we give the definition of rotation, we mention explicitly that we will freely avail ourselves of the concepts of **clockwise** direction and **counterclockwise** direction on a circle. Because a 180° angle is the straight angle, if a point $B$ is fixed on a circle with center $O$, then all the points $A$ so that $A$ is in the counterclockwise (respectively, clockwise) direction of $B$ and so that $0 < |\angle AOB| < 180°$ will lie in a half-plane of the line $L_{OB}$, and all the points $A$ so that $180° < |\angle AOB| < 360°$ lie in the opposite half-plane. In the counterclockwise case, these would be the upper half-plane and lower half-plane, respectively, in the following pictures.

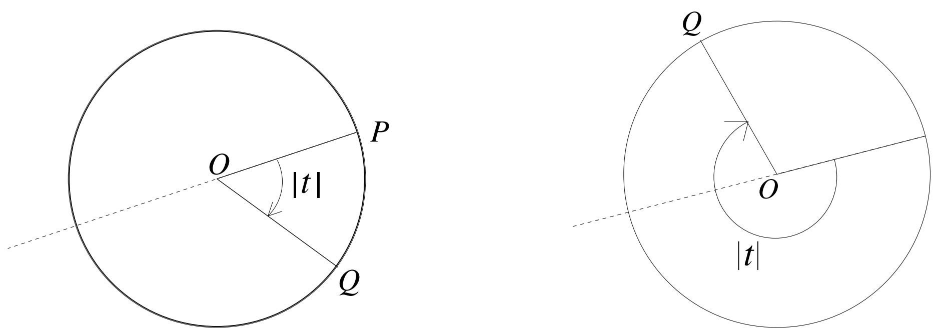



The whole discussion can be made more precise if we are willing to engage in more formalism and introduce some elaborate definitions, but that may not be the best use of class time for the learning of geometry at this juncture.

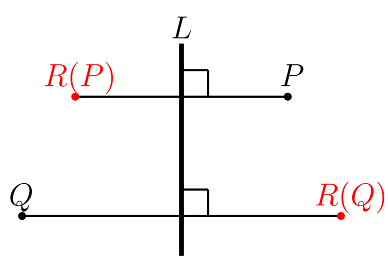
**Rotation.** The **rotation *Ro* of *t* degrees** $(-360 < t < 360)$ around a given point $O$, called the **center of the rotation**, is a transformation of the plane defined as follows. Given a point $P$, the point $Ro(P)$ is defined as follows. The rotation is counterclockwise or clockwise depending on whether the degree is positive or negative, respectively. For definiteness, we first deal with the case where $0 \le t < 360$. If $P = O$, then by definition, $Ro(O) = O$. If $P$ is distinct from $O$, then by definition, $Ro(P)$ is the point $Q$ on the circle with center $O$ and radius $|OP|$ so that $|\angle QOP| = t^\circ$ and so that $Q$ is in the counterclockwise direction of the point $P$. We claim that this assignment is unambiguous, i.e., there cannot be more than one such $Q$. Indeed, if $t = 180$, then $Q$ is the point on the circle so that $PQ$ is a diameter of the circle. If $t = 0$, then $Q = P$. Now if $0 < t < 180$, then all the $Q$'s in the counterclockwise direction of the point $P$ with the property $|\angle QOP| = t$ lie in the half-plane of the line $L_{OP}$ that contains $Q$, and if $180 < t < 360$, then all the $Q$'s in the clockwise direction of the point $P$ with the property $|\angle QOP| = t$ lie in the half-plane of the line $L_{OP}$ not containing $Q$. By Lemma 1 (page 127), there is only one such $Q$ (see the pictures below). Thus $Ro$ is **well-defined**, in the sense that the rule of assignment is unambiguous. Notice that if $t = 0$, then $Ro$ is the identity transformation $I$ of the plane.




Now suppose $-360 < t < 0$. Then by definition, we rotate the given point $P$ *clockwise* on the circle that is centered at $O$ with radius $|OP|$. Everything remains the same except that the point $Q$ is now the point on the circle so that $|\angle QOP| = |t|^\circ$ and $Q$ is in the clockwise direction of $P$ (see the pictures below). We define $Ro(P) = Q$.

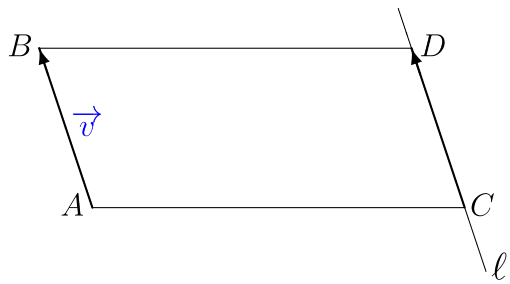



**Reflection.** The **reflection $R$ across a given line $L$**, where $L$ is called **the line of reflection**, assigns to each point on $L$ the point itself, and to any point $P$ not on $L$, $R$ assigns the point $R(P)$ which is **symmetric to it with respect to $L$**, in the sense that $L$ is the perpendicular bisector (page 126 ) of the segment joining $P$ to $R(P)$.

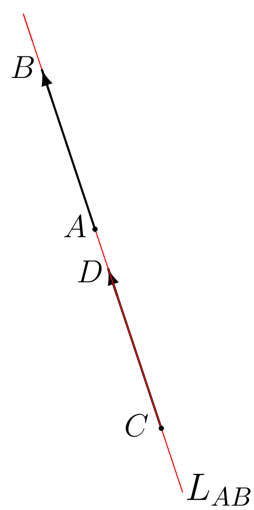



**Translation.** The **translation $T$ along a given vector $\overrightarrow{v}$** assigns the point $D$ to a given point $C$ in the following way. First, a vector is defined as in Grade 8, page

69 . Let the starting point and endpoint of $\overrightarrow{v}$ be $A$ and $B$, respectively. First assume $C$ does not lie on line $L_{AB}$. Draw the line $\ell$ parallel to line $L_{AB}$ passing through $C$. The line $L$ passing through $B$ and parallel to the line $L_{AC}$ then intersects line $\ell$ at a point $D$ ($L$ and $\ell$ must intersect because the Parallel Postulate says that through the point $C$, $L_{AC}$ is the only line parallel to $L$ and therefore $\ell$ is not parallel to $L$). By definition, $T$ assigns the point $D$ to $C$, i.e., $T(C) = D$.




If $C$ lies on the line $L_{AB}$, then the image $D$ is by definition the point on the line $L_{AB}$ so that the direction from $C$ to $D$ is the same as the direction from $A$ to $B$ and so that $|CD| = |AB|$. (Recall that we may regard $L_{AB}$ as the number line so that all the points on $L_{AB}$ are now numbers; then the direction from $C$ to $D$ being the same as the direction from $A$ to $B$ means $D$ is the number so that $D - C = B - A$.)




Observe that if $\overrightarrow{0}$ is the zero vector, i.e., the vector with 0 length, then the translation along $\overrightarrow{0}$ is the identity transformation $I$.

We now take a critical look at the preceding definitions.

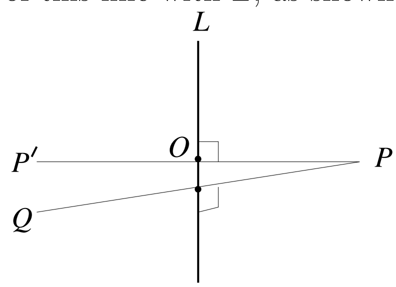
The definition of rotation is straightforward, but the definitions of translation and reflection raise some unanswered questions. In the definition of translation, the statement that we take $L$ to be the line passing through $B$ and parallel to $L_{AC}$ begs the question of how we know that there is such a line $L$. Recall that the Parallel Postulate as formulated on page 117 does not guarantee that there is such a line, only that there is no more than one such line. Fortunately, this issue will be quickly disposed of by the Corollary to Theorem 1 on page 135. The difficulty with the definition of reflection is, however, more knotty, and we will explain it as follows. Let a line $L$ be given and let $P$ be a point not lying on $L$. Let the reflection across $L$ be denoted by $R$. The definition of the point $R(P)$, to be denoted more simply by $P'$, is that $L$ is the perpendicular bisector of the segment $PP'$. Implicit in this definition is the fact that **(a)** there is such a point $P'$ so that $L$ is the perpendicular bisector of the segment $PP'$, and **(b)** there is only one such point $P'$. Neither is obvious at the moment. The need for (a) is obvious, but the need for (b) maybe less so. The fact is, if there is another point $Q$ distinct from $P'$ so that $L$ is the perpendicular bisector of $PQ$, then the definition of a reflection implies that we can also define $R(P) = Q$. This would then raise the question about which of the two points $R$ is going to assign to $P$, $P'$ or $Q$? (A transformation, by definition, assigns to a given point *a single point* without ambiguity.)

If we cannot verify that both (a) and (b) are valid, then the concept of a reflection is **not well-defined** on two levels. Given a line $L$ and a point $P$ in the plane, either the reflection $R$ across $L$ does not know which point to assign to $P$ (this would be the case if (a) fails), or there is more than one candidate for such a $P'$ so that the assignment of $R$ to $P$ becomes ambiguous (this would be the case if (b) fails).

We will resolve this difficulty by proving the following theorem, and in the process of doing this, we shall resolve the difficulty in the definition of translation as well.

**Theorem.** *Given a line $L$ and a point $P$, there is one and only one line passing through $P$ and perpendicular to $L$.*

Assuming this theorem, (a) is easily seen to be true because if there is such a line, we simply let $P'$ be the point on this line on the other side of $L$, so that $|PO| = |P'O|$, where $O$ is the intersection of this line with $L$, as shown.




Moreover, (b) is also true because, if there is another point $Q$ so that $L$ is also the perpendicular bisector of $PQ$, then in particular $PQ \perp L$. But we know there is only one such line, so the two lines $L_{PP'}$ and $L_{PQ}$ coincide and the point $Q$ falls on $L_{PP'}$. It follows that $Q$ and $P'$ are two points on the same half-line of the line $L_{PP'}$ with respect to $O$ and $|QO| = |P'O|$ $(= |PO|)$. Hence $Q = P'$ and (b) is also true.

We will prove the Theorem in the next section. Because the definitions of reflection and translation come after the definition of rotation, we will be able to avail ourselves, for the proof of this Theorem, of some properties that we will assume about rotations. To this end, and for the development of plane geometry as a whole, we now state our assumptions about rotations. These assumptions should impress the students as being completely unexceptional because of the hands-on experiences they have had with the basic rigid motions in grade 8. Precisely, we assume that:

Ro1. Rotations map lines to lines, rays to rays, and segments to segments.

Ro2. Rotations are **distance-preserving** (and therefore length-preserving), meaning that the distance between the images of two points is always equal to the distance between the original two points.

Ro3. Rotations are **degree-preserving**, meaning that the degree of the image of an angle is always equal to the degree of the original angle.

Note that, as in grade 8, assumption Ro1 guarantees that a rotation maps an angle to an angle (see page 76 ), so that assumption Ro3 makes sense.

Ultimately, we have to make the same assumptions about not just rotations, but also reflections and translations as well. Therefore, instead of giving a formal status to Ro1, Ro2, and Ro3, we make the following comprehensive assumption about all basic rigid motions that subsumes Ro1–Ro3.

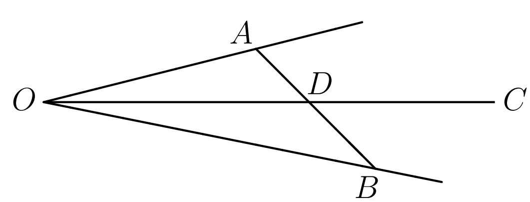
**(A7)** *The basic rigid motions* (rotations, reflections, and translations) *have the following properties:*

(*i*) *A basic rigid motion maps a line to a line, a ray to a ray, and a segment to a segment.*

(*ii*) *A basic rigid motion is distance-preserving and degree-preserving.*

Finally, the next assumption has to do with the intuitive fact that the angle bisector of $\angle A$ in a triangle $ABC$ must intersect side $BC$. Although your gut reaction may be "how can it be otherwise?", it is well to also recognize that, gut feelings notwithstanding, there is no way we can explain, *logically using what we know thus far,* why this must be the case. If we want to claim that this is true, the way to do so is to add another assumption, which will be our last.

**(A8) (Crossbar axiom)** *Given a* convex *angle $AOB$, then for any point $C$ in $\angle AOB$, the ray $R_{OC}$ intersects the segment $AB$* (at the point $D$ in the following figure).



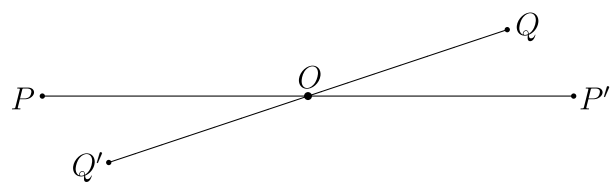

It is now clear that (A8) implies that the angle bisector of an angle in a triangle must intersect the opposite side.

## 2. Proofs of the first theorems

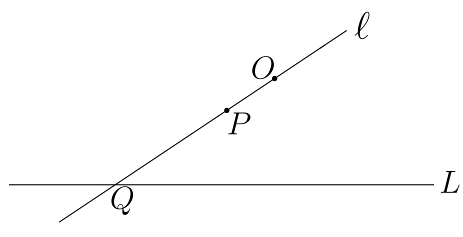
The main goal of this section is to prove the Theorem on page 131 using only assumptions Ro1–Ro3 on page 132. In the process, we will also justify the definition of a translation. The following theorem is a critical first step toward this goal.

**Theorem 1.** *Let $L$ be a line and $O$ be a point not lying on $L$. Let $\mathcal{R}$ be the 180 degree rotation around $O$. Then $\mathcal{R}$ maps $L$ to a line parallel to $L$ itself.*

Because a rotation of 180 degrees will be the main tool for the proofs of the first few theorems, we describe it explicitly. Let the center of rotation be $O$ and let $\mathcal{R}$ be the 180 degree rotation around $O$. Given a point $P$ distinct from $O$, let $P'$ denote the image $\mathcal{R}(P)$ of $P$ by $\mathcal{R}$ (see page 127 ). Then $P'$ is the point on the ray $R_{PO}$ so that $|P'O| = |PO|$. This is because $\angle POP'$ is a straight angle ($\mathcal{R}$ is a 180 degree rotation) and a rotation is distance-preserving (assumption Ro2 on page 132 ). Similarly, for any other point $Q$, the image $Q'$ of $Q$ by $\mathcal{R}$ is the point on $R_{QO}$ so that $|Q'O| = |QO|$.



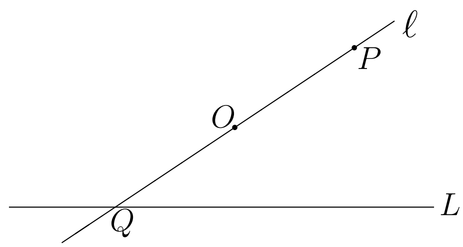

The truth of Theorem 1 depends on the simple observation that follows immediately from assumption (A1) on page 117 , to the effect that *two distinct lines are either parallel or intersect at exactly at one point*. (Recall: two lines are said to be *distinct* if there is one point on one that does not lie in the other.) Now consider the situation of Theorem 1 where a line $L$ and a point $O$ are given and $O$ does not lie on $L$. Let a line $\ell$ pass through $O$ and intersect $L$ at a point $Q$, as shown.




Now we make a second observation: *if $P$ is any point on the line $\ell$ not equal to $Q$, then $P$ does not lie on $L$.* This is because $L$ and $\ell$, being distinct lines, already have one point $Q$ in common and so the preceding observation says no other point can be common to both lines. In particular, $P$ does not lie on $L$, and the observation is proved.

With this second observation in place, we can now prove Theorem 1.

**Proof of Theorem 1.** First of all, we know that rotations map a line to another line (assumption Ro1 on page 132 ), so that with assumptions and notation as in Theorem 1, $\mathcal{R}$ maps the line $L$ to a line to be denoted by $\mathcal{R}(L)$. We have to show that $\mathcal{R}(L)$ and $L$ have no point in common. Thus, if $P$ is any point on $\mathcal{R}(L)$, we must show that $P$ does not lie on $L$. By definition of $\mathcal{R}(L)$, there is a point $Q$ of $L$ so that $P$ is the rotated image of $Q$ by $\mathcal{R}$ and, $\mathcal{R}$ being a $180°$ rotation, $P$, $O$, $Q$ lie on a line $\ell$ (see (*ii*) of Assumption (A6) on page 124 ).




The preceding observation now tells us that $P$—being a point of the line $\ell$ containing $O$ and $Q$ but not equal to $Q$—does not lie on $L$. This then proves Theorem 1.

Theorem 1 has an unexpected consequence. The Parallel Postulate assures us that, if $P$ is a point which does not lie on a given line $L$, there is at most one line passing through $P$ and parallel to $L$, but it leaves open the possibility that there may not be any line passing through $P$ and parallel to $L$. With Theorem 1 at our disposal, we now see that *there is such a line* because the said existence already follows from Theorem 1:

**Corollary.** *Given a line $L$ and point $P$ not on $L$, there is a line parallel to $L$ and passing through $P$.*

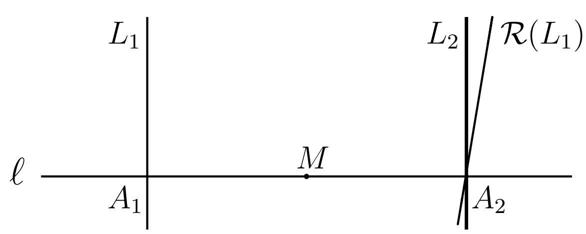
**Proof.** Indeed, referring to the preceding picture, we take a point $Q$ on $L$ and let $O$ be the midpoint of the segment $PQ$. If $\mathcal{R}$ is the 180 degree rotation around $O$, then Theorem 1 says the rotated image $\mathcal{R}(L)$ of $L$ is parallel to $L$. But since a rotation preserves length (assumption Ro1, page 132)), $\mathcal{R}$ maps $Q$ to $P$, so that $\mathcal{R}(L)$ in fact passes through $P$. The Corollary is proved.

As noted previously, this Corollary shows that our definition of translation is completely sound.

Theorem 1 is deceptive because it is not obvious how it can be put to use. We will see that it is in fact a central theorem with numerous interesting consequences, including the very fact we are after, namely, the Theorem on page 131 . The following is a first step toward this goal.

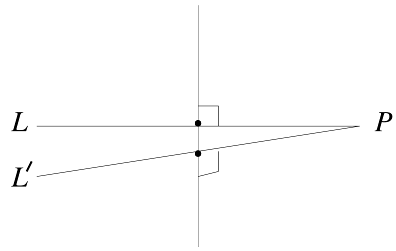
**Theorem 2.** *Two lines perpendicular to the same line are either identical or parallel to each other.*

**Proof.** Let $L_1$ and $L_2$ be two lines perpendicular to a line $\ell$ at $A_1$ and $A_2$, respectively. We have noted in an observation on page 126 that the line passing through a given point of a line and perpendicular to that line is unique. Thus if $A_1 = A_2$, $L_1$ and $L_2$ are identical. So suppose $A_1 \neq A_2$. We need to prove that $L_1 \parallel L_2$. Let $\mathcal{R}$ be the rotation of 180 degrees around the midpoint $M$ of $A_1A_2$. If we can show that the image of $L_1$ by $\mathcal{R}$ is $L_2$, then we know $L_2 \parallel L_1$ by virtue of Theorem 1.




To this end, note that $\mathcal{R}(L_1)$ contains $A_2$ because $\mathcal{R}(A_1) = A_2$. We are given that $L_1 \perp \ell$. Since $\mathcal{R}(A_1) = A_2$ and $\mathcal{R}(A_2) = A_1$, we see that $\mathcal{R}(\ell) = \ell$ (because of assumption (A1)). By assumption Ro3 on page 132, rotations map perpendicular lines to perpendicular lines. Thus we have $\mathcal{R}(L_1) \perp \ell$. It follows that each of $\mathcal{R}(L_1)$ and $L_2$ is a line that passes through $A_2$ and perpendicular to $\ell$. By the preceding observation about the uniqueness of the line perpendicular to a line $\ell$ at a given point of $\ell$, we see that, indeed, $\mathcal{R}(L_1) = L_2$ and therefore $L_1 \parallel L_2$. Theorem 2 is proved.

**Corollary 1.** *Through a point $P$ not lying on a line passes at most one line $L$ perpendicular to the given line.*




**Proof.** Suppose in addition to $L$, there is another line $L'$ passing through $P$ and also perpendicular to the given line. Since these lines are not parallel (they already have $P$ in common), they have to be identical, by Theorem 2. Thus $L = L'$. Corollary 1 is proved.

We will make a digression. Recall that earlier we introduced the concept of a rectangle as a quadrilateral whose adjacent sides are all perpendicular to each other. As a result of Theorem 2, we now have:

**Corollary 2.** *A rectangle is a parallelogram.*

Corollary 1 addresses one half of the concern about a reflection being well-defined. Now we prove the other half as well.

**Theorem 3.** *Given a point not lying on a line $\ell$, there is a line that passes through the point and perpendicular to $\ell$.*

**Proof.** Let $P$ be the point not lying on $\ell$. We have to show that there is a line passing through $P$ and perpendicular to $\ell$. Take any point $A \in \ell$ and let $L'$ be the line passing through $A$ and perpendicular to $\ell$ (see the observation on page 126). If $L'$ contains $P$, we are done, so we may assume that $L'$ does not contain $P$. By the Corollary to Theorem 1, there exists a line $L$ passing through $P$ and parallel to $L'$. Let $L$ intersect $\ell$ at $B$.


There is a line passing through $B$ and perpendicular to $\ell$ (see page 126 again); by Theorem 2, this line is parallel to $L'$ and must therefore coincide with $L$, by the Parallel Postulate. Thus $L \perp \ell$. This proves Theorem 3.

Theorem 3 and Corollary 1 to Theorem 2 together show that the Theorem on page 131 is valid, i.e., from a point outside a given line $L$, there is one and only one line passing through $P$ and perpendicular to $L$. As we pointed out above, this shows that the concept of reflection is well-defined.

At this point, we know that our definitions of all the basic rigid motions are well-defined. Thus *assumption* (A7) *on page 133 now makes sense and we will assume it forthwith.*

We note that (A7) is not a very useful assumption if there are not "plenty" of basic rigid motions, and we proceed to address this concern. First, rotations. As a result of assumption (A6) ($i$) and the definition of a rotation, there are "plenty" of rotations in the following sense:

**Lemma 2.** *Given a point and a degree $t$ so that $-360 \leq t \leq 360$, there is a rotation of degree $t$ around the point.*

Analogously, the same can be said about reflections as a result of Theorem 3 and the definition of a reflection:

**Lemma 3.** *Given a line in the plane, there is a reflection across that line.*

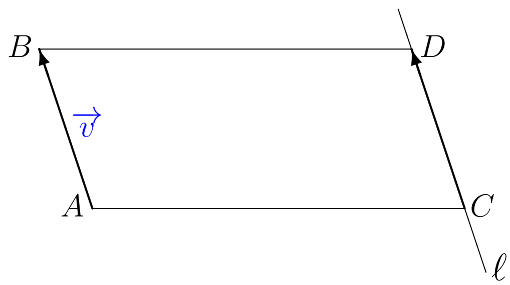
Finally, because we know that given a line $L$ and a point not on $L$, there is always a line passing through that point and parallel to $L$ (Corollary to Theorem 1, page

135 ), the definition of the translation along any vector is well-defined. Thus there are also "plenty" of translations:

**Lemma 4.** *Given any vector, there is a translation along that vector.*

In summary, Lemmas 2, 3, and 4 guarantee that, indeed, *there are "plenty" of basic rigid motions* for any occasion. They will be the main tools for proving theorems in plane geometry.

We end this section by making an interesting observation about the concept of a translation $T$ along a given vector $\overrightarrow{v}$. Recall the definition: suppose the vector $\overrightarrow{v}$ has starting point $A$ and endpoint $B$, then if $C$ does not lie on $L_{AB}$, the image $D = T(C)$ is by definition the intersection of the line $\ell$ that is parallel to $L_{AB}$ and the line passing through $B$ parallel to $L_{AC}$.




We are going to prove that $|CD| = |AB|$. Granting this for the moment, we now observe that the translation $T_{AB}$ along the vector $\overrightarrow{AB}$ has the following intuitive interpretation: it moves every point the same distance as that from $A$ to $B$ and "in the same direction" as $\overrightarrow{AB}$.

The fact that $|CD| = |AB|$ follows from a general theorem: opposite sides of a parallelogram are equal. This is because, by construction, the opposite sides of the quadrilateral $BACD$ are parallel: $L_{BA} \parallel L_{CD}$ and $L_{BD} \parallel L_{AC}$. Thus $BACD$ is a parallelogram and $|CD| = |AB|$. It remains therefore to prove the following theorem.

**Theorem 4.** *Opposite sides of a parallelogram are equal.*

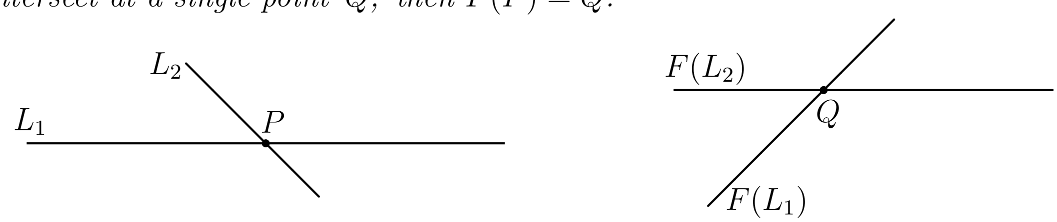
Theorem 4 together with the Corollary 2 to Theorem 2 imply:

*Opposite sides of a rectangle are equal.*

This reconciles the usual definition in school mathematics of a rectangle (a quadrilateral with four right angles and equal opposite sides) with our definition of a rectangle (a quadrilateral with four right angles).

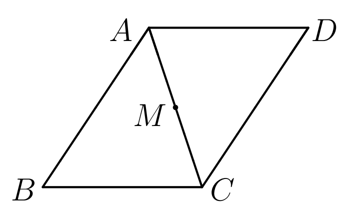
The idea of the proof of Theorem 4 is to exploit Theorem 1, for the most practical of reasons: at this point, what other tools have we got? Of course, the presence of parallel lines in a parallelogram already suggests that something like Theorem 1 should be relevant. It will be obvious from the proof of Theorem 4 below why the following lemma is needed.

**Lemma 5.** *Let $F$ be a transformation of the plane that maps lines to lines. Suppose two distinct lines $L_1$ and $L_2$ intersect at $P$ and the image lines $F(L_1)$ and $F(L_2)$ intersect at a single point $Q$, then $F(P) = Q$.*




**Proof of Lemma 5.** Since $P$ is a point in $L_1$, we see that $F(P)$ is a point on $F(L_1)$, by the definition of the image of $L_1$ by $F$. Similarly, $F(P)$ lies on the line $F(L_2)$. Therefore $F(P)$ lies in the intersection of $F(L_1)$ and $F(L_2)$. But by hypothesis, the latter intersection is exactly the point $Q$. So $F(P) = Q$ and Lemma 5 is proved.

**Proof of Theorem 4.** Given parallelogram $ABCD$, we have to prove that $|AD| = |BC|$ and $|AB| = |CD|$. It suffices to prove the former as the proof of the latter is similar. Let $M$ be the midpoint of the diagonal $AC$ and we will use Theorem 1 to explore the implications of the 180 degree rotation $\mathcal{R}$ around $M$.



Because $|MA| = |MC|$ and rotations preserve distance (assumption (A7) on page 133), we have $\mathcal{R}(C) = A$ so that $\mathcal{R}(L_{BC})$ is a line passing through $A$ and parallel to $L_{BC}$ (by Theorem 1). Since the line $L_{AD}$ has exactly the same two properties by assumption, the Parallel Postulate implies that $\mathcal{R}(L_{BC}) = L_{AD}$. Similarly, $\mathcal{R}(L_{AB}) = L_{CD}$. Thus, using the usual symbol $\cap$ to denote intersection (for two sets $A$ and $B$, $A \cap B$ denotes all the elements belonging to both $A$ and $B$), we have:

$$\mathcal{R}(L_{BC}) \cap \mathcal{R}(L_{AB}) \;=\; L_{AD} \cap L_{CD} = \{D\}$$

On the other hand, $L_{BC} \cap L_{AB} = \{B\}$. By Lemma 5, we have

$$\mathcal{R}(B) = D$$

Recall we also have $\mathcal{R}(C) = A$. Therefore $\mathcal{R}$ maps the segment $BC$ to the segment joining $D$ (which is the image of $B$) to $A$ (which is the image of $C$), by the property that a rotation maps segments to segments (see assumption (A7) on page 133). The latter segment has to be the segment $DA$, by (A1) (page 117). Thus $\mathcal{R}(BC) = DA$, so that by assumption (A7) that rotations preserve distance (page 133), we have $|BC| = |AD|$, as desired.

**Corollary.** *The angles of a parallelogram at opposite vertices are equal.*

The proof is implicit in the proof of Theorem 4: we already have $\mathcal{R}(\angle ABC) = \angle CDA$, so simply use $(ii)$ of assumption (A7) on page 133 to conclude the proof.

## 3. Congruence criteria for triangles

The main concern of this section is the proof of the three basic criteria for triangle congruence: SAS, ASA, and SSS. We begin by elucidating the concept of congruence.

We need the concept of *composing transformations*. Let $F$ and $G$ be transformations of the plane. We define a new transformation $F \circ G$, called the **composition of $G$ and $F$**, to be the rule which assigns to each point $P$ of the plane the point $F(G(P))$. Schematically, we have;

$$P \;\xrightarrow{G}\; G(P) \;\xrightarrow{F}\; F(G(P))$$

i.e., we first let $G$ send $P$ to $G(P)$, and then let $F$ send the point $G(P)$ to $F(G(P))$. (For those familiar with the concept of composite functions, this should be déjà vu.) Notice the peculiar feature of the notation: the symbol $F \circ G$ suggests that $F$ comes before $G$ if we read from left to right as usual, but in fact the definition itself, which assigns to $P$ the point $F(G(P))$, requires that $G$ acts first. There is reason to be careful about the order of the transformations in a given composition, for the following reason. We say two **transformations $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ are equal**, in symbols $\boldsymbol{F_1 = F_2}$, if for *every* point $P$, it is true that $F_1(P) = F_2(P)$. Then simple examples would show that, in general, $F \circ G \neq G \circ F$ for two transformations $F$ and $G$ of the plane.

The composition of more than two transformations is defined similarly. For example, if $F$, $G$, $H$, $K$ are transformations, then **the composition $\boldsymbol{F \circ G \circ H \circ K}$** is defined to be the rule which assigns to each point $P$ the point $F(G(H(K(P))))$.

**Definition.** *A* **congruence** *in the plane is a transformation of the plane which is equal to the composition of a finite number of basic rigid motions.*

If $\mathcal{S}$ is congruent to $\mathcal{S}'$, we write $\boldsymbol{\mathcal{S} \cong \mathcal{S}'}$. The definition of congruence immediately implies that a composition of congruences is still a congruence. One can also verify, with a bit more effort, that if $\mathcal{S}$ is congruent to $\mathcal{S}'$, then also $\mathcal{S}'$ is congruent to $\mathcal{S}$. This eases the need to be extra careful about whether $\mathcal{S}$ is congruent to $\mathcal{S}'$ or the other way around, $\mathcal{S}'$ is congruent to $\mathcal{S}$; the two statements are equivalent.

Because each basic rigid motion is assumed to satisfy the properties of (A7) (page 133), we would expect that so does a congruence. This is the content of the next lemma whose proof is straightforward.

**Lemma 6.** *A congruence*

(*i*) *maps lines to lines, rays to rays, and segments to segments,*

(*ii*) *is distance-preserving and degree-preserving.*

To demonstrate the usefulness of this definition of congruence, we can now *explain* why the three classical criteria for triangle congruence—SAS, ASA, SSS—are true.

We begin with SAS and ASA,

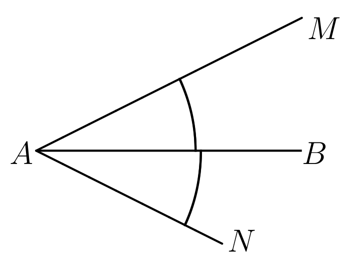
**Theorem 5 (SAS).** *Given two triangles $ABC$ and $A_0B_0C_0$ so that* $|\angle A| = |\angle A_0|$, $|AB| = |A_0B_0|$, *and* $|AC| = |A_0C_0|$. *Then the triangles are congruent.*

**Theorem 6 (ASA).** *Given two triangles $ABC$ and $A_0B_0C_0$ so that* $|AB| = |A_0B_0|$, $|\angle A| = |\angle A_0|$, *and* $|\angle B| = |\angle B_0|$. *Then the triangles are congruent.*

The proofs of these two theorems are very similar. Because we have already given an informal proof of ASA back in grade 8 (page 82 ),[27] we will only give the proof of Theorem 5 (SAS) here.

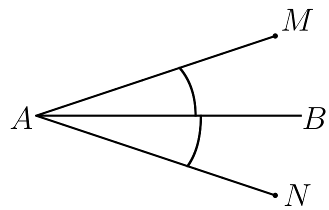
We begin with two simple observations on the behavior of angles under a reflection. They are nothing more than variations on the theme of Lemma 1 (page 127). As we shall see, they will be useful for other purposes as well.

**Lemma 7.** *Given two equal angles $\angle MAB$ and $\angle NAB$, suppose they have one side $AB$ in common and $M$ and $N$ are on opposite sides of the line $L_{AB}$. Then the reflection across the line $L_{AB}$ maps $\angle NAB$ to $\angle MAB$ (and also maps $\angle MAB$ to $\angle NAB$).*




**Lemma 8.** *Suppose two angles $\angle MAB$ and $\angle NAB$ are equal, and they have one side $AB$ in common. Assume further that the* segments *$AM$ and $AN$ are equal. Then either $M = N$ (if $M$ and $N$ are on the same side of $L_{AB}$) or the reflection across $L_{AB}$ maps $N$ to $M$ (if $M$ and $N$ are on opposite sides of $L_{AB}$).*

[27] That proof is essentially correct. It should also be mentioned that, instead of proving ASA as on page 82 , one can also prove ASA by invoking SAS.



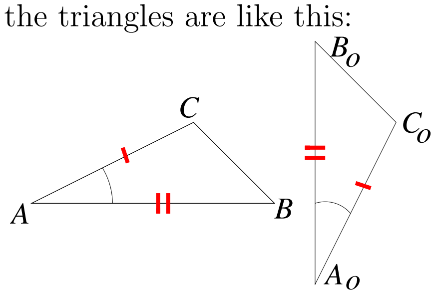

For the proof of Lemma 7, observe that the reflection $R$ across $L_{AB}$ maps $\angle NAB$ to $\angle N_0AB$, where $N_0 = R(N)$, so that $\angle N_0AB$ and $\angle MAB$ are now equal angles with one side $R_{AB}$ in common and the other side lying in the same half-plane. So $\angle N_0AB = \angle MAB$, by Lemma 1 (page 127). This proves Lemma 7. As to Lemma 8, suppose $M$ and $N$ are on the same side of $L_{AB}$. By the same Lemma 1, we know that the *rays* $AM$ and $AN$ coincide. But since $|AM| = |AN|$, necessarily $M = N$. Now if $M$ and $N$ are on opposite sides of $L_{AB}$, then Lemma 7 shows that the reflection across $L_{AB}$ maps the ray $R_{AN}$ to the ray $R_{AM}$. Since a reflection preserves distance, the reflection maps the segment $AN$ to a segment of length equal to $|AM|$, and therefore maps $N$ to $M$ by the preceding argument. This proves Lemma 8.

We are now in a position to begin the **proof of SAS.** Note that Larry Francis has created an animation for this proof:

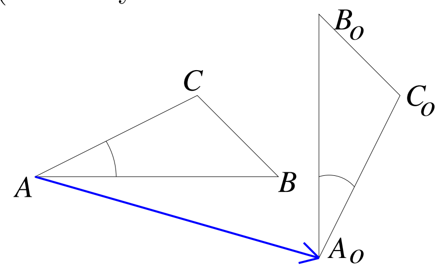
Side-Angle-Side Congruence by basic rigid motions

This animation complements the verbal proof below.

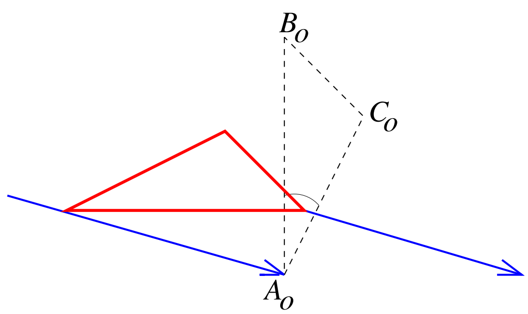
A general remark is that throughout the proofs in the remainder of this section, we will be making implicit use of the fact that *there are "plenty" of basic rigid motions at our disposal in the precise sense of Lemmas 2–4 on page 138*. Suppose we are given triangles $ABC$ and $A_0B_0C_0$ in the plane so that $\angle A$ and $\angle A_0$ are equal, and furthermore, $|AB| = |A_0B_0|$ and $|AC| = |A_0C_0|$. We have to explain why the triangles are congruent. Let us say the triangles are like this:

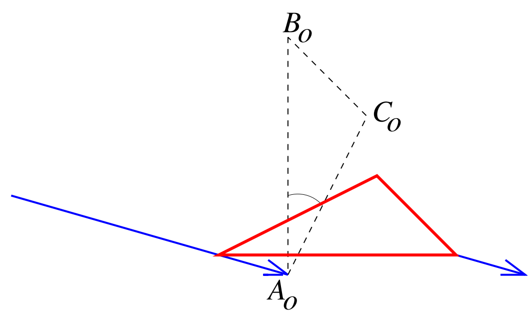



By our definition of congruence, this means we must exhibit a sequence of basic rigid motions so that their composition brings (let us say) $\triangle ABC$ to coincide exactly with $\triangle A_0B_0C_0$. For ease of comprehension, we will first prove the theorem for the pair of triangles in the above picture and leave a discussion of other variations to the end. We will first move vertex $A$ to $A_0$ by a translation $T$ along the vector from $A$ to $A_0$, denoted by $\overrightarrow{\boldsymbol{AA_0}}$ (shown by the blue vector below).

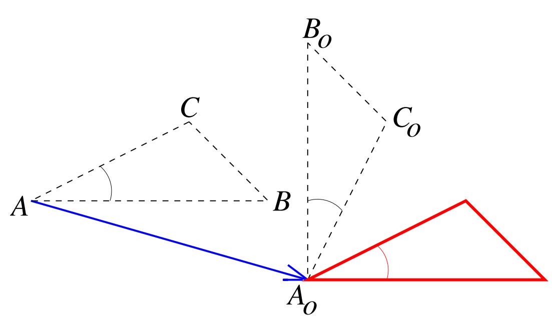



The effect of $T$ is to slide $\triangle ABC$ along $\overrightarrow{AA_0}$. We show the image of two stages of $\triangle ABC$ in transition: $\triangle ABC$ is shown in red and the right blue arrow in each picture indicates how much further the red triangle has yet to go. Because $T$ also translates $\triangle A_0B_0C_0$, we show the *original position* of $\triangle A_0B_0C_0$ in dashed lines to remind us of where we are ultimately heading.

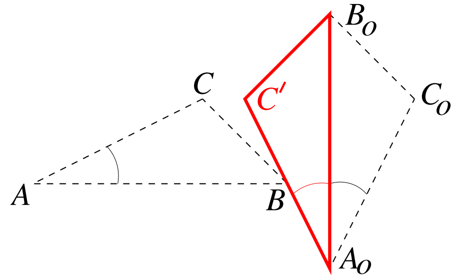


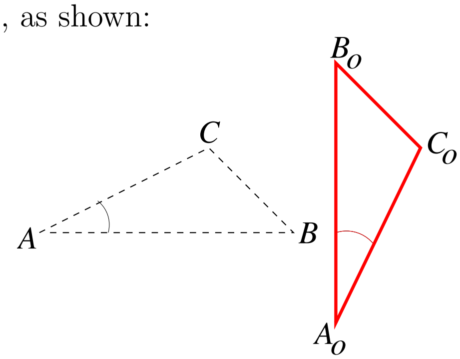

Here is the final position of $\triangle ABC$ (shown in red). We use dashed lines to indicate the original positions of $\triangle ABC$ and $\triangle A_0B_0C_0$.




Next, we will use a rotation to bring the horizontal side of the red triangle (which is the translated image of $AB$ by $T$) to $A_0B_0$. If the angle between the horizontal red side and $A_0B_0$ is $t$ degrees (in the picture above, $t = 90$), then a rotation of $t$ degrees (in this case counterclockwise) around $A_0$ will map the horizontal ray issuing from $A_0$ to the ray $R_{A_0B_0}$. Call this rotation $\mathcal{R}$. Now it is given that $|AB| = |A_0B_0|$, and we know a translation preserves lengths (assumption (A7), page 133). So the horizontal side of the red triangle has the same length as $A_0B_0$ and therefore $\mathcal{R}$ will map the horizontal side of the red triangle to the side $A_0B_0$ of $\triangle A_0B_0C_0$, as shown.



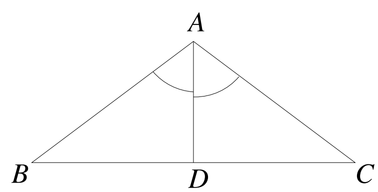

Two of the vertices of the red triangle already coincide with $A_0$ and $B_0$ of $\triangle A_0B_0C_0$. We claim that after a reflection across line $L_{A_0B_0}$ the third vertex of the red triangle will be equal to $C_0$. Indeed, the two marked angles with vertex $A_0$ are equal since basic rigid motions preserve degrees of angles (assumption (A7), page 133) and, by hypothesis, $\angle CAB$ and $\angle C_0A_0B_0$ are equal. Moreover, the left side of the red triangle with $A_0$ as endpoint has the same length as $A_0C_0$ because basic rigid motions preserve length ((A7) again), and by hypothesis $|AC| = |A_0C_0|$. Therefore our claim follows from Lemma 8, page 143. Thus after a reflection across $L_{A_0B_0}$, the red triangle coincides with $\triangle A_0B_0C_0$, as shown:




Thus the desired congruence for the two triangles $ABC$ and $A_0B_0C_0$ in this particular picture is the composition of a translation, a rotation, and a reflection.

It remains to address the other possibilities and how they affect the above argument. If $A = A_0$ to begin with, then the initial translation would be unnecessary. It can also happen that after the translation $T$, the image $T(AB)$ (which corresponds to the horizontal side of the red triangle on page 145) already coincides with $A_0B_0$. In that case, the rotation $\mathcal{R}$ would be unnecessary. Finally, if after the rotation the image of $C$ is already on the same side of $L_{A_0B_0}$ as $C_0$, then Lemma 1 (page 127) implies that the image of $C$ and $C_0$ already coincide and the reflection would not be needed. In any case, Theorem 5 is proved.

We next take up the third major criterion for triangle congruence, *SSS*. To this end we prove the following theorem, which is interesting in its own right. If $\triangle ABC$ is an isosceles triangle so that $|AB| = |AC|$, then it is common to refer to $\angle B$ and $\angle C$ as its **base angles**, $\angle A$ as its **top angle**, and $BC$ as its **base**.




We will also refer to the line joining the midpoint of a side of a triangle to the opposite vertex as a **median** of the side, and the line passing through the opposite vertex and perpendicular to this side as **the altitude on this side.** Note that sometimes the segment from the vertex to the point of intersection of this line with the (line containing the) side is called the *median* and the *altitude*, respectively.

**Theorem 7.** (*a*) *An isosceles triangle has equal base angles.* (*b*) *In an isosceles triangle, the perpendicular bisector of the base, the angle bisector of the top angle, the median from the top vertex, and the altitude on the base all coincide.*

**Proof.** Referring to the preceding picture, let $|AB| = |AC|$ in $\triangle ABC$, and let the angle bisector of the top angle $\angle A$ intersect the base $BC$ at $D$.[28] Let $R$ be the reflection across the line $L_{AD}$. Since $|\angle BAD| = |\angle CAD|$, and since $|AB| = |AC|$, we have $R(B) = C$ by Lemma 8 (page 143). Now it is also true that $R(D) = D$ and $R(A) = A$ because $D$ and $A$ lie on the line of reflection of $R$, so $R(BD) = CD$ and $R(BA) = CA$ because a reflection maps a segment to a segment (by assumption (A7), page 133). Consequently, $R(\angle B) = \angle C$. Since a reflection preserves the degree

[28]The fact that the angle bisector of $\angle A$ intersects $BC$ is implied by assumption (A8) on page 133.

of angles (again by assumption (A7), page 133 ), we have $|\angle B| = |\angle C|$. This proves part (a). For part (b), observe that since $L_{AD}$ is the line of reflection and $R(B) = C$,

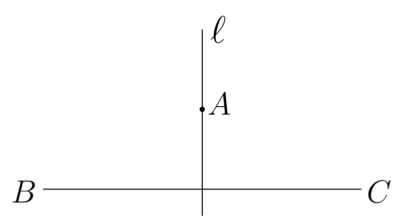
$$R(\angle ADB) = \angle ADC \quad \text{and} \quad R(BD) = (CD)$$

Therefore $|\angle ADB| = |\angle ADC| = 90°$, and $|BD| = |CD|$, so that $L_{AD}$ is the perpendicular bisector of $BC$. Since $L_{AD}$ is, by construction, also the angle bisector of $\angle A$, every statement in (b) follows. The proof is complete.

As an immediate corollary, we have the following useful characterization of the perpendicular bisector of a segment:

**Corollary.** *A point is on the perpendicular bisector of a segment if and only if it is equidistant from the endpoints of the segment.*

**Proof.** Let the segment be $BC$ and let the point be $A$. If $A$ is on the perpendicular bisector $\ell$ of $BC$, then by the definition of the reflection $\mathcal{R}$ across $\ell$, $\mathcal{R}(B) = C$ and $\mathcal{R}(A) = A$.



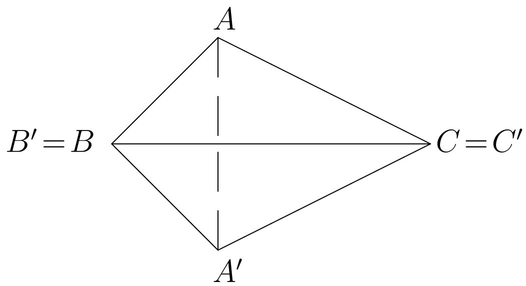

Thus $\mathcal{R}(AB) = AC$, and since reflection is distance-preserving (assumption (A7) page 133 ), $|AB| = |AC|$ and $A$ is equidistant from the endpoints $B$ and $C$. Conversely, suppose $|AB| = |AC|$. Thus triangle $ABC$ is isosceles and the angle bisector of $\angle A$ is the perpendicular bisector of $BC$, by Theorem 7. But the angle bisector of $\angle A$ passes through $A$, so the perpendicular bisector of $BC$ passes through $A$. The proof is complete.

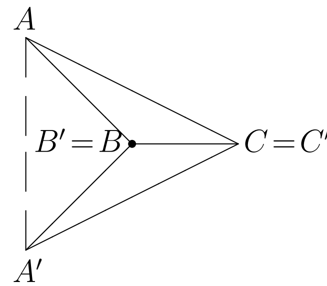
We now turn to the third major congruence criterion for triangles: SSS.

**Theorem 8 (SSS).** *Two triangles with three pairs of equal sides are congruent.*

**Proof.** Suppose triangles $ABC$ and $A'B'C'$ are given so that $|AB| = |A'B'|$, $|AC| = |A'C'|$, and $|BC| = |B'C'|$.

**Case 1.** We begin by assuming that the triangles satisfy an additional restrictive assumption: $B = B'$ and $C = C'$, and we will prove that there is a basic rigid motion that maps $\triangle A'B'C'$ to $\triangle ABC$. Either $A$ and $A'$ are on the same side of the line $L_{BC}$ or on opposite sides; first assume they are on opposite sides. Here are two of the possibilities, but our proof will be valid in all cases.




By hypothesis, $|AB| = |A'B'|$, so $B$ is equidistant from $A$ and $A'$; by the Corollary to Theorem 7 (page 148 ), $B$ lies on the perpendicular bisector of $AA'$. For the same reason, $C$ lies on the perpendicular bisector of $AA'$. Because two points determine a line ((A1), page 117 ), $L_{BC}$ is the perpendicular bisector of $AA'$. Thus the reflection $R$ across $L_{BC}$ maps $A'$ to $A$, $B$ to $B$ and $C$ to $C$ (see the definition of reflection on page 129 ). Thus $R(\triangle A'B'C') = \triangle ABC$. This then proves the theorem under the stated restrictions that $B = B'$ and $C = C'$ and $A$, $A'$ being on opposite sides of $L_{BC}$. Now suppose $A$, $A'$ are on the same side of $L_{BC}$. Still with $R$ as the reflection across $L_{BC}$, let $R(A') = A_0$, $R(B') = B_0$, and $R(C') = C_0$. Then $R(\triangle A'B'C') = \triangle A_0B_0C_0$, and the latter has the property that $B = B_0$, $C = C_0$, $A$, $A_0$ are on opposite sides of $L_{BC}$, and $|AB| = |A_0B|$, $|AC| = |A_0C|$ (because a reflection preserves distance, by assumption (A7) on page 133). The preceding argument then shows that

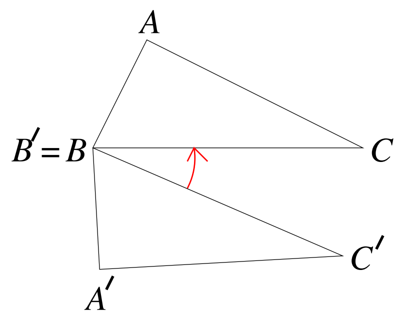
$$\triangle ABC \;=\; R(\triangle A_0B_0C_0)$$

In view of $\triangle A_0B_0C_0 = R(\triangle A'B'C')$ and the fact that $R \circ R$ is the identity transformation, we have

$$\triangle ABC \;=\; R(\triangle A_0B_0C_0) \;=\; R(R(\triangle A'B'C')) \;=\; \triangle A'B'C'$$

Thus $\triangle A'B'C'$ coincides with $\triangle ABC$ in the first place. Therefore the theorem is true if, in addition to the equality of three pairs of sides, $B = B'$ and $C = C'$.

**Case 2.** Suppose we assume only that $B = B'$ but $C \neq C'$. Because $|BC| = |B'C'|$, a suitable rotation $Ro$ around $B$ will bring $B'C'$ to $BC$. Then the triangle $Ro(\triangle A'B'C')$ and $\triangle ABC$ share a side $BC$, so that by Case 1, there is a basic rigid motion $F$ (which is either a reflection or the identity) so that $F(Ro(\triangle A'B'C')) = \triangle ABC$.




**Case 3.** Finally, we treat the general case. In view of Case 2, we may assume that triangles $ABC$ and $A'B'C'$ do not even share a vertex. Let $T$ be the translation along the vector $\overrightarrow{B'B}$. Then $T(B') = B$, so that $T(\triangle A'B'C')$ and $\triangle ABC$ share a vertex $B$. Depending on whether $T(C')$ is equal to $C$ or not, we are in either Case 1 or Case 2. Thus there is some basic rigid motion $F$ and some rotation $Ro$ ($Ro$ would be the rotation of 0 degrees if $T(C') = C$), we have $F(Ro(T(\triangle A'B'C'))) = \triangle ABC$. This proves Theorem 8.

Now that we have the three most important criteria for triangle congruence, it is time to take stock of where we stand in terms of geometry instruction in high school. It is well to reiterate the philosophy behind the geometry standards in the CCSSM: whereas the usual geometry curriculum in grades 8–12 is incoherent due to a sharp break between the *recreational* study of translations, rotations, and reflections in middle school and the introduction of axioms and (seemingly) rigorous proofs in high school, the CCSSM smooth over this break by amplifying on the *mathematical* significance of translations, rotations, and reflections and using them to directly access high school geometry. In the process, we also bypass the deadly litany of the proofs of boring elementary theorems that attend the early part of every axiomatic development. The preceding three sections are devoted to a fairly detailed description of how this new transition can be accomplished. Now that some basic facts (e.g., parallelograms have equal opposite sides, isosceles triangles have equal base angles, etc.)

together with SAS, ASA, SSS are at our disposal, classroom instruction can turn to the traditional presentations of theorems in Euclidean geometry if one so wishes. For details of an alternate approach at this point, please see the reference on page 115 .

***It should be clear that the CCSSM do not pursue so-called transformational geometry as a goal in itself.*** In the CCSSM, the basic rigid motions are given a place of prominence only because (1) these rigid motions are already a time-honored part of the existing middle-school geometry curriculum, (2) they serve to reveal that "congruence" means more than "same size and same shape", and (3) they serve to bridge the transition from middle school to high school. The basic rigid motions therefore are a means to an end in the CCSSM, but by no means an end in itself.

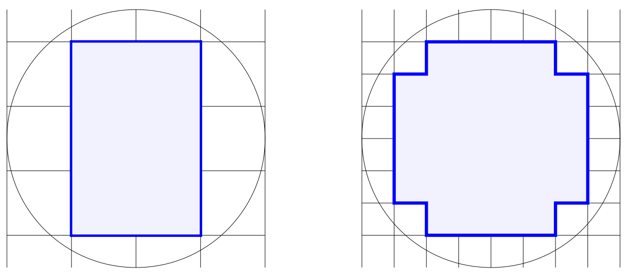
Of course, high school geometry is not just about congruences of triangles. There are two major topics we have not touched on here, similarity and circles, but we hope that what has been done for congruence is a sufficient indication of the potential of the CCSSM approach to high school geometry. Once again, we refer to the second reference on page 115 for more details on these two topics.

## 4. The number $\pi$

The goal of this section is to bring closure to the discussion of why it is pedagogically preferable to define the number $\pi$ as the area of the unit disk rather than as the ratio of the circumference over the diameter (see page 56 ). At this point, we will slightly extend the intuitive definition of the area of a disk (page 57 ) by giving a general definition of the **area of a plane region** as the limit of the areas of *approximating* polygons that are inside the region (this is a definition that can be made completely correct by tightening the phrasing). We make use of this fact as follows. First, by a **square grid**, we mean a collection of horizontal and vertical lines so that the distance between any two horizontal lines is equal to the distance between any two vertical lines, and by **a square of the square grid**, we mean a square whose sides are on two adjacent horizontal lines and two adjacent vertical lines of the grid. The length of a side of the squares in a grid is called the **mesh of the grid**. When a circle and a grid $\mathcal{G}$ are given, it makes sense to talk about all the squares of $\mathcal{G}$ lying inside the circle; the collection of all such squares is called the **inner polygon** of $\mathcal{G}$

when the circle is clearly understood.

We need so much terminology for a reason, as we proceed to explain. Fix a *unit circle*. If we have a sequence of square grids $\mathcal{G}_n$ so that the mesh of $\mathcal{G}$ goes to 0 as $n \to \infty$, then it is also intuitively clear that as $n \to \infty$, the (region inside the) inner polygon $P_n$ of $\mathcal{G}_n$ gets closer and closer to the unit disk. Below are two examples of inner polygons (colored in blue) which indicate why the (region inside an) inner polygon approximates the unit disk better and better when the mesh of the grid decreases.

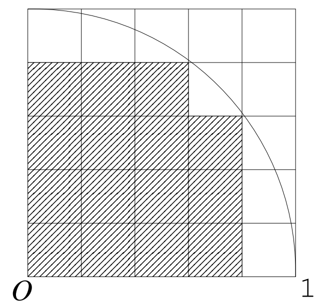


By definition, the limit of the area $|P_n|$ of the inner polygon $P_n$, as $n \to \infty$ (thus the mesh of the grid goes down to 0) is the **area of the unit disk**. Recalling that $\pi$ is the area of the unit disk, we have:

$$\pi \;=\; \lim_{n\to\infty} |P_n|$$

We will now produce an inner polygon for the unit circle by using graph paper. We start by drawing a quarter unit circle on a piece of graph paper. In principle, you should get the best graph paper possible because we are going to use the grid of the graph paper to directly estimate $\pi$. (The squares in the grid of cheap graph paper are usually not really squares.) So to simplify matters, suppose a quarter of a unit circle is drawn on a piece of graph paper so that the radius of length 1 is equal to 5 (sides of the) small squares, as shown. (Now as later, we shall use **small squares** to refer to the squares of the grid.)

The square of area 1 then contains $5^2$ small squares. We want to estimate how many small squares are contained in this quarter circle. The shaded polygon consists of 15 small squares of the grid. There are 7 small squares each of which is partially inside the quarter circle. Let us estimate the best we can how many small squares altogether are inside the quarter circle. Among the three small squares in the top row, a little more than 2 small squares seem to be inside the quarter circle; let us say 2.1 small




squares. By symmetry, the three small squares in the right column also contribute 2.1 small squares. As to the remaining lonely small square near the top right-hand corner, there is about 0.5 of it inside the quarter circle . Altogether the non-shaded small squares contribute $2.1 + 2.1 + 0.5 = 4.7$ small squares, so that the total number of small squares inside the quarter circle is, approximately,

$$15 + 4.7 = 19.7$$

The unit circle therefore contains about

$$4 \times 19.7 \ = \ 78.8 \quad \text{small squares}$$

Now $\pi$ is the area of the unit circle, and we know that the area of 25 small squares is equal to 1. So the total area of 78.8 small squares is

$$\frac{78.8}{25} \ = \ 3.152$$

Our estimate of $\pi$ is that it is roughly equal to 3.152. Taking the value of $\pi$ to be 3.14159, accurate to 5 decimal digits, the percentage error of this estimate is approximately equal to

$$\frac{3.152 - 3.14159}{3.14159} \ \sim \ 0.33\%$$

While a relative error of 0.33% is very impressive, this experiment is not convincing because the amount of guesswork needed to arrive at the final answer is too high. With a very fine and accurate grid (this is where you spend money to get good graph paper), one can reasonably get the *linear* unit 1 to be equal to anywhere between 25 to 50 sides of small squares so that the unit area would consist of between $25^2$ to $50^2$

small squares. Then the percentage of guesswork needed to estimate what happens to the small squares near the circle will be greatly reduced (though the counting of the total number of small squares can get dizzying).

In general, with the unit 1 equal to $n$ small squares, then $n^2$ small squares have a total area of 1. If there are, after some guessing, $k$ small squares in a quarter circle, then there are $4k$ small squares in the unit circle. Thus the area of the unit disk is

$$\pi \approx \frac{4k}{n^2}$$

The relative error rarely exceeds 1%. This should be both impressive and instructive to students. This is also the reason it is pedagogically better to define the number $\pi$ by area, as it is nearly impossible to get such a good estimate of $\pi$ using the definition of $\pi$ as the ratio of circumference to diameter.

It is recommended that all students do this activity so that they get a firm conception of what $\pi$ is, namely, a number between 3 and 4 whose value *they themselves* can approximate closely if they wish.

High school students should also be exposed to an informal derivation of the volume formula of a sphere of radius $r$ using Cavalieri's Principle. As is well known, the volume formula in question is $\frac{4}{3}\pi r^3$. However, in order not to add to the size of this already long document, we will simply refer to the second reference on page 115 for details.

## 5. Pedagogical implications

One of the problems encountered by beginners in geometry is their inability to reconcile the chasm between *intuition* and *formalism* in the prevailing presentations of the subject. The two basic concepts of *congruence* and *similarity* come across as either formal and abstract, or intuitive but irrelevant. In the axiomatic presentations, congruence and similarity are defined precisely but only for polygons and, as such, they are divorced from the intuitive way these terms are normally used. In the other extreme, congruence is "same size and same shape", and similarity is "same shape but not necessarily the same size". Because mathematics demands precision, the informality of these definitions raises the question, never addressed, about what these concepts have to do with *mathematics*. Students cannot use these intuitive definitions to prove theorems; instead, they are taught to use SAS and SSS for both congruence and similarity, AA for similarity, etc. At the end, these concepts become synonymous with rote procedures and, by extension, proving theorems in geometry also becomes a rote procedure.

The potential benefit of defining congruence using reflections, rotations, and translations is that they transform an abstract concept into one that is concrete and tactile. This is the whole point of the eighth grade geometry standards, which ask for the use of manipulatives, especially transparencies, to model reflections, rotations, and translations, i.e., to model congruence. It is for this reason that we used reflections, rotations, and translations to *prove* all three criteria of triangle congruence—SAS, ASA, and SSS—even when there is always the option to use SAS to prove ASA and SSS. We hope that the two earlier sections, Sections 2 and 3, have given a convincing demonstration that *theorem-proving in geometry does not have to be an exercise in formalism and abstraction.* Congruence is something students can relate to in a tactile manner just by moving a transparency over a piece of paper or a cardboard geometric figure across the blackboard. In the same way, we can also ground the learning of similarity in such tactile experiences.

There is an additional advantage in this approach that has been mentioned and deserves to be mentioned again. In Sections 2 and 3, we have given the complete logical development of the first few theorems of plane geometry by making strong use of basic rigid motions. In addition, we have proved in this short space the three basic congruence criteria for triangle congruence: SAS, ASA, and SSS. Because most of the theorems in plane geometry before the introduction of similarity depend only on these three criteria, this fact allows a transition into the traditional way of proving theorems at this point, without further use of basic rigid motions *if so desired*. The use of dilation to treat similarity can likewise be limited to the initial stage, if so desired; again see the following article for further details:

H. Wu, Teaching Geometry in Grade 8 and High School
According to the Common Core Standards

The professional judgment of the practitioners in geometry is that, at least initially, geometric intuition is mostly built on tactile experiences rather than abstract formalism. The goal of these standards is to provide a sound foundation for the learning of geometry by maximizing the use of such tactile experiences.