Preface

and

Teaching Geometry (Section 4.1)

from

Teaching School Mathematics: Pre-Algebra

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American Mathematical Society, 2016

Preface

Training has no shortcuts.

Golden State Warriors
Ramp Run video,
October 24, 2012
([GoldenState])

This volume and its companion volume—*Teaching School Mathematics: Algebra* ([Wu-Alg])—address the mathematics that is generally taught in grades 5–9. They are not student texts, however, because they have been written expressly for teachers, especially middle school teachers. These two volumes are designed *not* to show you how mathematics is really just common sense and lots of fun, but to help you teach the mathematics of middle school in a way that meets the minimal standards of human communication. In other words, problems are solved without recourse to tricks or any *ad hoc* sleight-of-hand, every step is explained logically using only concepts and skills already developed, and every concept is clearly defined so that no clever guessing is needed for its understanding. There may be an added bonus in that the mathematical development of these volumes parallels that of the Common Core State Standards for Mathematics ([CCSSM]) for middle school.

These volumes differ from the usual presentations found in standard school textbooks (and professional development materials as well) in substantial ways. First and foremost, the presentations in the standard textbooks, be they traditional or reform, are riddled with mathematical errors, thanks to *Textbook School Mathematics* (**TSM**).[1] While the Table of Contents bears a superficial resemblance to what you normally find in school textbooks and other professional development materials, there are major differences in terms of precision, sequencing, and reasoning. It is hoped that these volumes will lead you to rethink some of this material even if you believe you already know it very well.

[1]This is the name given to the *mathematics* in almost all standard school mathematics textbooks of roughly the past four decades. It is notable for being antithetical to the five principles listed on pages xv ff. A more elaborate discussion of TSM can be found in [Wu2013b] and [Wu2015].

The first major departure from TSM in these volumes is the treatment of fractions and rational numbers. Fractions (and rational numbers) are the backbone of K–12 mathematics and are therefore the centerpiece of not only these two volumes, but also the other volumes written for teachers: [Wu2011a] and [Wu-HighSchool]. Contrary to the prevailing norm in mathematics education, these volumes will ask you to spread the message that:

(1) Fractions are numbers that you can compare to see which is bigger, and can add, subtract, multiply, and divide.
(2) The number line is home for all (real) numbers, including whole numbers, fractions, and rational numbers.
(3) Fractions of a fixed denominator, when viewed as multiples of the corresponding unit fraction, are just like whole numbers, at least in terms of addition and subtraction.
(4) Students should get to know what a fraction is and what it means to add, subtract, multiply, and divide fractions before they perform the formal procedures of fraction arithmetic.
(5) The *least common denominator* is not needed for adding fractions, and there is no compelling mathematical reason to insist that fractions be always reduced to lowest terms.
(6) Finite decimals are a special class of fractions.
(7) Everything we need to know about fractions, including multiplication and division, can be explained using the definition of a fraction as a point on the number line.

(These emphases were first put forth in [Wu1998], and can be found in complete detail in [Wu2002]; they are also present in [Jensen].)

A second major departure lies in the heavy emphasis placed on geometry in the middle school curriculum, especially on giving precise definitions for the concepts of congruence and similarity. According to TSM, congruence means *same size and same shape* and similarity means *same shape but not necessarily the same size*. As mathematics, this is unacceptable because "same size" and "same shape" are words that can mean different things to different people, whereas mathematics only deals with clear and unambiguous information. What these volumes promote is a different approach to the teaching of these concepts. Take congruence, for example. First make sure that you know what translations, reflections, and rotations are, then devise hands-on activities for your students to familiarize themselves with these transformations, and, finally, teach them that, by definition, *two geometric figures are congruent if one figure can be carried onto the other by the use of a finite number of translations, reflections, and rotations.* Conceptually the same thing can be said about similarity. These volumes will help you acquire the requisite knowledge you need to teach congruence and similarity differently—and better.

The heavy emphasis on geometry all through both volumes is motivated by the fact that—contrary to what TSM would have you believe—familiarity with similar triangles is absolutely crucial to the learning of linear equations in algebra, particularly the concept of the **slope** of a line (see Chapter 3 of [NMP2], [Wu2010b], and [Wu2010c]). Students' understanding of the concept of slope is a main stumbling block in beginning algebra (see, e.g., [Postelnicu]), and one of the contributions of these volumes is a different approach to the definition of slope that is more intuitive and makes entirely obvious why certain lines have negative slope (see Section 4.3 in [Wu-Alg]).

While the geometric topics taken up are, with but one exception,[2] what one normally finds in the standard middle school curriculum—translations, reflections, rotations, congruence, length, area, volume, etc.—they are not taken up as fun, optional activities. Rather, these are topics that are essential for the learning of algebra and, to that end, are put to use in [Wu-Alg] for substantive logical reasoning in the discussion of the graphs of linear equations, linear functions, linear inequalities, and quadratic functions. For example, having a correct definition of the slope of a line makes it possible for teachers to *explain*, and for students to *understand* (rather than merely memorize), why the graph of a linear equation $ax + by = c$ is a line (see Section 4.4 in [Wu-Alg]). The absence of this reasoning in TSM has made the writing down of the equation of a line that satisfies certain geometric data a fearsome task to many students of algebra. But teachers who have been exposed to this reasoning will begin to see how they might teach the graphing of linear equations differently and liberate their students from this fear, because reasoning can now replace rote memorization.

Beyond the implications for the teaching of algebra, the other reason for the emphasis on geometry in the middle school curriculum is that translations, rotations, reflections, and dilations provide a much more accessible introduction to the staple of a rigorous high school course on geometry: the study of triangles and circles (cf. Volumes I and II of [Wu-HighSchool]). Because the learning of these transformations can be made more accessible and greatly expedited through the use of hands-on geometric experiments, the hands-on experiences serve to demystify congruence and similarity for students. At a time when the school geometry curriculum is beset by issues of fragmentation (because of the disconnect between middle school geometry and high school geometry) and meaningless abstraction (as a result of the rote application of the axiomatic method in a school setting), the middle course offered in these volumes is one potential solution to this pressing curricular problem. For a more detailed discussion of these ideas, see Section 4.1 on page 229.

[2]The one exception is the concept of *dilation*.

The final major departure from TSM in these volumes is the emphasis put on the careful use of symbols. The concept of a "variable" is at present the scourge of middle school mathematics that bars any meaningful entry into algebra. In mathematics, "variable" is no more than an *informal* piece of terminology that serves to remind us of an element in the domain of a function. Yet in TSM and the education literature, "variable" has been elevated to the status of a *mathematical* concept. The inevitable result of such an aberration is to make introductory algebra unlearnable. The whole of the companion volume [Wu-Alg] will testify to the fact that when careful attention is given to the correct use of symbols, rather than to the contortions involved in trying to make sense of "variable", every foundational concept and skill in introductory algebra (what is an equation? what does it mean to solve an equation? what is an expression? etc.) gains in clarity and conceptual simplicity, and algebra becomes once again a potentially learnable subject.

Although these two volumes (an expansion of [Wu2010b] and [Wu2010c]) have been used in my professional development institutes since 2006, it has been difficult to convince teachers to put such a mathematical development directly to use in their classrooms. Their reluctance is entirely understandable because doing so would entail the need to develop new classroom lessons—and probably new curricular units—on their own. It would also require them to teach *against* the existing curriculum of TSM. For example, according to TSM, fractions are best understood through the use of analogies and metaphors (compare the critique in pages 34–39 of [Wu2008]), the concept of a "variable" is central to middle school mathematics (page 102 of [NCTM]), and similar triangles are irrelevant to the learning of school algebra (look up almost any school algebra textbook in K–12 in the past four decades). It is unfair to ask teachers to singlehandedly defy such an entrenched tradition.

This situation has changed somewhat with the advent of the Common Core State Standards for Mathematics (CCSSM) (see [CCSSM]). The CCSSM have come to substantial agreement with the main advocacies of these volumes,[3] especially the three major departures from TSM mentioned above. A recent article in *Education Week* ([Heitin]) indicates that, perhaps, educators have finally come around to embracing the main emphases on the teaching of fractions in (1)–(7) above (one can gain a little historical perspective on this issue by reading Chapter 24 of [Wu2011a]). It should now be easier to convince teachers to learn and apply the content of these volumes (and to convince their administrators to allow them to do so) because the CCSSM are now being implemented in most states. This fact acquires additional significance because on the one hand, school textbooks in general have not risen to the challenge of the CCSSM as of

[3]The document [Wu2010b] is the same document as the one cited as *"Wu, H. 'Lecture Notes for the* 2009 *Pre-Algebra Institute,' September* 15, 2009*"* on page 92 of [CCSSM].

November 2015, and on the other, there seems to be no other complete *mathematical* exposition of middle school mathematics that is consistent with the CCSSM—this is especially true for fractions, negative numbers, and geometry. My hope is that these volumes can double as a stopgap measure at a time when the implementation of the CCSSM seems not too sure of its mathematical footing.

An original impetus for the writing of these volumes was to help solve our nation's severe mathematics education crisis.[4] Back in 2004 when this work was first conceived, the CCSSM did not exist, but the glaring defects of TSM could not be ignored. There are good reasons to believe that the writing of the CCSSM was inspired by this same crisis. It is finally time to banish from schools the jumbled, chaotic, and even downright anti-mathematical presentations that characterize and pervade TSM. To this end, the present volumes strive to improve mathematics teaching by emphasizing, throughout, the following five *fundamental principles* (compare [Wu2011b]):

(I) *Precise definitions are essential.* In mathematics, precise definitions are the bedrock on which all logical reasoning rests, because mathematics does not deal with vaguely conceived notions. Yet definitions are looked upon with something close to disdain by most teachers (and students) as just "more things to memorize". Such a fundamental misconception of the basic structure of mathematics can only come from the TSM we all remember from our own schooling and now teach again to our students, and from the flawed professional development we provide for our teachers. In these volumes, we will respect this fundamental characteristic of mathematics by offering—and employing—precise definitions for every concept, including those that are commonly used, yet remain undefined, in TSM: *fractions, decimals, sum of fractions, product of fractions, division of fractions, ratio, percent, rate, equation, congruence, similarity, slope of a line, graph of an inequality, polygon, length, area,* etc.

(II) *Every statement must be supported by reasoning.* There are no unexplained assertions in these volumes.[5] If something is true, a logical explanation will be given. Although it takes some effort to learn the logical language used in mathematical reasoning, in the long run the presence of reasoning in all we do has the advantage of disarming disbelief and removing the stress of learning-by-rote. It also has the salutary effect of putting the learner and the teacher on the same footing, because the ultimate arbiter of truth will no longer be the teacher's or the textbook's authority, but the compelling rigor of the reasoning.

[4] See, for example, [Askey], [RAGS], and [NMP1].

[5] Except those few explicitly designated as such, because their proofs require advanced mathematics.

(III) *Mathematical statements are precise.* In mathematics, there is no room for imprecision because imprecision leads to misunderstanding and therefore errors. TSM, however, is rife with imprecision, saying things such as "the pizza is the whole" in the study of fractions. This leads to misconceptions about the "whole" being a *shape* (the circle), whereas what is meant mathematically is that the whole is the *area* of the pizza. TSM also defines *percent* to be *out of a hundred.* This then leaves students confused as to whether *percent* is an "action" or a number. If it is an "action", how does one add and divide "actions", and if it is a number, what kind of number is it? It is difficult to imagine how mathematics learning can take place when learners' minds are beset by such confusion. Another example is TSM's claim that "multiplication and division are inverse operations; they undo each other". But given two numbers such as 2 and 3, we have $2 \times 3 = 6$ and $2 \div 3 = \frac{2}{3}$. TSM does not explain in which way 6 and $\frac{2}{3}$ undo each other. What is meant is that if we fix a number k ($\neq 0$), then the operation of multiplying a given number by k followed by the operation of dividing the resulting number by k leaves the given number unchanged; in this sense, multiplication and division indeed undo each other. It would seem, however, that even this much precision is unattainable by TSM. This is another reason why TSM is unlearnable.

This lack of precision is by no means limited to elementary school mathematics; it pervades the K–12 curriculum. On the high school level, for example, the definition that $3^{-x} = 1/3^x$ is too often offered amidst a flurry of heuristic arguments that leave the readers with the impression that the equality $3^{-x} = 1/3^x$ has been *proved*. Such persistent ambiguities consequently leave many students as well as teachers confused about the difference between a definition and a theorem.

(IV) *Mathematics is coherent.* The concept of *mathematical coherence* is often brought up in educational discussions nowadays, but it is not something that can be understood through verbal descriptions any more than the transcendental serenity of the adagio in the Schubert C major quintet can be appreciated through the reading of an essay praising its beauty. Very crudely speaking, the coherence of mathematics refers to the fact that the body of knowledge that is mathematics has a tightly-knit structure, but the only way one can get to know and appreciate this structure is by wading into its details. For example, the concept of similarity in Section 4.7 on page 320 relies on a knowledge of multiplying and dividing fractions (Sections 1.5 and 1.6 on pages 56 and 70, respectively) and congruence (Section 4.5 on page 287), and is itself used in a crucial way for the definition of slope (Section 4.3 in [Wu-Alg]). Another example is the omnipresence of the theorem on equivalent fractions in the discussion of almost every topic in fractions, when TSM would have you believe that it is only useful for simplifying fractions. Yet another example is the key role played by congruence not only in the definition of similarity (Section 4.7 on page 320) but also in the considerations of length, area, and volume (see Chapter 5). As a final example, you will notice that the division of whole numbers, the division of fractions (Section 1.6 on page 70), and the division of rational numbers (Section 2.5 on page 174) are conceptually identical.

The coherence of mathematics makes mathematics more teachable and more learnable. This can be easily understood by an analogy: whereas one can pore over a page from a phone book without any recollection of what has been read afterwards, almost all readers have vivid memories of *Don Quixote*—all one thousand pages of it—even after only one reading, because it tells a coherent story.

Although coherence is difficult to describe, the *lack of coherence* can be more easily illustrated. A striking example of the failure of coherence in TSM is the common explanation of the theorem on equivalent fractions, which states that $\frac{m}{n} = \frac{km}{kn}$ *for all fractions* $\frac{m}{n}$ *and for all positive integers* k. TSM would have you believe that this is true because

$$\frac{m}{n} \;=\; 1 \times \frac{m}{n} \;=\; \frac{k}{k} \times \frac{m}{n} \;=\; \frac{km}{kn}.$$

However, the last step depends on knowing how to multiply fractions, and the multiplication of fractions is a topic that comes late in the development of the subject.[6] When the reasoning for *the* basic theorem in fractions—the theorem on equivalent fractions—is given in terms of something more complex and, in any case, not yet available, how can we expect students to learn? Unfortunately, such subversions of logic abound in TSM.

(V) *Mathematics is purposeful.* Mathematics is goal-oriented, and every concept or skill in the standard curriculum must be there for a purpose. Teachers who recognize the purposefulness of mathematics gain an extra tool for making their lessons more compelling and, therefore, more learnable. When congruence and similarity are taught with no apparent purpose except to do "fun activities",[7] students lose sight of the mathematics and may wonder why they are required to learn it. However, as noted above, the concept of congruence lies behind the concept of similarity, and both are needed to make sense of basic issues in algebra such as linear equations of two variables and their graphs, e.g., why is the graph of such an equation a (straight) line? Students are more likely to feel motivated to learn if presented with a curriculum that actually offers explanations of why its basic facts are worth learning.

[6] Multiplication is the most subtle among the four arithmetic operations on fractions. Its definition is nontrivial; the proof of the product formula is sophisticated; and its relationship with the area of a rectangle (with fractional sides) is subtle. See Section 1.5 on page 56.

[7] This has been happening all too often lately as a result of the misunderstanding of the CCSSM propagated by people immersed in TSM.

Middle school mathematics is the bridge that leads from fairly concrete concepts about numbers in elementary school to more abstract concepts in algebra, geometry, and trigonometry in high school. Our nation's curriculum is traditionally weak in middle school; one can almost say that it has been delinquent in its failure to provide careful guidance for students' transition from the concrete to the abstract. The teaching of TSM has become the norm in those years. Instead of giving precise instruction on the correct use of symbols and explaining the need for the idea of generality in students' next step on their mathematical journey, TSM harps on the alleged profundity of the fictitious concept of a "variable"; instead of guiding students' tentative first steps to think abstractly about negative numbers, TSM redirects them to *replace* abstract thinking by analogies and heuristic patterns. By contrast, this volume and its companion volume [Wu-Alg] take this bridge seriously. They confront the necessary abstractions without compromise, but they do so by building on the foundation of elementary school mathematics (cf. [Wu2011a]). I hope these volumes will initiate change by making you more aware of the overriding importance of this bridge in students' mathematics learning trajectory. Ultimately, the goal of these volumes is to help you teach your students better.

Acknowledgements. This volume and its companion volume [Wu-Alg] evolved from the lecture notes ([Wu2010b] and [Wu2010c]) for the Pre-Algebra and Algebra summer institutes that I used to teach to middle school mathematics teachers from 2004 to 2013. My ideas on professional development for K–12 mathematics teachers were derived from two sources: my understanding as a professional mathematician of the *minimum* requirements of mathematics (see the five fundamental principles on pages xv ff.) and the blatant corrosive effects of TSM on the teaching and learning of mathematics. Those summer institutes therefore placed a special emphasis on improving teachers' content knowledge. I would not have had the opportunity to try out these ideas on teachers but for the generous financial support from 2004 to 2006 by the Los Angeles County Office of Education (LACOE), and from 2007 to 2013 by the S. D. Bechtel, Jr. Foundation. Because of the difficulty I have had with funding by government agencies—they did not (and perhaps still do not) consider the kind of content-based professional development I insist on to be worthy of support—my debt to Henry Mothner and Tim Murphy of LACOE and Stephen D. Bechtel, Jr. is enormous.

Through the years, I have benefited from the help of many dedicated teachers; to Bob LeBoeuf, Monique Maynard, Marlene Wilson, and Betty Zamudio, I owe the corrections of a large number of linguistic infelicities and typos, among other things. Winnie Gilbert, Stefanie Hassan, and Sunil Koswatta were my assistants in the professional development institutes, and their comments on the daily lectures of the institutes could not help

but leave their mark on these volumes. In addition, Sunil created some animations (referenced in Chapters 2 and 4) at my request. Phil Daro graciously shared with me his insight on how to communicate with teachers. Sergei Gelfand made editorial suggestions on these volumes—including their titles—that left an indelible imprint on their looks as well as their user-friendliness. R. A. Askey read through a late draft with greater care than I had imagined possible, and he suggested many improvements as well as corrections. I shudder to think what these volumes would have been like had he not caught those errors. Finally, Larry Francis helped me in multiple ways. He created animations for me that can be found in Chapter 4. He is also the only person who has read almost as many drafts as I have written. (He claimed to have read twenty-seven, but I think he overestimated it!) He met numerous last minute requests with unfailing good humor, and he never ceased to be supportive; more importantly, he offered many fruitful corrections and suggestions.

To all of them, it gives me great pleasure to express my heartfelt thanks.

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Berkeley, California
March 15, 2016

CHAPTER 4

Experimental Geometry

4.1. Overview

In this chapter, we will be engaged mainly in an *informal* study of the geometry of the plane, supplemented by a gentle introduction to proofs of geometric theorems. The specific mathematical goals are twofold: to make a working knowledge of similar triangles an integral part of the middle school curriculum, and to set up the intuitive foundation for a more precise discussion of the concepts of congruence and similarity in high school geometry.

These are not quite the usual emphases in the conventional middle school curriculum, and you may wonder about the call for change. A meaningful explanation requires us to take a broad overview of three critical issues that directly impact the teaching of middle school and high school geometry:

(1) The continuing crisis in the teaching of high school geometry.
(2) The role of similar triangles in the study of linear equations of two variables.
(3) The discontinuity between the middle school and high school geometry curricula in TSM.[1]

A more expansive discussion follows.

(1) *The continuing crisis in the teaching of high school geometry.*

The teaching of high school geometry has been broken for more than four decades, if not much longer. Until the 1990's, it was always taught à la Euclid, starting with axioms. For the first month or two, such a course would be devoted to a mind-numbing litany of definitions, axioms, and proofs of boring, obvious statements that were offered as theorems. A well-known but notorious fact is that learning geometry in most classrooms became synonymous with regurgitating memorized two-column proofs. Worse still, those proofs were often constructed according to—not the demands of mathematical reasoning—but the idiosyncratic demands

[1]See page xi for the definition.

of a teacher's *grading policy.*[2] Nevertheless, the learning of axiomatic geometry was supposed to be all about reasoning and mathematical rigor.

The untenable situation inevitably triggered radical reactions. For the past twenty years or so, it has been common practice to teach high school geometry with *no* proofs but only engage students in hands-on activities to verify the validity of geometric theorems. The reliance on hands-on activities was aided by the serendipitous emergence of increasingly versatile geometry computer software that made experimentations with geometric figures—such as observing that the three medians of a triangle continue to meet at a point even when the vertices are randomly altered—not only painless but even "fun". We will further address this aspect of the crisis on page 234.

Students' ability to reason needs careful long-term nurturing for it to develop; it cannot be turned on and off like a faucet.

There are at least three reasons for this crisis. Foremost is the fact that, in TSM, reasoning is absent everywhere except in high school geometry where proofs are explicitly demanded. Not having learned how to reason anywhere else in TSM, students are suddenly asked—in 9th or 10th grade—to write proofs, and *for everything* to boot. The cynical mindset behind the design of such a math curriculum would seem to regard students' ability to reason as something one can turn on and off like a faucet. Unfortunately, the ability to reason needs careful long-term nurturing for it to develop: students cannot go directly from a Proof-Free Zone to a Proof-Only Zone and be expected to acquit themselves respectably. It therefore came to pass that in most high school geometry courses that did not reject proofs outright, the rote-teaching of two-column proofs and the attendant rote-learning by students became the norm.

Axiomatization is an organizational afterthought. It is a learning tool for the mathematically sophisticated, but usually not for beginners.

Until reasoning is insisted upon *everywhere* in the K–12 math curriculum and not just in geometry, it would be out of the question to talk about geometry *education* in schools. On the other hand, if students are accustomed to reasoning, geometric proofs then become part of the normal mathematical activities, and high school geometry would not be the fiasco that it is now.

A second reason for the crisis is the rigid adherence to the *axiomatic development* of school geometry. The axioms of any axiomatic system are in general abstract, and the nature of

[2] Some of these practices are documented in the article, "When good teaching leads to bad results" ([Schoenfeld]). As a side remark, however, one should raise the question: what kind of an education system would consider such teaching to be "good"?

an axiomatic development makes matters worse by hurling multiple abstractions at learners right from the start, when they are ill-prepared for such a mathematical onslaught. And the immediate payoff? A succession of theorems that are glaringly obvious but which can only be proved by *abstract, formal* reasoning (see, e.g., Chapter 2 of [Hartshorne]), as anyone who has gone though a standard course in plane geometry knows only too well. Even professional mathematicians often find this kind of tedious reasoning to be daunting. What complicates things further is the fact that the axiomatic system for plane geometry is among the most complex in classical mathematics. It is pedagogically unsound to confront school students with something this difficult at the beginning of their journey in mathematics. Mathematics educators should realize that axiomatization is an organizational afterthought. It is a learning tool for mathematically sophisticated learners, but not for others, and certainly *not for K–12 students in something as complicated as geometry.*

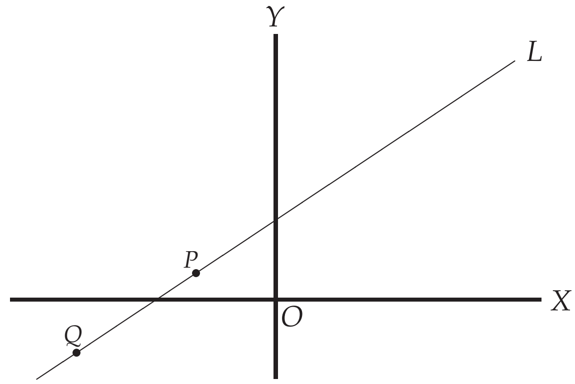
The undesirability of the axiomatic treatment in high school geometry can also be seen, for example, in the way the concepts of **congruence** and **similarity**—the two cornerstones of school geometry—are typically taught. First students are told in K–8 that *congruence* is "same size and same shape", and that *similarity* is "same shape but not necessarily the same size". Then in high school, all that is forgotten because both concepts now have to be tied down to *abstract* axioms about triangles. The decision to teach high school geometry axiomatically therefore necessitates an unwarranted disruption of students' learning trajectory: whereas congruence and similarity are taught, in middle school, as (imprecise) metaphors that apply to all geometric figures, they suddenly become abstractions in high school that apply only to triangles and polygons. This is not the way we want to promote student learning.

We can do better for our K–12 students. We will do better in this volume as well as in the companion volume [Wu-Alg].

A final reason for the crisis is the vicious cycle created by TSM: today's geometry teachers were themselves brought up in TSM geometry and—because our colleges do not see fit to help them overcome TSM by teaching them correct school mathematics—they have no choice but to teach the same TSM geometry to their own students. In due course, some of these students will become teachers and take their turn inflicting TSM geometry on *their own students.* In this way, from one generation to the next, the crisis becomes self-replicating.

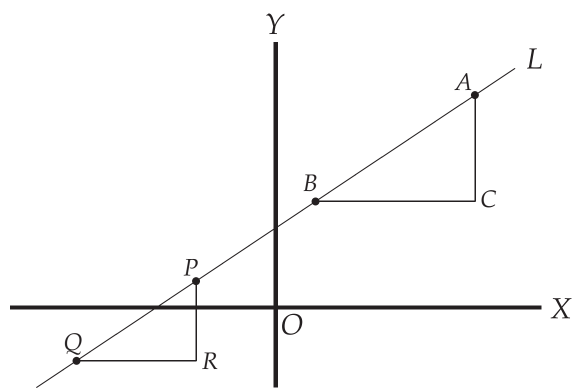
(2) *The role of similar triangles in the study of linear equations of two variables.*

TSM makes believe that geometry is connected to school algebra *only* through the setting up of a coordinate system in the plane or space and the drawing of the graph of an equation or a function. What is hidden in TSM is the fact that a solid foundation in introductory algebra has to be built on a knowledge of similar triangles. TSM defines the *slope* of a (nonvertical) line L in the coordinate plane in the following way: let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be distinct points on L. Then the **slope** of L is the ratio $\dfrac{p_2 - q_2}{p_1 - q_1}$.



The first thing you should ask is whether this definition of slope is *well-defined,* i.e., whether it makes sense. The answer is "not yet", because if $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are two *other* points, also on L, is the slope of L equal to $\dfrac{a_2 - b_2}{a_1 - b_1}$? In other words, *which* of these ratios should be the slope of L:

$$\frac{p_2 - q_2}{p_1 - q_1} \quad \text{or} \quad \frac{a_2 - b_2}{a_1 - b_1} \ ?$$



After all, there are an infinite number of pairs of such points A and B on L, and if the preceding ratios are not equal, which of these ratios should be called "the slope of the line L"? Fortunately, it turns out that for any points A and B on L, it is always the case that

$$\frac{p_2 - q_2}{p_1 - q_1} = \frac{a_2 - b_2}{a_1 - b_1}. \tag{4.1}$$

Therefore any such ratio formed from two distinct points on L can be used as the definition of the slope of L. The fact that (4.1) is true requires the proof that $\triangle ABC$ is similar to $\triangle PQR$ ($\triangle$ stands for *triangle*).[3] Unfortunately, TSM does not bother to let students know about equation (4.1).

It is difficult to solve problems related to *slope* without the explicit knowledge that slope can be computed by *choosing any two points on the line* that suit one's purpose, but this very fact is missing in TSM. The natural consequence of this omission is students' well-known difficulty with learning all aspects of the geometry of linear equations. Unable to confidently base their work with slope on mathematical reasoning, they are forced to memorize—often without success—the four forms of the equation of a line (two-point, point-slope, slope-intercept, and standard) by brute force.

It may not be obvious, but this concern about the *correct* definition of slope is fundamental to students' learning of algebra. A recent survey ([Postelnicu-Greenes]) of students' understanding of (straight) lines in introductory algebra shows that the most difficult problems for them are those requiring the *identification of the slope of a line from its graph*. Think about this for a second: how can a straightforward, simple skill of computing a division,

$$\frac{p_2 - q_2}{p_1 - q_1},$$

be the *most difficult problem* for students learning about lines unless, of course, they don't even understand what they are supposed to compute? So they haven't the foggiest idea of what *slope* is because this concept, as taught in TSM, makes no sense to them. Any redesign of the geometry curriculum must therefore tackle the issue of helping students acquire a working knowledge of similar triangles *before* they take up the algebra of linear equations.

"Curricular coherence" is a concept that is gaining currency in present-day education discussions. If such discussions are to be taken seriously, then the first order of business would have to be the cementing of this curricular rupture between the study of similar triangles and the algebra of linear equations.

(3) *The discontinuity between the middle school and high school geometry curricula in TSM.*

A main topic of this chapter is how to use translations, reflections, and rotations to define the concept of *congruence*. Here are two observations:

> (a) In TSM, translations, reflections, and rotations are taught in the middle school curriculum, without mentioning congruence, as tools for art appreciation (symmetries), but in high school, they are tagged on—as an

[3] See Section 4.3 in [Wu-Alg] for the details.

> afterthought—at the end of the geometry course to supplement the concept of congruence. The relationship between congruence on the one hand and translations, reflections, and rotations on the other seems tenuous at best.
>
> (b) In K–8, *congruence* is "same size and same shape", and *similarity* is "same shape but not necessarily the same size". In high school, *congruence* and *similarity* are defined anew without any reference to "size" or "shape", *but only for polygons* in terms of degrees of angles and lengths of segments. There is no more reference to "size", "shape", or any geometric figure that is not a polygon.

This blatant chasm between middle school and high school geometry is presumably not an example of "curricular coherence".

Any reasonable school geometry education therefore must ease the progression from middle school to high school by bridging the preceding discontinuity and minimizing unnecessary abstractions. As mentioned above, it must also introduce similar triangles into middle school to support the teaching of algebra. These are the problems confronting any reasonable attempt at revamping the school geometry curriculum. Unfortunately, the only such attempt we have on record would seem to be the radical solution offered in the 1990's that replaced the teaching of high school geometry with hands-on activities alone containing *no* proofs at all. See, e.g., [Serra]. Clearly, if a main issue with high school geometry is the pervasive lack of reasoning, then one should not tackle this issue by abandoning reasoning altogether. After all, two wrongs do not make a right.

The present volume, together with Volumes I and II of [Wu-HighSchool], directly address these three critical issues in the teaching of school geometry.

This chapter begins by outlining a series of activities in geometry that are designed to foster the acquisition of geometric intuition. In the process, it also acquaints the reader with some working vocabulary and concepts in geometry. The chapter then goes on to introduce translations, reflections, rotations, and dilations, mainly through the use of transparencies and drawings. It culminates in the precise definitions of *congruence* and *similarity*, as well as the explanations for some basic theorems related to similarity that make possible a correct definition of the slope of a line. The emphasis throughout will be on the applications of the concepts of congruence and similarity and not so much on the internal logical structure that underlies these concepts. The intention is to lay a robust *intuitive* foundation for a more precise and more proof-oriented high school course in geometry. Indeed, such a course can be developed—with precise assumptions and proofs—as a *direct continuation* of this intuitive treatment. See Volumes I and II of [Wu-HighSchool].

Developing students' geometric intuition is important because, without it, the learning of geometry will forever stay on a formal level and therefore be easily forgotten. This chapter will not be overly concerned with *total* precision or *total* accuracy; all the concepts introduced can be formally defined and all proofs can be logically tightened in a course for high school. Rest assured, however, that except for a few missing details, every concept in this chapter has been defined correctly and every proof is valid. The reason for sidestepping total precision in an introductory treatment is that the requisite precision of formal definitions can sometimes rob a simple concept (such as the "direction of a translation") of its intuitive appeal. The main purpose of this chapter is therefore to make sure that the underlying intuition is in place before formal definitions are introduced. The latter part of the chapter shows how to use these concepts to make some simple, logical deductions on the basis of this intuitive foundation. It is worth repeating that—just as in the teaching of calculus—there is nothing wrong with the strategy of emphasizing how to *use* the concepts and skills correctly before confronting the intricacies of their internal structure.

Having extolled the virtue of intuition, we wish to also sound the alarm that a teacher cannot teach geometry (or any topic) knowing only its intuitive content without a firm grounding in its theoretical underpinnings. This explains the insistent presence of a significant amount of mathematical reasoning in what follows.

This and the following chapter (Chapter 5), with occasional exceptions, present geometry in a way that can be taught (and learned), *as is*, in middle school provided one gives the presentation some obvious pedagogical embellishments. The exceptions will be pointed out in due course (see, for example, the discussions on pages 290 and 324). The implicit agenda behind these two chapters is, of course, an attempt to solve the above-mentioned curricular problem in middle school.

It remains to point out that the Common Core State Standards for Mathematics ([CCSSM]) have adopted the same course of action regarding the teaching of middle school and high school geometry. In so doing, these Standards give all students an opportunity to learn school algebra properly for the first time by removing TSM's illegitimate approach to slope from the school curriculum. In addition, they also provide a more intuitive approach to high school geometry. By the same token, these Standards have also lent a sense of urgency to the need for all middle school teachers to put TSM behind them and learn something about congruence and similar triangles. This chapter has been written with such a need in mind.

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