Fractions, Decimals, and Rational Numbers

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September 24, 2008; *revised*, June 1, 2014

Contents

Introduction This is a slightly revised version of the report, written in the early part of 2007 at the request of the Learning Processes Task Group of the National Mathematics Advisory Panel (NMP), on the curricular aspects of the teaching and learning of fractions.[1] For a reason to be explained presently, my

[1](*June 1, 2014.*) It is a pleasure for me to thank David Collins for a large number of corrections.

decision was to focus the report on the difficulties of teaching fractions and rational numbers **in grades 5–7**. It contains a detailed description of the most essential concepts and skills together with comments about the pitfalls in teaching them. What may distinguish this report from others of a similar nature is the careful attention given to the *logical underpinning and inter-connections* among these concepts and skills. It is in essence a blueprint for a textbook series in grades 5–7. It would serve equally well as the content of an extended professional development institute on fractions, decimals and rational numbers.

The concluding section on **Comments on fractions research**, beginning on p. 33, discusses the research literature on the teaching of fractions, decimals, and rational numbers. Because the learning of these topics is integral to the learning of algebra, to be explained presently, *I will make a few comments on the teaching of algebra as well.* It may be that, for some readers, reading these **Comments** should precede the reading of the main body of this report.

The teaching of fractions in the U.S. is spread roughly over grades 2–7. In the early grades, grades 2–4 more or less, students' learning is mainly on acquiring the vocabulary of fractions and using it for descriptive purposes. It is only in grades 5 and up that serious learning of the *mathematics* of fractions takes place. In those years, students begin to put the isolated bits of information they have acquired into a mathematical framework and learn how to compute extensively with fractions. This learning process may be likened to the work of a scientist in studying a new phenomenon. The initial exploration of fractions may be taken to be the "data-collecting phase": just take it all in and worry about the meaning later. In time, however, the point will be reached where, unless the data are put into a theoretical framework and organized accordingly, they would get out of control. So it is that when students reach the fifth or sixth grade, they have to learn a *precise mathematical concept* of a fraction and make logical sense of the myriad skills that come with the territory.

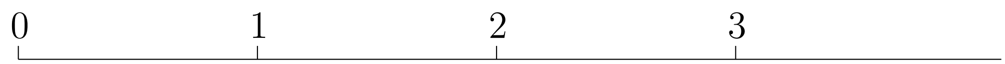
Students' fear of fractions is well documented (cf. [Ashcraft]), but to my knowledge, there is no such pervasive fear in the early vocabulary-acquiring stage. In the second stage, however, this fear is real and seems to develop around the time they learn how to add fractions using the least common denominator. From a curricular perspective, this fear can be traced to at least two sources. The first is the loss of a natural reference point when students work with fractions. In learning to deal with the mathematics of *whole numbers* in grades 1–4, children always have a natural reference point: their fingers. The modeling of whole numbers on one's finger is both powerful and accurate. But for fractions, the curricular decision in the U.S. is to use a pizza or a pie as the reference point. Unfortunately, while pies may be useful in the lower grades to help with the vocabulary-learning aspect of fractions, they are a very awkward model for fractions bigger than 1 or for any arithmetic operations with fractions. For example, how do you multiply two pieces of pie? ([Hart]). Such difficulties probably cause classroom instructions as well as textbooks to concentrate on those fractions which are less than 1 and have single digit numerators and denominators. Needless to say, such artificial restrictions are a distraction to the learning of fractions in general. A second source that contributes to the fear of fractions is the inherent abstract nature of the concept of a fraction. Fractions are in fact a child's first excursion into abstract mathematics. A hard-won lesson in mathematics research of the past two centuries is that when dealing with abstractions, precise definitions and precise reasoning are critical to the prevention of errors and to the clarification of one's thoughts. Because the generic U.S. K-12 mathematics curriculum has not been emphasizing definitions or reasoning for decades, the teaching of fractions in grades 5–7 is almost set up for failure. Students' morbid fear of the subject is the inevitable consequence.

The fear of fractions would be of little concern to us were it not for the fact that, for many reasons, fractions are crucial for the learning of algebra (see, for example, [Wu1]). Because algebra is the gateway to the learning of higher mathematics and because learning algebra is now considered to be the new civil right in our technological age, the goal of NMP is to improve the learning of algebra in our nation. With this in mind, removing the two sources of the fear of fractions then becomes a national mandate. It is for this reason that this report is focussed on the teaching and learning of fractions in grades 5–7. There is another curricular reason that makes this topic fully deserving of our full attention. For almost all school students — the exceptions being future math majors in college — what they learn about fractions and decimals in grades 5–7 is *all* they will ever learn about these numbers for the rest of their lives. When one considers the role these numbers play in the life of the average person, it is nothing short of our basic civic duty to eliminate this fear from our national discourse.

The following exposition in (A)–(Q) describes a way to teach fractions that meets the minimum mathematical requirements of precision, accuracy, providing a definition for every concept, and sequencing topics in a way that makes reasoning possible. More importantly, it is also placed in a context that is suitable for use in grades 5 and up. I try to navigate a course that is at once mathematically correct and pedagogically feasible. For example, the number line is used as a natural reference point for fractions, in the same way that fingers serve as a reference point for whole numbers. I believe that the comparison of the number line to fingers is apt in terms of efficacy and conceptual simplicity. It will be noted that the use of the number line has the immediate advantage of conferring coherence on the study of numbers in school mathematics: decimals are rightfully restored as fractions of a special kind, and positive and negative fractions all become points on the number line. In particular, whole numbers are now points on the number line too and ***the arithmetic of whole numbers, in this new setting, is now seen to be entirely analogous to the arithmetic of fractions.*** We hope this will lay to rest the idea that "Children must adopt new rules for fractions that often conflict with well-established ideas about whole number" ([Bezuk-Cramer], p. 156). Such coherence provides a more effective platform for learning these numbers, because simplicity is easier to learn than unnecessary complexities. It must be said that this coherence has been largely absent from school mathematics for a long time.

(A) Definition Mathematics requires that every concept has a precise definition. In the informal and exploratory stage of learning (roughly grades K-4), such precision may not be necessary for the learning of fractions. *For grades 5 and up, there is no choice: there has to be a definition of a fraction.* By and large, school mathematics (if textbooks are any indication, regardless of whether they are traditional or reform) does not provide such a definition, so that teachers and students are left groping in the dark about what a fraction is. A fraction, to almost all teachers or students, is a piece of a pie or pizza. This is not helpful in the learning of mathematics unless one can figure out how to multiply or divide two pieces of a pie, or how a pie can help solve problems about speed or ratio.

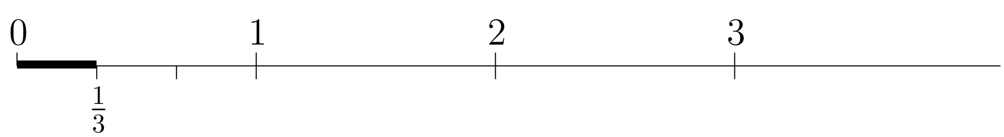
A usable definition of a fraction, say those with denominator 3, can be given as follows. We begin with the *number line.* So on a line which is (usually chosen to be) horizontal, we pick a point and designate it as 0. We then choose another point to the right of 0 and, by reproducing the distance between 0 and this point, we get an infinite sequence of equi-spaced points to the right of 0. Next we denote all these points by the nonzero whole numbers 1, 2, 3, ... in the usual manner. Thus all the whole numbers $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ are now displayed on the line as equi-spaced points increasing to the right of 0, as shown:



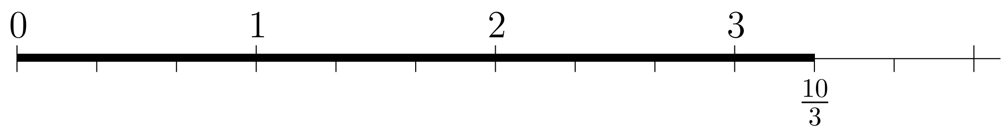
A horizontal line with an infinite sequence of equi-spaced points identified with $\mathbf{N}$ on its right side is called the **number line**. By definition, a **number** is just a point on the number line. In sections (A)–(K), we will use only the number line to the right of 0. We will make use of the complete number line starting in (L) when we get to negative numbers.

Fractions are a special class of numbers constructed in the manner below. If a and b are two points on the number line, with a to the left of b, we denote the segment from a to b by $[\boldsymbol{a}, \boldsymbol{b}]$. The points a and b are called the **endpoints** of $[a, b]$. The special case of the segment $[0, 1]$ occupies a distinguished position in the study of fractions; it is called the **unit segment**. The point 1 is called the **unit**. As mentioned above, 0 and 1 determine the points we call the whole numbers. So if 1 stands for an orange, 5 would be 5 *oranges*, and if 1 stands for 5 pounds of rice, then 6 would be 30 pounds of rice. And so on.

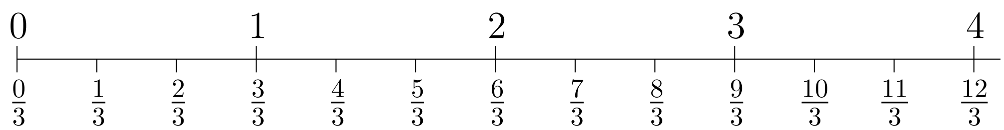
We take as our "whole" the unit segment $[0, 1]$. The fraction $\frac{1}{3}$ is therefore one-third of the whole, i.e., if we divide $[0, 1]$ into 3 equal parts, $\frac{1}{3}$ stands for one of the parts. One obvious example is the thickened segment below, and we use the right endpoint of this segment as the standard representation of $\frac{1}{3}$



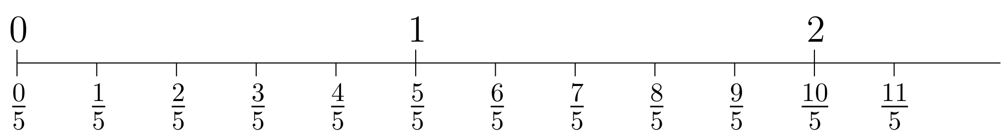
We next divide, not just $[0, 1]$, but every segment between two consecutive whole numbers — $[0, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, etc. — into three equal parts. Then these division points, together with the whole numbers, form an infinite sequence of equi-spaced points, to be called **the sequence of thirds**. In general, a fraction $\frac{m}{3}$ for some whole number m, which intuitively stands for "m copies of thirds", has the **standard representation** consisting of m adjoining short segments abutting 0, where a short segment refers to a segment between consecutive points in the sequence of thirds. Since we may identify this standard representation of $\frac{m}{3}$ with its right endpoint, we denote the latter simply by $\frac{m}{3}$. The case of $m = 10$ is shown below:



Having identified each standard representation of $\frac{m}{3}$ with its right endpoint, each point in the sequence of thirds now acquires a name, as shown below. These are exactly the fractions with denominator equal to 3.



If we consider all the fractions with denominator equal to n, then we would be led to **the sequence of *n*-ths**, which is the sequence of equi-spaced points resulting from dividing each of $[0, 1]$, $[1, 2]$, $[2, 3]$, $\ldots$, into n equal parts. The fraction $\frac{m}{n}$ is then the m-th point to the right of 0 in this sequence. For example, the fractions with denominator equal to 5 are now displayed as shown:



This definition of a fraction, compared with the usual one as a piece of pie, is in fact simpler: we have replaced a round pie by a segment (the unit segment), and every student will tell you that it is far easier to divided a segment into 5 parts of equal length than to divide a circle into 5 congruent parts. It is also far more flexible, in the sense that by specifying the unit 1 to be an apple, $\frac{1}{3}$ will be a third of the apple, and if we designate the unit 1 to be a mile, 54 will be 54 miles. Finally, and this is most important, all fractions, proper or improper, can be displayed with ease on the number line, thereby affording a platform for

all *fractions to be treated equally. One can speculate, and this is something researchers can work on, that the reason students' conception of fractions is limited to* proper fractions with single digit numerators and denominators *is because the pie model compels teachers and textbooks to work within* one single *pie. It is a mathematical judgment, which can be amply justified, that such a limited conception of fractions automatically limits students' conceptual understanding of the subject. The superiority of this definition in every aspect related to the teaching of fractions will be borne out in the rest of this article.*

Those fractions whose denominators are all positive powers of 10, e.g.,

$$\frac{1489}{100}, \quad \frac{24}{100000}, \quad \frac{58900}{10000},$$

are called **decimal fractions**, but they are better known in a different notation. It has been recognized for a long time that, with the number 10 understood, there is no reason to write it over and over again so long as we can keep track of zeros, namely 2, 5, and 4, respectively, in this case. These fractions are therefore abbreviated to

$$14.89, \quad 0.00024, \quad 5.8900$$

respectively. The rationale of the notation is clear: the number of digits to the right of the so-called **decimal point** keeps track of the number of zeros in the respective denominators, 2 in 14.89, 5 in 0.00024, and 4 in 5.8900. In this notation, these numbers are called **finite** or **terminating decimals**. In context, we usually omit any mention of "finite" or "terminating" and just say "decimals". Notice the convention that, in order to keep track of the 5 zeros in $\frac{24}{100000}$, three zeros are added to the left of 24 to make sure that there are 5 digits to the right of the decimal point in 0.00024. Note also that the 0 in front of the decimal point is only for the purpose of clarity, and is optional.

One would like to think that 5.8900 is the same as 5.89. For this we have to wait for the next section.

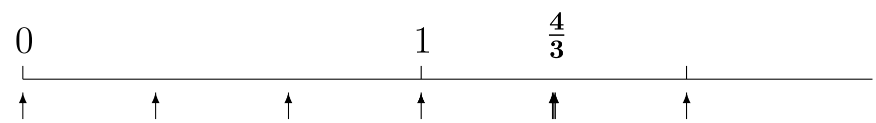
(B) Equivalent fractions This is the single most important fact about fractions. It says that *for all whole numbers k, m, and n (so that $n \neq 0$ and $k \neq 0$),*

$$\frac{m}{n} = \frac{km}{kn}$$

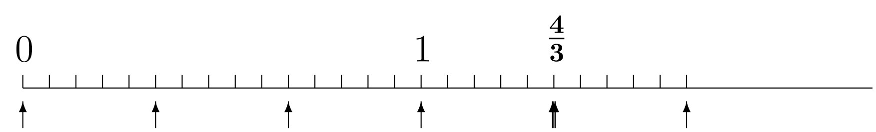
This fact can be proved very simply now that a precise definition of a fraction is available. The reasoning for the special case

$$\frac{5 \times 4}{5 \times 3} = \frac{4}{3}$$

will be seen to hold in general. First locate $\frac{4}{3}$ on the number line:



We divide each of the segments between consecutive points in the sequence of thirds into 5 equal parts. Then each of the segments $[0, 1]$, $[1, 2]$, $[2, 3]$, $\ldots$ is now divided into 15 equal parts and, in an obvious way, we have obtained the sequence of fifteenths on the number line:



The point $\frac{4}{3}$, being the 4-th point in the sequence of thirds, is now the 20-th point in the sequence of fifteenths (20 being equal to 5×4). The latter is by definition the fraction $\frac{20}{15}$, i.e., $\frac{5 \times 4}{5 \times 3}$. Thus $\frac{4}{3} = \frac{5 \times 4}{5 \times 3}$.

The first application of equivalent fractions is to bring closure to the discussion in the last section about the decimal 5.8900. Recall that we had, by definition,

$$\frac{58900}{10000} = 5.8900$$

We now show that $5.8900 = 5.89$ and, more generally, *one can add or delete zeros to the right end of the decimal point without changing the decimal.* Indeed,

$$5.8900 = \frac{58900}{10000} = \frac{589 \times 100}{100 \times 100} = \frac{589}{100} = 5.89$$

where the next to the last equality makes use of equivalent fractions. The reasoning is of course valid in general, e.g.,

$$12.7 = \frac{127}{10} = \frac{127 \times 10000}{10 \times 10000} = \frac{1270000}{100000} = 12.70000$$

Another useful consequence of equivalent fractions is the following **Fundamental Fact of Fraction-Pairs (FFFP)**:

> Any two fractions may be symbolically represented as two fractions with the same denominator.

The reason is simple: if the fractions are $\frac{m}{n}$ and $\frac{k}{\ell}$, then because of equivalent fractions, we have

$$\frac{m}{n} = \frac{m\ell}{n\ell} \quad \text{and} \quad \frac{k}{\ell} = \frac{nk}{n\ell}$$

Now they share the denominator $n\ell$.

Why should we pay attention to FFFP? If any two fractions can be written as fractions with the same denominator, e.g., $\frac{a}{n}$ and $\frac{b}{n}$, then they are put on the same footing, in the sense that in the sequence of n-ths, these two fractions are in the a-th and b-th positions. For example, one can tell right away that $\frac{a}{n}$ is to the left of $\frac{b}{n}$ if $a < b$. Such considerations will play an important role below.

Equivalent fractions naturally brings up the issue of whether students should always reduce each fraction to lowest terms. Implicit in this statement is the assumption that every fraction is equal to a unique fraction in lowest terms. (While this assumption is true and believable, it is nonetheless the case that its proof is quite nontrivial, depending as it does on the Euclidean algorithm.) What is more pertinent is the fact that, as a fraction, $\frac{12}{9}$ is every bit as good as $\frac{4}{3}$, and in general $\frac{nk}{n\ell}$ is every bit as good as $\frac{k}{\ell}$. An insistence on always having a fraction in its lowest terms is thus a preference but not a mathematical necessity. Moreover, it is sometimes not immediately obvious whether a fraction is in lowest terms or not, e.g., $\frac{68}{51}$. A more flexible attitude towards unreduced fractions would consequently make for a better mathematics education for school students.

(C) Fraction as division For any two whole numbers m and n, with $n \neq 0$, we define **the division of m by n** as follows:

$\boldsymbol{m \div n}$ is the length of one part when a segment
of length $\boldsymbol{m}$ is partitioned[2] into $\boldsymbol{n}$ equal parts.

Why this definition? Because students coming into fifth grade or thereabout only know about the meaning of "9 divided by 3", "28 divided by 7", or in general, "m divided by n *when m is a multiple of n*". But now we are talking about the division of arbitrary positive integers such as "5 divided by 7" or "28 divided by 9". Such divisions are conceptually distinct from fifth graders' previous encounters with the concept of "division". A major weakness in the school mathematics literature is the failure to draw attention to this sharp distinction between these two kinds of division and give a precise definition of the general case (see above).

With this definition understood, a critical point in the development of the concept of a fraction is the *proof* of the following

Theorem *For any two whole numbers m and n, $n \neq 0$,*

$$\frac{m}{n} = m \div n$$

This is called **the division interpretation of a fraction**. The proof is simplicity itself. To partition $[0, m]$ into n equal parts, we express $m = \frac{m}{1}$ as

$$\frac{nm}{n}$$

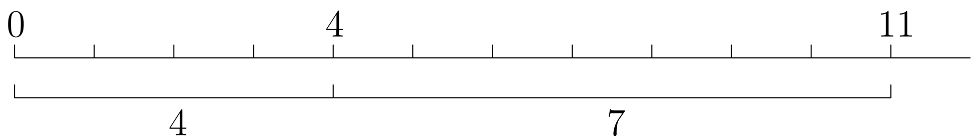
That is, $[0, m]$ is equal to nm copies of $\frac{1}{n}$, which is also n copies of $\frac{m}{n}$. So 1 part in a partitioning of $[0, m]$ into n equal parts is $\frac{m}{n}$.

This theorem allows for the solutions of problems such as, "Nine students chip in to buy a 50-pound sack of rice. They are to share the rice equally by weight. How many pounds should each person get?" More importantly, this theorem is the reason we can now retire the division symbol $\div$ and use $\frac{m}{n}$ exclusively to denote "m divided by n" *when m, n are whole numbers.*

(D) Adding fractions The addition of fractions cannot be different, conceptually, from the addition of whole numbers because every whole number is a

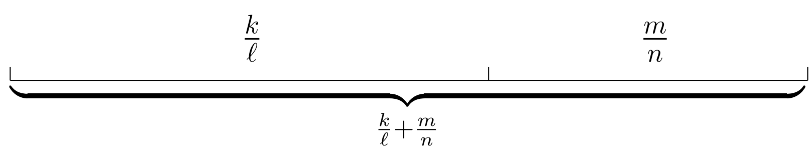
[2] To avoid the possibly confusing appearance of the word "divide" at this juncture, we have intentionally used "partition" instead.

fraction. So how do we add whole numbers when whole numbers are considered points on the number line? Consider, for example, the addition of 4 to 7. In terms of the number line, this is just the total length of the two segments joined together end-to-end, one of length 4 and the other of length 7, which is of course 11, as shown.



We call this process the **concatenation** of the two segments. Imitating this process, we define, given fractions $\frac{k}{\ell}$ and $\frac{m}{n}$, their **sum** $\boldsymbol{\frac{k}{\ell} + \frac{m}{n}}$ by

$$\frac{k}{\ell} + \frac{m}{n} \;=\; \textit{the length of two concatenated segments, one of length } \tfrac{k}{\ell}\textit{, followed by one of length } \tfrac{m}{n}$$



It is an immediate consequence of the definition that

$$\frac{k}{\ell} + \frac{m}{\ell} = \frac{k+m}{\ell}$$

because both sides are equal to the length of $k+m$ copies of $\frac{1}{\ell}$. More explicitly, the left side is the length of k copies of $\frac{1}{\ell}$ combined with m copies of $\frac{1}{\ell}$, and is therefore the length of $k+m$ copies of $\frac{1}{\ell}$, which is exactly the right side. Because of FFFP, the general case of adding two fractions with unequal denominators is immediately reduced to the case of equal denominators, i.e., to add

$$\frac{k}{\ell} + \frac{m}{n}$$

where $\ell \neq n$, we use FFFP to rewrite $\frac{k}{\ell}$ as $\frac{kn}{\ell n}$ and $\frac{m}{n}$ as $\frac{\ell m}{\ell n}$. Then

$$\frac{k}{\ell} + \frac{m}{n} = \frac{kn}{\ell n} + \frac{lm}{\ell n} = \frac{kn + \ell m}{\ell n}$$

The first application of fraction addition is the explanation of the addition algorithm for (finite) decimals. For example, consider

$$4.0451 + 7.28$$

This algorithm calls for

(α) lining up 4.0451 and 7.28 by their decimal point,

(β) adding the two numbers as if they are whole numbers and get a whole number, to be called N, and

(γ) putting the decimal point back in N to get the answer of $4.0451 + 7.28$.

We now supply the reasoning for the algorithm. First of all, we use equivalent fractions[3] to rewrite the two decimals as two with the same number of "decimal digits", i.e., $4.0451 + 7.28 = 4.0451 + 7.2800$. This corresponds to ($\alpha$). Then,

$$\begin{aligned} 4.0451 + 7.28 &= \frac{40451 + 72800}{10^4} \\ &= \frac{113251}{10^4} \qquad \text{(corresponds to } (\beta)\text{)} \\ &= 11.3251 \qquad \text{(corresponds to } (\gamma)\text{)} \end{aligned}$$

The reasoning is of course completely general and is applicable to any other pair of decimals.

A second application is to get the so-called complete expanded form of a (finite) decimal. For example, given 40.1297, we know it is the fraction

$$\frac{401297}{10^4}$$

But

$$401297 = (4 \times 10^5) + (1 \times 10^3) + (2 \times 10^2) + (9 \times 10^1) + (7 \times 10^0)$$

[3] A little reflection would tell you that we are essentially using FFFP here.

We also know that, by equivalent fractions, $\frac{4\times 10^5}{10^4} = 40$, $\frac{1\times 10^3}{10^4} = \frac{1}{10}$, etc. Thus

$$40.1297 = 40 + \frac{1}{10} + \frac{2}{10^2} + \frac{9}{10^3} + \frac{7}{10^4}$$

This expression of 40.1297 as a sum of descending powers[4] of 10, where the coefficients of these powers are the digits 4, 1, 2, 9, and 7, is called the **complete expanded form** of 40.1297.

A third application of fraction addition is to introduce the concept of mixed numbers. We observe that, in order to locate fractions on the number line, it is an effective method to use division-with-remainder on the numerator. For example, we have

$$\frac{187}{14} = \frac{(13\times 14)+5}{14} = \frac{13\times 14}{14} + \frac{5}{14} = 13 + \frac{5}{14}$$

and therefore $\frac{187}{14}$ is beyond 13 but not yet 14, because the sum $13 + \frac{5}{14}$, as a concatenation of two segments of lengths 13 and $\frac{5}{14}$, clearly exhibits the fraction $\frac{187}{14}$ as a point on the number line about one-third beyond the number 13. The sum $13 + \frac{5}{14}$ is usually abbreviated to $13\frac{5}{14}$ by omitting the + sign and, as such, it is called a **mixed number**.

(E) Comparing fractions By definition, given two fractions $\frac{k}{\ell}$ and $\frac{m}{n}$, we say $\frac{m}{n}$ is **less than** $\frac{k}{\ell}$ or $\frac{k}{\ell}$ is **bigger than** $\frac{m}{n}$, if the point $\frac{m}{n}$ is to the left of the point $\frac{k}{\ell}$ on the number line. In symbols: $\boldsymbol{\frac{m}{n} < \frac{k}{\ell}}$.

It is a rather shocking realization that in the usual presentation of fractions, one that does not use the number line, there is *no definition* of what it means for one fraction to be bigger than another.

To tell which of two given fractions is bigger, the following is useful.

[4]Here we use the exponential notation for convenience. "Descending" if you think of $\frac{1}{10}$ as 10^{-1}, etc.

Cross-multiplication algorithm *Given fractions* $\frac{k}{\ell}$ *and* $\frac{m}{n}$, $\frac{k}{\ell} > \frac{m}{n}$ *is equivalent to* $kn > \ell m$.

Here is the formal proof. By FFFP, we can rewrite $\frac{k}{\ell}$ and $\frac{m}{n}$ as $\frac{kn}{\ell n}$ and $\frac{\ell m}{\ell n}$, respectively. The algorithm can be read off from this observation. It should be pointed out that exactly the same reasoning proves a similar algorithm for equality:

$$\frac{k}{\ell} = \frac{m}{n} \quad \textit{is equivalent to} \quad kn = \ell m$$

This is also referred to as the **cross-multiplication algorithm**.

We can also *compare decimals*. For example, which of 0.0082 and 0.013 is bigger? By definition, we need to compare $\frac{82}{10000}$ and $\frac{13}{1000}$. By FFFP, we compare instead $\frac{82}{10000}$ and $\frac{130}{10000}$. Clearly, 130 copies of 1/10000 is more than 82 copies of 1/10000, so $0.013 > 0.0082$.

Note that this is far from a mindless algorithm that minimizes students' number sense or their understanding of place value. Decimals such as 0.0082 or 0.013 are merely symbols, and the first priority in doing mathematics is to inquire about the meanings of the symbols in question. Therefore going back to the original fractions $\frac{82}{10000}$ and $\frac{13}{1000}$ serves exactly the purpose of finding out the meanings of 0.0082 and 0.013. The fact that this reduces the comparison of decimals to the comparison of whole numbers is precisely what the subject of decimals should be about: decimals are nothing but whole numbers in disguise.

(F) Subtracting fractions Suppose $\frac{k}{\ell} > \frac{m}{n}$, then a segment of length $\frac{k}{\ell}$ is longer than a segment of length $\frac{m}{n}$. The subtraction $\boldsymbol{\frac{k}{\ell} - \frac{m}{n}}$ is by definition the length of the remaining segment when a segment of length $\frac{m}{n}$ is taken from one end of a segment of length $\frac{k}{\ell}$.

The same reasoning as in the case of addition, using FFFP, then yields

$$\frac{k}{\ell} - \frac{m}{n} = \frac{kn - \ell m}{\ell n}$$

Consider the subtraction of $17\frac{2}{5} - 7\frac{3}{4}$. One can do this routinely by converting the mixed numbers into fractions:

$$17\frac{2}{5} - 7\frac{3}{4} = \frac{85+2}{5} - \frac{28+3}{4} = \frac{87}{5} - \frac{31}{4} = \frac{87 \times 4 - 31 \times 5}{5 \times 4} = \frac{193}{20}.$$

However, there is another way to do the computation:

$$\begin{aligned} 17\frac{2}{5} - 7\frac{3}{4} &= (16 + 1\frac{2}{5}) - (7 + \frac{3}{4}) \\ &= (16-7) + (1\frac{2}{5} - \frac{3}{4}) \\ &= 9 + \frac{13}{20} \\ &= 9\frac{13}{20} \end{aligned}$$

(G) Multiplying fractions The colloquial expression *two-thirds of a* 9.5 *fluid oz. of juice* can be given a precise meaning: it is the totality of two parts when 9.5 fluid oz. of juice is divided into three equal parts (by volume). In general, we define $\boldsymbol{\frac{m}{n}}$ **of a number** to mean the totality of m parts when that number is partitioned into n equal parts according to this unit. More explicitly, if the number is a fraction $\frac{k}{\ell}$, then we partition the segment $[0, \frac{k}{\ell}]$ into n parts of equal length, and $\frac{m}{n}$ *of* $\frac{k}{\ell}$ is the length of the concatenation of m of these parts. Then we define the **product** or **multiplication** of two fractions by

$$\frac{k}{\ell} \times \frac{m}{n} \;=\; \frac{k}{\ell} \text{ of a segment of length } \frac{m}{n}$$

This definition justifies what we do everyday concerning situations such as drinking "two-thirds of a 9.5 fluid oz. of juice": we would compute the amount as $\frac{2}{3} \times 9.5$ fluid oz. This number, as we know *by habit*, is equal to

$$\frac{2 \times 9.5}{3} = \frac{19}{3} = 6\frac{1}{3}$$

fluid oz. But now, we have to give the reason behind this computation: this is what we call the **product formula**:

$$\frac{k}{\ell} \times \frac{m}{n} = \frac{km}{\ell n}$$

Here is the proof: Let us partition $[0, \frac{m}{n}]$ into ℓ equal parts. By the definition of the product $\frac{k}{\ell} \times \frac{m}{n}$, it suffices to show that the length of k concatenated parts is $\frac{km}{\ell n}$. By equivalent fractions,

$$\frac{m}{n} = \frac{\ell m}{\ell n} = \frac{m + \cdots + m}{\ell n} = \underbrace{\frac{m}{\ell n} + \cdots + \frac{m}{\ell n}}_{\ell}$$

This directly exhibits $\frac{m}{n}$ as the concatenation of ℓ parts, each part of length $\frac{m}{\ell n}$. The length of k concatenated parts is thus $\frac{km}{\ell n}$, as desired.

As a *logical consequence* of the product formula, one shows that the area of a rectangle whose sides have fractional lengths is the product of the lengths. This fact, together with the original definition of fraction multiplication, are the two principal interpretations of fraction multiplication.

The product formula explains the multiplication algorithm of decimals. Consider for example

$$1.25 \times 0.0067$$

The algorithm calls for

(α) multiply the two numbers as if they are whole numbers by ignoring the decimal points,

(β) count the total number of decimal digits of the two decimal numbers, say p, and

(γ) put the decimal point back in the whole number obtained in (α) so that it has p decimal digits.

We now justify the algorithm using this example, noting at the same time that the reasoning in the general case is the same.

$$\begin{aligned}
1.25 \times 0.0067 &= \frac{125}{10^2} \times \frac{67}{10^4} \\
&= \frac{125 \times 67}{10^2 \times 10^4} \qquad \text{(product formula)} \\
&= \frac{8375}{10^2 \times 10^4} \qquad \text{(corresponding to } (\alpha)) \\
&= \frac{8375}{10^{2+4}} \qquad \text{(corresponding to } (\beta)) \\
&= 0.008375 \qquad \text{(corresponding to } (\gamma))
\end{aligned}$$

(H) Dividing fractions We teach children that $\frac{36}{9} = 4$ because 4 is the whole number so that $4 \times 9 = 36$. This then is the statement that *36 divided by 9 is the whole number which, when multiplied by 9, gives 36.* In symbols, we may express the foregoing as follows: $\frac{36}{9}$ is by definition the number k which satisfies $k \times 9 = 36$. Similarly, $\frac{72}{24}$ *is the whole number which satisfies* $\frac{72}{24} \times 24 = 72$. In general,

> Given whole numbers a and b, with $b \neq 0$ and a being a multiple of b, then **the division of** a **by** b, in symbols $\frac{a}{b}$, is the whole number c so that the equality $cb = a$ holds.

The preceding definition of division *among whole numbers* is important for the understanding of division among fractions, because once we replace "whole number" by "fraction", this will be essentially the definition of the division of fractions. However, there is a caveat. In the definition in case a and b are whole numbers, the division $\frac{a}{b}$ makes sense only when a is a multiple of b. Our first task in approaching the division of fractions is to show that, if a and b are fractions, $\frac{a}{b}$ always makes sense so long as b is nonzero. The following theorem accomplishes this goal.

Theorem *Given fractions A and B ($B \neq 0$), there is a fraction C, so that $A = CB$. Furthermore, there is only one such fraction.*

The proof is simplicity itself. Let $A = \frac{k}{\ell}$ and $B = \frac{m}{n}$, then the fraction C defined by $C = \frac{kn}{\ell m}$ clearly satisfies $A = CB$. (Since B is nonzero, m is nonzero. Therefore $\ell m \neq 0$ and this fraction C makes sense.) This proves that such a C exists. If there is another fraction C' that also satisfies $A = C'B$, then

$$\frac{k}{\ell} = C' \times \frac{m}{n}$$

Multiply both sides by $\frac{n}{m}$ yields $\frac{kn}{\ell m} = C'$. So $C' = C$, as desired.

The proof of the theorem shows explicitly how to get the fraction C so that $CB = A$: If $A = \frac{k}{\ell}$ and $B = \frac{m}{n}$, then the proof gives C as

$$C \;=\; \frac{kn}{\ell m} \;=\; \frac{k}{\ell} \times \frac{n}{m}$$

Now we are in a position to define fraction division:

If A, B, are fractions ($B \neq 0$), then the **division of $\boldsymbol{A}$ by $\boldsymbol{B}$**, *or the* **quotient of $\boldsymbol{A}$ by $\boldsymbol{B}$**, *denoted by* $\frac{\boldsymbol{A}}{\boldsymbol{B}}$, *is the unique fraction C (as guaranteed by the Theorem) so that $CB = A$.*

If the given fractions are $\frac{k}{\ell}$ and $\frac{m}{n}$, then the preceding comment implies that

$$\frac{\frac{k}{\ell}}{\frac{m}{n}} = \frac{k}{\ell} \times \frac{n}{m}$$

This is the famous **invert and multiply rule** for the division of fractions. Observe that it has been proved as a consequence of the precise definition of division.

We now bring closure to the discussion of the arithmetic of decimals by taking up the division of decimals. The main observation is that *the division of decimals is reduced to the division of whole numbers*, e.g., the division

$$\frac{21.87}{1.0925}$$

becomes, upon using invert and multiply,

$$\frac{21.8700}{1.0925} = \frac{\frac{218700}{10^4}}{\frac{10925}{10^4}} = \frac{218700}{10925}$$

This reasoning is naturally valid for the division of any two decimals. We did not obtain the desired answer to the original division, however, because it is understood that we should get a decimal for the answer, not a fraction. It turns out that in almost all cases, the answer is an infinite decimal.

(J) Complex fractions Further applications of the concept of division cannot be given without introducing a certain *formalism* for computation about **complex fractions**, which are by definition *the fractions obtained by a division* $\frac{A}{B}$ *of two* fractions A, B ($B > 0$). We continue to call A and B the **numerator** and **denominator** of $\frac{A}{B}$, respectively. Note that any complex fraction $\frac{A}{B}$ is just a fraction (more precisely, the fraction C in the Theorem), so all that we have said about fractions apply to complex fractions, e.g., if $\frac{A}{B}$ and $\frac{C}{D}$ are complex fractions, then (see section (G)),

$$\frac{A}{B} \times \frac{C}{D} \quad \text{is} \quad \frac{A}{B} \textit{ of (the quantity) } \frac{C}{D}$$

Such being the case, why then do we single out complex fractions for a separate discussion? It is not difficult to give the reason. Consider, for example, an addition of fractions of the following type:

$$\frac{1.2}{31.5} + \frac{3.7}{0.008}$$

First of all, such an addition is not uncommon, and secondly, this is an addition of complex fractions because $1.2 = \frac{12}{10}$, $31.5 = \frac{315}{10}$, etc. Now, the addition can be handled by the usual procedures for fractions, but school students are taught to do the addition by treating the decimals as if they were whole numbers, and *directly* apply the addition algorithm for fractions to get the same answer:

$$\frac{(1.2 \times 0.008) + (3.7 \times 31.5)}{31.5 \times 0.008} = \frac{116.5596}{0.252} = \frac{\frac{1165596}{10000}}{\frac{2520}{10000}} = \frac{1165596}{2520}$$

What this does is to make use of the formula $\frac{k}{\ell} + \frac{m}{n} = \frac{kn+m\ell}{\ell n}$, by letting $k = 1.2$, $\ell = 31.5$, $m = 3.7$, and $n = 0.008$. However, this formula has only been proved to be valid for *whole numbers* k, ℓ, m, and n, whereas 1.2, 31.5, etc., are not whole numbers. On the face of what has been proved, such an application of $\frac{k}{\ell} + \frac{m}{n} = \frac{kn+m\ell}{\ell n}$ is illegitimate. But the simplicity of the above computation is so attractive that it provides a strong incentive to extend this formula to any *fractions* k, ℓ, m, n. Similarly, we would like to be able to multiply the following complex fractions as if they were ordinary fractions by writing

$$\frac{0.21}{0.037} \times \frac{84.3}{2.6} \;=\; \frac{0.21 \times 84.3}{0.037 \times 2.6}$$

regardless of the fact that the product formula $\frac{k}{\ell} \times \frac{m}{n} = \frac{km}{\ell n}$ has only been proved for whole numbers k, ℓ, m, n.

It is considerations of this type that force us to take a serious look at complex fractions.

Almost all existing textbooks allow computations with complex fractions to be performed as if they were ordinary fractions *without a word of explanation.* The situation demands improvement.

Here is a brief summary of the basic facts about complex fractions that figure prominently in school mathematics: Let $A, \ldots, F$ be *fractions*, and we assume further that they are nonzero where appropriate in the following. Then, using "$\Longleftrightarrow$" to denote "is equivalent to", we have:

(a) If $C \neq 0$, then

$$\frac{AC}{BC} = \frac{A}{B}$$

(b) $\dfrac{A}{B} > \dfrac{C}{D}$ (resp., $\dfrac{A}{B} = \dfrac{C}{D}$) $\quad\Longleftrightarrow\quad AD > BC$ (resp., $AD = BC$).

(c) $\dfrac{A}{B} \pm \dfrac{C}{D} = \dfrac{(AD) \pm (BC)}{BD}$

(d) $\dfrac{A}{B} \times \dfrac{C}{D} = \dfrac{AC}{BD}$

(e) $\dfrac{A}{B} \times \left(\dfrac{C}{D} \pm \dfrac{E}{F}\right) = \left(\dfrac{A}{B} \times \dfrac{C}{D}\right) \pm \left(\dfrac{A}{B} \times \dfrac{E}{F}\right)$

The proofs of (a)–(e) are straightforward, but the important thing is to identify the concept of a *complex fractions* and make students aware of it. It should not be assumed that (a)–(e) are needed only to make certain computations easier. On the contrary, when students come to algebra, they would recognize that almost all the arguments related to division or rational expressions make use of complex fractions (a fact which, again, is almost never mentioned in textbooks).

(K) Percent, ratio, and rate At this point, students should be able to handle any word problem about fractions. However, textbooks and the education literature do not supply them with the requisite definitions so that they are left to navigate in the dark as to what they are dealing with. As a consequence, the problems related to percent, ratio, and rates become notorious in school mathematics for being difficult. The following gives explicit definitions of these concepts.

Note that every single one of these definitions requires the concept of a complex fraction.

A **percent** is a complex fraction whose denominator is 100. By tradition, a percent $\frac{N}{100}$, where N is a fraction, is often written as $N\%$. By regarding $\frac{N}{100}$ as an ordinary fraction, we see that the usual statement $N\%$ **of a quantity** $\frac{m}{n}$ is exactly $N\% \times \frac{m}{n}$ (see the discussion in (G)).

Now, the following are three standard questions concerning percents that students traditionally consider to be difficult:

(i) What is 5% of 24?

(ii) 5% of what number is 16?

(iii) What percent of 24 is equal to 9?

The answers are simple consequences of what we have done *provided we follow the precise definitions.* Thus, (i) 5% of 24 is $5\% \times 24 = \frac{5}{100} \times 24 = \frac{6}{5}$. For (ii), let us say that 5% of a certain number y is 16, then again strictly from the definition given above, this translates into $(5\%)y = 16$, i.e., $y \times \frac{5}{100} = 16$. By the definition of division, this says

$$y = \frac{16}{\frac{5}{100}} = 16 \times \frac{100}{5} = 320$$

Finally, (iii). Suppose $N\%$ of 24 is 9. This translates into $N\% \times 24 = 9$, or $\frac{N}{100} \times 24 = 9$, which is the same as $N \times \frac{24}{100} = 9$. By the definition of division again, we have

$$N = \frac{9}{\frac{24}{100}} = 9 \times \frac{100}{24} = 37.5$$

So the answer to (iii) is 37.5%.

Next we take up the concept of ratio, and it is unfortunately one that is encrusted in excessive verbiage. By definition, given two fractions A and B, where $B \neq 0$ and both refer to the same unit (i.e., they are points on the same number line), the **ratio of A to B**, sometimes denoted by $\boldsymbol{A : B}$, is the complex fraction $\frac{\boldsymbol{A}}{\boldsymbol{B}}$.

In connection with ratio, there is a common expression that needs to be made explicit. To say that **the ratio of boys to girls in a classroom is 3 to 2** is to say that if B (resp., G) is *the number of* boys (resp., girls) in the classroom, then the ratio of B to G is $\frac{3}{2}$.

In school mathematics, the most substantial application of the concept of division is to problems related to *rate*, or more precisely, *constant rate*. The precise definition of the general concept of "rate" requires more advanced mathematics, and in any case, it is irrelevant whether we know what a *rate* is or not. What is relevant is to know the precise meaning of "constant rate" in specific situations, and the most common of these situations are enumerated in the following. The most intuitive among the various kinds of rate is *speed*, and we proceed to define constant speed without giving a detailed discussion in order to save space.

A motion is of **constant speed** v if the distance traveled, d, from time 0 to any time t is $d = vt$. Equivalently, in view of (H), a motion is of constant speed if there is a fixed number v, so that for any positive number t, the distance d (feet, miles, etc.) traveled in any time interval of length t (seconds, minutes, etc.) starting from time 0 satisfies

$$\frac{d}{t} = v$$

In the language of school mathematics, speed is the "rate" at which the work of moving from one place to another is done. Other standard "rate" problems which deserve to be mentioned are the following. One of them is painting (the exterior of) a house. The rate there would be the number of square feet painted per day or per hour. A second one is mowing a lawn. The rate in question would be the number of square feet mowed per hour or per minute. A third is the work done by water flowing out of a faucet, and the rate is the number of gallons of water coming out per minute or per second. In each case, the concept of **constant rate** can be precisely defined as in the case of constant speed. For example, a **constant rate of lawn-mowing** can be defined as follows: if A is the total area that has been mowed after T hours starting from time 0, then there is a constant r (with unit square-feet-per-hour) so that $A = rT$, and this equality is valid no matter what T is.

For example, assume that water from a faucet flows at a constant rate. If a tub with a capacity of 20 gallons can be filled with water in 3 minutes, how long does it take to fill a container of 26 gallons? Let us say the rate of water flow is r gallons per minute and it takes t minutes to fill the container. By definition of constant rate, we have $20 = r \times 3$ and $26 = r \times t$. From the first equation, we get $r = \frac{20}{3}$, and from the second, $26 = \frac{20}{3}\,t$. Therefore $t = 26 \times \frac{3}{20} = 3.9$ minutes, or 3 minutes and 54 seconds. *Notice that there is absolutely no mention of "setting up a proportion" in this solution; there is no such concept as "setting up a proportion" in mathematics.*

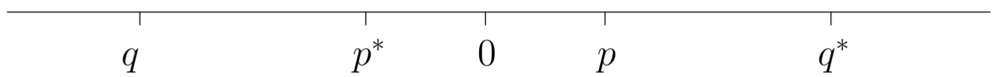
(L) Negative numbers Recall that a **number** is a point on the number line. We now look at all the numbers as a whole. Take any point p on the number line which is not equal to 0; such a p could be on either side of 0 and, in particular, it does not have to be a fraction. Denote its mirror reflection on the opposite side of 0 by p^*, i.e., p and p^* are equidistant from 0 and are on opposite sides of 0. If $p = 0$, let

$$0^* = 0$$

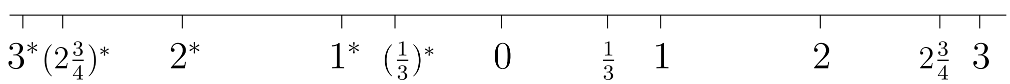
Then for any points p, it is clear that

$$p^{**} = p$$

This is nothing but a succinct way of expressing the fact that reflecting a nonzero point across 0 twice in succession brings it back to itself (if $p = 0$, of course $0^{**} = 0$). Here are two examples of reflecting two points p and q in the manner described:



Because the fractions are to the right of 0, the numbers such as 1^*, 2^*, or $(\frac{9}{5})^*$ are to the left of 0. Here are some examples of the reflections of fractions (remember that fractions include whole numbers):



The set of all the fractions and their mirror reflections with respect to 0, i.e., the numbers $\frac{m}{n}$ and $(\frac{k}{l})^*$ for all whole numbers k, l, m, n ($l \neq 0$, $n \neq 0$), is called the **rational numbers**. Recall that the whole numbers are a sub-set of the fractions. The set of whole numbers and their mirror reflections,

$$\ldots 3^*,\ 2^*,\ 1^*,\ 0,\ 1,\ 2,\ 3, \ldots$$

is called the **integers**. Then, using "$\subset$" to denote "is a subset of", we have:

whole numbers $\subset$ integers $\subset$ rational numbers

We now extend the **order** among numbers from fractions to all numbers: for any x, y on the number line, $x < y$ means that x is to the left of y. An equivalent notation is $y > x$.

Numbers which are to the right of 0 (thus those x satisfying $x > 0$) are called **positive**, and those which are to the left of 0 (thus those that satisfy $x < 0$) are **negative**. So 2^* and $(\frac{1}{3})^*$ are **negative fractions**, while all nonzero fractions are positive. **The number 0 is, by definition, neither positive nor negative.**

As is well-known, a number such as 2^* is normally written as -2 and $(\frac{1}{3})^*$ as $-\frac{1}{3}$, and that the "$-$" sign in front of -2 is called the **negative sign**. The reason we employ this $*$ notation and have avoided mentioning the negative sign up to this point is that the negative sign, having to do with the operation of subtraction, simply will not figure in our considerations until we begin to subtract rational numbers. Moreover, the terminology of "negative sign" carries certain psychological baggage that may interfere with learning rational numbers the proper way. For example, if $a = -3$, then there is nothing "negative" about $-a$, which is 3. It is therefore best to hold off introducing the negative sign until its natural arrival in the context of subtraction in the next section.

(M) Adding rational numbers A fact not mentioned in the brief discussion of fractions up to this point is that the addition and multiplication of fractions satisfy the associative and commutative laws (of addition and multiplication, respectively) and the distributive law. For the arithmetic operations on rational numbers, these laws come to the forefront. The rational numbers are simply "expected" to satisfy the associative, commutative, and distributive laws. With this in mind, we make three **fundamental assumptions** about the addition of rational numbers. The first two are entirely noncontroversial:

(A1) *Given any two rational numbers x and y, there is a way to add these to get another rational number $x+y$ so that, if x and y are* fractions, *$x+y$ is the same as the usual sum of fractions. Furthermore, this addition of rational numbers satisfies the associative and commutative laws.*

(A2) $x + 0 = x$ *for any rational number x.*

The last assumption explicitly prescribes the role of all negative fractions:

(A3) *If x is any rational number, $x + x^* = 0$.*

On the basis of (A1)–(A3), we can prove in succession how addition can be done. Let s and t be any two *positive* fractions. By (A1),

$s + t =$ the old addition of fractions.

In general, (A1)–(A3) imply that

$s^* + t^* = (s + t)^*$, e.g., $3^* + 8^* = 11^*$.

$s + t^* = (s - t)$ if $s \geq t$, e.g., $7 + 4^* = (7 - 4)^* = 3$.

$s + t^* = (t - s)^*$ if $s < t$, e.g., $2 + 8^* = (8 - 2)^* = 6^*$.

Because $s^* + t = t + s^*$, by the commutative law of addition, we have covered all possibilities for the addition of rational numbers.

The second equality above becomes especially interesting if we write it backwards:

$$s - t = s + t^* \quad \textit{when } s \geq t$$

The fraction subtraction $s - t$ now becomes the addition of s and t^*. Because the sum $s + t^*$ makes sense regardless of whether s is bigger than t or not, this equality prompts us to **define**, in general, the subtraction between any two rational numbers x and y to mean:

$$\boldsymbol{x - y \overset{\text{def}}{=} x + y^*}$$

Note the obvious fact that, when x, y are fractions and $x > y$, the meaning of $x - y$ coincides with the meaning of subtracting fractions as given in section (F). This concept of subtraction between two rational numbers is therefore an extension of the old concept of subtraction between two fractions.

As a consequence of the definition of $x - y$, we have

$$0 - y = y^*$$

because $0 + y^* = y^*$. Common sense dictates that we should **abbreviate $\boldsymbol{0 - y}$ to $\boldsymbol{-y}$**. So we have

$$-y = y^*$$

It is only at this point that we can abandon the notation of y^* and replace it by $-y$. Many of the preceding equalities will now assume a more familiar appearance, e.g., from $x^{**} = x$ for any rational number x, we get

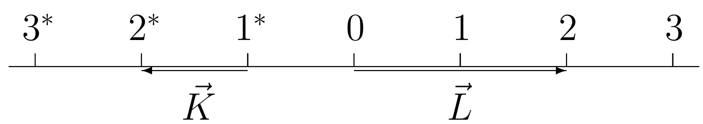
$$-(-x) = x$$

and from $x^* + y^* = (x + y)^*$, we get

$$-(x + y) = -x - y$$

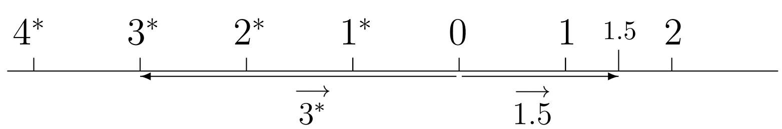
In the school classroom, it would be a good idea to also teach a more concrete approach to adding rational numbers. To this end, define a **vector** to be a segment on the number line together with a designation of one of its two endpoints as a **starting point** and the other as an **endpoint**. We will continue to refer to the length of the segment as the **length** of the vector, and call the vector

left-pointing if the endpoint is to the left of the starting point, **right-pointing** if the endpoint is to the right of the starting point. The **direction** of a vector refers to whether it is left-pointing or right-pointing. We denote vectors by placing an arrow above the letter, e.g., $\vec{A}$, $\vec{x}$, etc., and in pictures we put an arrowhead at the endpoint of a vector to indicate its direction. For example, the vector $\vec{K}$ below is left-pointing and has length 1, with a starting point at 1^* and an endpoint at 2^*, while the vector $\vec{L}$ is right-pointing and has length 2, with a starting point at 0 and an endpoint at 2.



Observe that two vectors being equal means exactly that they have the same starting point, the same length, and the same direction.

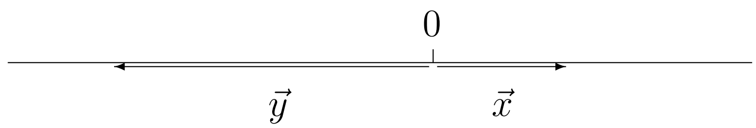
For the purpose of discussing the addition of rational numbers, we can further simplify matters by restricting attention to a special class of vectors. Let x be a rational number, then we define the vector $\vec{x}$ to be the vector with its starting point at 0 and its endpoint at x. It follows from the definition that, *if x is a nonzero fraction, then the segment of the vector $\vec{x}$ is exactly* $[0, x]$. Here are two examples of vectors arising from rational numbers:



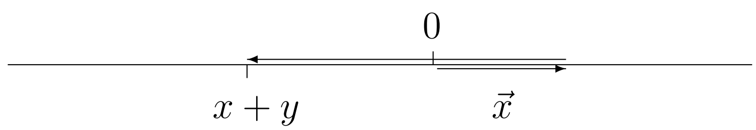
In the following, we will concentrate only on those vectors $\vec{x}$ where x is a rational number, so that all vectors under discussion will be understood to have their starting point at 0. We now describe how to add such vectors. Given $\vec{x}$ and $\vec{y}$, where x and y are two rational numbers, the **sum** vector $\vec{x} + \vec{y}$ is, by definition, the vector whose starting point is 0, and whose endpoint is obtained as follows:

slide $\vec{y}$ along the number line until its starting point (which is 0) is at the endpoint of $\vec{x}$, then the endpoint of $\vec{y}$ in this new position is by definition the endpoint of $\vec{x} + \vec{y}$.

For example, if x and y are rational numbers, as shown:



Then, by definition, $x + y$ is the point as indicated,



We are now in a position to define the addition of rational numbers. The **sum** $\boldsymbol{x + y}$ for any two rational numbers x and y is by definition the endpoint of the vector $\vec{x} + \vec{y}$. In other words,

$$x + y = \text{ the endpoint of } \ \vec{x} + \vec{y}$$

Put another way, $x + y$ is defined to be the point on the number line so that its corresponding vector $\overrightarrow{(x+y)}$ satisfies:

$$\boldsymbol{\overrightarrow{(x+y)} = \vec{x} + \vec{y}}$$

Suffice it to say that at this point, the exact computation of the addition of rational numbers can be carried out and the previous information about $s + t$, $s^* + t$, $s + t^*$, and $s^* + t^*$ can be retrieved.

(N) Multiplying rational numbers We take the same approach to multiplication as addition, namely, we make the **fundamental assumptions** that

(M1) *Given any two rational numbers x and y, there is a way to multiply them to get another rational number xy so that, if x and y are fractions, xy is the usual product of fractions. Furthermore, this multiplication of rational numbers satisfies the associative, commutative, and distributive laws.*

(M2) *If x is any rational number, then $1 \times x = x$.*

We note that (M2) must be an assumption because, for instance, we do not know as yet what $1 \times 5^*$ means. The equally "obvious" fact that

(M3) $0 \times x = 0$ *for any rational number* x.

turns out to be provable.

We want to know explicitly how to multiply rational numbers. Thus let x, y be rational numbers. What is xy? If $x = 0$ or $y = 0$, we have just seen from (M3) that $xy = 0$. We may therefore assume both x and y to be nonzero, so that each is either a fraction, or the negative of a fraction. Letting s, t be *nonzero* fractions, then the following can be proved:

$$(-s)t = -(st)$$
$$s(-t) = -(st)$$
$$(-s)(-t) = st$$

Since we already know how to multiply s and t, we have exhausted all the possibilities of the product of rational numbers.

The last item, that if s and t are fractions then $(-s)(-t) = st$, is such a big part of school mathematics education that it is worthwhile to go over at least a special case of it. When students are puzzled by this phenomenon, the disbelief centers on how *anything like this could be true*. The pressing need in such a situation is to win the psychological battle, e.g., use a simple example to demonstrate that *this phenomenon has to happen*. With this in mind, we will give the reasoning of why

$$(-1)(-1) = 1$$

Now, even the most hard-nosed skeptic among students would concede that a number x is equal to 1 if it satisfies $x - 1 = 0$. We will therefore prove that

$$(-1)(-1) \ - 1 = 0$$

Recall that, by definition of subtraction, $(-1)(-1) - 1 = (-1)(-1) + (-1)$. By the distributive law,

$$(-1)(-1) - 1 = (-1)(-1) + 1 \times (-1) = [(-1) + 1](-1) = 0 \times (-1) = 0$$

This then shows that, *if we believe in the distributive law for rational numbers*, it must be that $(-1)(-1) = 1$. Therefore the critical issue behind the fact of *negative* $\times$ *negative* $=$ *positive* is that the distributive law holds for rational numbers.

(P) Dividing rational numbers The concept of the division of rational numbers is the same as that of dividing whole numbers or dividing fractions. As before, we begin such a discussion with the proof of a theorem that is the counterpart of the theorem in section (H).

Theorem *Give rational numbers x, y, with $y \neq 0$. Then there is one and only one rational number z such that $x = zy$.*

The proof is similar. *What does this theorem really say*? It says that if we have a nonzero rational number y, then *any* rational number x can be expressed as a unique **(rational) multiple** of y, in the sense that $x = zy$ for some rational number z. This number z is what is called **the division of x by y**, written as

$$\frac{x}{y}$$

$\frac{x}{y}$ is also called the **quotient of x by y**. In other words, for two rational numbers x and y, with $y \neq 0$,

> $\frac{x}{y}$ *is by definition the unique rational number z so that $x = zy$.*

We can now clear up a standard confusion in the study of rational numbers. The following equalities are tacitly assumed to be true in pre-algebra or algebra,

$$\frac{3}{-7} = \frac{-3}{7} = -\frac{3}{7}$$

We now supply the explanation. First we claim that, if $C = -\frac{3}{7}$, then dividing 3 by -7 yields C, i.e.,

$$\frac{3}{-7} = C$$

This would be true, by definition, if we can prove $3 = C \times (-7)$, and this is so because

$$C \times (-7) = (-\frac{3}{7}) \times (-7) = (\frac{3}{7})(7) = 3$$

where we have made use of $(-a)(-b) = ab$ for all fractions. This then proves $\frac{3}{-7} = -\,\frac{3}{7}$. In a similar manner, we can prove $\frac{-3}{7} = -\,\frac{3}{7}$. More generally, the same reasoning supports the assertion that if k and ℓ are whole numbers and $\ell \neq 0$, then

$$\frac{-k}{\ell} = \frac{k}{-\ell} = -\,\frac{k}{\ell}$$

and

$$\frac{-k}{-\ell} = \frac{k}{\ell}$$

We may also summarize these two formulas in the following statement: *for any two integers a and b, with $b \neq 0$,*

$$\frac{-a}{b} = \frac{a}{-b} = -\,\frac{a}{b}$$

These equalities are well-nigh indispensable in everyday computations with rational numbers. In particular, it implies that

every rational number can be written as the quotient of two integers.

We can further refine this to read:

every rational number can be written as the quotient of two integers so that the denominator is a whole number.

Thus, the rational number $-\,\frac{9}{7}$ is equal to $\frac{-9}{7}$ or $\frac{9}{-7}$, and the former is the preferred choice for ease of computation.

(Q) Comparing rational numbers Recall the definition of $x < y$ between two rational numbers x and y: it means x is to the left of y on the number line.

In this section, we mention several basic inequalities that are useful in school mathematics. We begin with a basic observation about numbers. Given any two numbers x and y, then either they are the same point, or if they are distinct, one is to the left of the other, i.e., x is to the left of y, or y is to the left of x. These three possibilities are obviously mutually exclusive. In symbols, this becomes:

Given two numbers x and y, then one and only one of the three possibilities holds: $x = y$, $x < y$, or $x > y$.

This is called the **trichotomy law**. It is sometimes useful for determining how two numbers stand relative to each other, e.g., if we can eliminate $x < y$ or $x > y$, then necessarily, $x = y$.

The basic inequalities we are after are as follows. Here, x, y, z are rational numbers and the symbol "$\iff$" stands for "is equivalent to":

(i) For any x, y, $x < y \iff -x > -y$.

(ii) For any x, y, z, $x < y \iff x + z < y + z$.

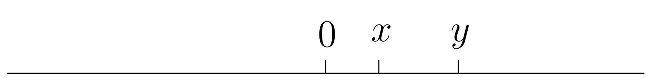
(iii) For any x, y, $x < y \iff y - x > 0$.

(iv) For any x, y, z, if $z > 0$, then $x < y \iff xz < yz$.

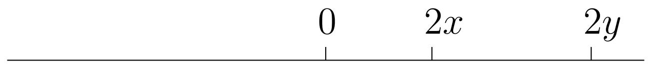
(v) For any x, y, z, if $z < 0$, then $x < y \iff xz > yz$.

(vi) For any x, $x > 0 \iff \frac{1}{x} > 0$.

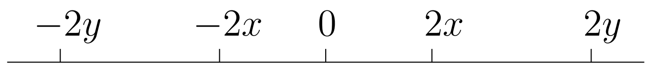
Of these, (v) is the most intriguing. We give an intuitive argument of "$z < 0$ and $x < y$ imply $xz > yz$" that can be refined to be a correct proof. Consider the special case where $0 < x < y$ and $z = -2$. So we want to understand why $(-2)y < (-2)x$. By section (N), $(-2)y = -2y$ and $(-2)x = -2x$. Thus we want to see, intuitively, why $-2y < -2x$. From $0 < x < y$, we get the following picture:



Then $2x$ will continue to be to the left of $2y$, but both are pushed further to the right of 0:



If we reflect this picture across 0, we get the following:



We see that $-2y$ is now to the left of $-2x$, so that $-2y < -2x$, as claimed.

Obviously, this consideration is essentially unchanged if the number -2 is replaced by any negative number z.

Comments on fractions research These comments attempt to put in perspective the preceding detailed description of the basic skills and concepts in the subject of fractions. Why give such details? This has to do with the state of mathematics education in year 2007.

The difficulty with the research on the learning of fractions is that students' learning cannot be divorced from the instruction they receive. If they are taught fractions in a mathematically incorrect way, then it stands to reason that their understanding of fractions would be faulty. *Garbage in, garbage out.* This law of nature cannot be denied. Which of the following should then be blamed for students' underachievement in subsequent assessments of their mathematics learning: students' own misconceptions or the defective instruction they received? Any research that does not attempt to decouple the two cannot lay claim to a whole lot of validity. Because this is an important point that has been traditionally overlooked in education research, we will try to make it absolutely clear by way of an analogy.

Suppose a university in a foreign country has designed a program to train English interpreters, and it decided at some point to have an evaluation of the effectiveness of this program. The evaluator discovered that all the students coming out of the program spoke English with an unacceptably heavy accent, and he was determined to uncover the flaws in the program itself. He looked through the admissions criteria, the courses offered, the requirements for graduation, the availability of language lab facilities, the credentials of the instructors, and so on. He found not a few glitches and made his recommendations accordingly. All the recommendations were duly implemented, but five years later the same evaluator found no improvement in the outcome: the graduates continued to speak with the same heavy accent. He was about to admit defeat, until he attended a few training sessions and found that all the instructors themselves spoke English with the same objectionable accent.

Our claim is that research on the improvement of student learning in fractions has to be built on a foundation of mathematically correct instructions. Otherwise such research would become one on "the effects of handicapped learning", in the sense that it would be a study on *how students fail to learn fractions when they are given defective information on this subject.* One hesitates to declare such a study to be worthless, but for the good of the nation, it may be more profitable to first focus our energy on teaching fractions correctly before launching any such research.

As I mentioned at the outset, this report is not about the teaching and learning of fractions per se, only about what happens in grades 5–7. It is critically important to keep this restricted objective in mind in the following discussion of the research literature. What I will be looking for is not the quality of the research—I am not competent to pass judgment on education research—only whether any research has been done that seems relevant to improving the teaching and learning of fractions in grades 5–7. With this understood, then at least the following articles and monographs on fractions, [Behr et al.], [Lamon], [Litwiller-Bright], [Morrow-Kenny], and [Sowder-Schappelle] do not seem to have a direct bearing on the objective I have in mind, because they appear not to be aware of the need to teach fractions correctly. I hasten to amplify on the latter judgment, which would undoubtedly seem excessively harsh to some. Fractions have been taught probably for as long as school mathematics has been taught. Except for a brief period in the New Math era when some mathematicians did take a close look at fractions (cf. [NCTM1972]), the subject has been taught more or less the same way, defining a fraction as a piece of a pie and over-using single digit numbers, but never attempting to treat the subject as part of mathematics. Due to the long separation of educators from mathematicians in the past decades, educators have had no access to valid mathematical input for a very long time (Cf. [Wu4]). Under the circumstances, educators' lack of awareness of how fractions could be taught ***as mathematics*** is perfectly understandable. As of year 2007, the idea is still a novelty in mathematics education that school mathematics can be taught with due attention to the need of precision, the support by logical reasoning for every assertion, the need of clear-cut definition for each concept introduced, and a coherent presentation of concept and skills in the overall context of mathematics. It should not be a surprise, therefore, that the education research literature reflects such a lack of awareness.

Instead of trying to give a summary of the articles in the cited sources, not to mention numerous others, let me concentrate on the fairly representative article by T. E. Kieren on pp. 31-65 of [Sowder-Schappelle]. The conception of a *mathematical* presentation of fractions in Kieren's article is that of a "static definition followed by given algorithms" (p. 36). And what of this definition? It is not clear that he has any *definition* of a fraction in the sense of section (A) above[5] beyond partitioning a given geometric figure into parts of equal *size*.[6] The need of presenting fractions as a precisely defined concept and explaining each skill logically is not part of his pedagogical picture. What Kieren proposes instead is lots of story-telling and lots of activities for students to engage in so that, through them, students gain experiential and informal knowledge of fractions. In this way of teaching, informal knowledge replaces mathematical knowledge per se. The alternative, according to Kieren, is "to develop an algorithm and specify the practice", without intuitive understanding (p. 40). This is of course the old skill-versus-understanding dichotomy, but we know all too well by now that such a dichotomy is not what mathematics is about. See sections (A)–(Q) above.

As mentioned in the Introduction, fractions are young kids' first excursion into abstractions: they face the bleak future of no longer having the good old standby of counting-on-fingers to help them learn fractions as this practice used to help them learn whole numbers. They need extra support, and they won't get it so long as we try to duck the issue of what a fraction is and fail to supply ample reasoning for every skill in addition to picture-drawing and allied activities. But if a student's conception of a fraction is just a piece of a pie or part of a square, learning about multiplying and dividing pieces of pies and squares can be an excruciating experience. The unending anecdotal data together with results from standardized tests should be sufficient evidence that, to many, the experience *is* indeed excruciating.

It is an intriguing question how to judge students' learning processes if they are fed extremely defective information. For example, students generically do not know what it means to multiply two fractions, except to operationally multiply the numerators and denominators. Indeed, in the way the multiplication of fractions is taught — by traditional or reform methods — in schools (again to judge by textbooks and the educational literature), they are never told what multiplication is. Consequently, when a word problem comes along that calls for multiplying fractions, they do not recognize it unless they resort to the rote process of "watching for key words". The fact that multiplication can be precisely defined and then the formula $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ rigorously *explained* (see section

[5]Recall once again that we are now discussing the teaching and learning of fractions beyond grade 4.

[6]It is not clear what "size" means. Area? So we have a problem right away with imprecision of language.

(G)) is, up to this point, news in school mathematics. In the absence of such an understanding of the mathematics underlying fractions, the multiplication and division of fractions are concepts that are difficult to teach, and therefore much more difficult to learn. The struggle in the education literature to cope with these issues is partially recorded in, e.g., [Litwiller-Bright] and [Sowder-Schappelle]. So again, how to judge students' learning processes in critical topics such as these?

The most sustained, and also one of the best-known research projects about the learning of fractions is the Rational Numbers Project, partly summarized in [Behr et al.]. Its main goal seems to be to address the fragmented picture of a fraction (with all its multifaceted "personalities" that float in and out of a given mathematical discussion on fractions) by promoting *cognitive* connections among these personalities through the appropriate use of problems, hands-on activities, and contextual presentations. This project, like others, was unaware of the existence of a coherent *mathematical* presentation of fractions that provides a logical framework to accommodate all these personalities as part of the mathematical structure ([Jensen], [Wu2], [Wu5]). In a sense, the purpose of the long and detailed description of such a logical development in sections (A) to (Q) above is to counteract the implicit message in the education research literature that such a mathematical structure does not exist. I hope the education community will begin to accept the fact that one cannot promote the learning of fractions by addressing *only* the pedagogical, cognitive, or some other learning issues because, above all else, the *mathematical* development of the subject must be accorded its position of primacy. Our students must be taught correct mathematics before we can begin to consider their learning processes ([Wu4]).

As an illustration of how teaching affects learning, consider a popular example of students' non-learning in the subject of decimals: many believe that $0.0009 > 0.002$ because $9 > 2$. Now a common way to teach students about decimals is to say that the decimal notation is an extension of the base-ten system of writing whole numbers to the writing of other numbers, including numbers between 0 and 1, between 1 and 2, and so on. Thus 1.26 is 1 and two-tenths and 6 hundredths. Students are then asked to see that 0.002 is 2 thousandths while 0.0009 is 9 ten-thousandths, and since 2 thousandths is surely greater than 9 ten-thousandths, they should see that $0.0009 < 0.002$. This is correct as far as it goes, but let us consider the cognitive load overall. There are at least two issues.

First, there is the problem of unnecessary complexity. Students are familiar with whole numbers, but fractions are pieces of a pie and are therefore different. Into this mix we introduce decimals as yet a third kind of number because attaching tenths, hundredths, etc., to whole numbers makes a decimal neither a whole number nor a piece of pie. This is not even mentioning the fact that "tenths", "hundredths", and "thousandths" are very unpleasant words to school students. Such unnecessary confusion throws students off: how to deal with three kinds of numbers? In the way we introduced decimals in section (A), we already put whole numbers and fractions on the number line. So when decimals are singled out as a special class of fractions, there is no added cognitive load at all.

A second issue is the chasm that exists between the world of verbal descriptions of tenths, hundredths, thousandths, etc., and the world of exact computations and symbolic representations. The words "hundredths", "thousandths", etc., used in the verbal definition of a decimal, are hardly the right vehicle to convey precision and clarity. Take 1.26 for example. There is a big difference between

1.26 is 1 and 2 tenths and 6 hundredths

and

$$1.26 = 1 + \frac{2}{10} + \frac{6}{100}$$

The verbal definition of* 1.26 *masks the fact that fraction addition is involved in the seemingly user-friendly description. In particular, we see from the symbolic representation that the common way of treating decimals separately from fractions does not make any sense: one must know what fractions are and how to add fractions before a decimal can be defined. With this understood, we now gain a new appreciation of the definition of a decimal given in section (A) and come to recognize that it is indeed the correct one. (Historically, that was in fact how decimals were introduced.)

Now suppose students are taught that a decimal is a fraction with a denominator that is a power of 10, as in section (A). Then for the comparison of 0.0009 and 0.002, they would learn to first write down the definitions of these numbers:

$$\frac{9}{10000} \quad \text{and} \quad \frac{2}{1000}$$

Then knowing FFFP, they rewrite them as

$$\frac{9}{10000} \quad \text{and} \quad \frac{20}{10000}$$

So the comparison becomes one between 20 parts and 9 parts of the same thing (in fact, $\frac{1}{10000}$, to be precise). Obviously, 20 parts is larger, i.e., $0.0009 < 0.002$. Research should be conducted to confirm or refute the anecdotal evidence that students find such a conception of a decimal much more accessible.

There is one aspect of the learning of fractions that is unquestionably important from a mathematical standpoint, but which to my limited knowledge has not received adequate research attention. It is the hypothesis that by gradually teaching students to freely use symbols in their discussion of rational numbers, we can improve their achievement in algebra (cf. [Wu1] and [Wu3]). Because beginning algebra is just generalized arithmetic, this hypothesis is valid not only from the mathematical perspective, but from the historical perspective as well. In teaching fractions, the opportunity to make use of letters to stand for numbers is available at every turn, starting with the statement of equivalent fractions (section (B)), to the formula for adding fractions (section (D)), to the cross-multiplication algorithm (section (E)), to the formula for subtracting fractions (section (F)), to the product formula (section (G)), to the correct definition of dividing fractions (section (H)), to the rules on complex fractions (section (J)), etc., etc. What I would like to advocate is that we capitalize on the opportunity to make students feel at ease with symbols. Some would object to this kind of teaching because it "confuses" algebra with arithmetic. However, such a "confusion" is a deliberate mathematical and pedagogical decision. Students cannot be thrust into the symbolic environment cold and be expected to perform, and the present failure in the learning of algebra bears eloquent witness to the futility of such an expectation. It would be of some value to obtain data on this hypothesis.

Considerations of the use of symbols lead us naturally to the concept of a "variable" in algebra. Using a "variable" is supposed to mark students' rite of passage in the learning of algebra, but when a "variable" is presented to students as "a quantity that varies", then this passage can be rough going. Informal surveys among students and teachers of algebra reveal that they are all mystified by the concept of something that varies. We should therefore make it very clear that, in mathematics, there is no such concept as "a quantity that varies" and, moreover, a "variable" is an *informal* piece of terminology rather than a formally defined concept. The crux of the matter is not about terminology but about correct usage of symbols, and this is why teaching the use of symbols in fractions bears on the learning of algebra.

The basic protocol in the use of symbols is that the meaning of each symbol *must be clearly quantified* (*specified*). An equality such as "$xy - yx = 0$" has no meaning when it stands alone. Such an equality is sometimes solemnly analyzed in algebra textbooks as an "open sentence", but in fact it is simply a *mistake* in mathematics. Each symbol must be quantified, period. For example, if x and y are complex numbers, then $xy - yx = 0$ is always true. If x and y are matrices, then this sentence is meaningless because the multiplication xy or yx may not even be defined. If x and y are $n \times n$ square matrices where $n > 1$, then $xy - yx = 0$ is sometimes true and sometimes false. And so on. The quantification of a symbol is therefore of critical importance. In the context of school algebra, if a symbol stands for a collection of numbers and this collection has more than one element, then it would be permissible to refer to this symbol as a "variable". In other words, while there is no formal concept with this name, at least in this case most mathematicians would informally use "variable" to refer to such a symbol. If a symbol stands for a specific number, then that symbol would be called a **constant**. Both kinds of symbols come up naturally in the discussion of fractions. Students who are carefully guided through the use of symbols when learning fractions would therefore have the advantage of getting to know what a "variable" really means and will not be subject to the fruitless soul-searching regarding "a quantity that varies" when they come to algebra. This will not be a small advantage.

It was reported in the 1970s and 1980s that incoming algebra students had trouble interpreting "variables" as letters (cf. [Küchemann]), and some of them were quoted as saying "letters are stupid; they don't mean anything" ([Booth]). It seems likely that these students' teachers did not have a clear conception of what a "variable" really means or how symbols (letters) should be properly used. We recall the dictum stated at the beginning of this section that learning cannot be divorced from instruction.

A *correct* use of symbols would also eliminate a standard misconception of what **solving an equation** means. Consider the problem of solving a linear equation such as

$$4x - 1 = 7 - 3x$$

As we said, every symbol must be quantified. So what is the quantification that comes with this equality? It is this: *find all numbers x so that* $4x-1 = 7-3x$. The most important aspect of this quantification is that, since x is a number, $4x$ and $-3x$ are simply numbers and, as such, we can apply to them the usual arithmetic operations without any second thoughts. For example, *suppose there is such an x so that* $4x - 1 = 7 - 3x$. Then since the numbers $4x - 1$ and $7 - 3x$ are equal, adding the same number $3x$ to both of them would produce two numbers that are also equal. Thus $(4x - 1) + 3x = (7 - 3x) + 3x$, and doing arithmetic as usual, we immediately obtain $7x-1 = 7$. Still with the same number x, adding 1 to both $7x - 1$ and 7 gives $7x = 8$. Multiply $7x$ and 8 by $\frac{1}{7}$ then leads to $x = \frac{8}{7}$.

What we have proved is this:

(†) **if** a number x satisfies $4x - 1 = 7 - 3x$, then $x = \frac{8}{7}$.

We arrived at this conclusion by performing *ordinary arithmetic operations* on numbers, no more and no less, and we could do that because knowing x is a specific number, we are entitled to apply all we know about arithmetic to the task. This is one illustration of why we want to carefully quantify each symbol.

One may believe that we have already "solved" the equation. After all, did we not get $x = \frac{8}{7}$? But no, we have not solved the equation at all because all we have done, to repeat the statement (†), is merely to show that *if* a number x satisfies $4x - 1 = 7 - 3x$, then $x = \frac{8}{7}$. What we mean by getting a solution of the equation $4x - 1 = 7 - 3x$ is in fact the converse statement:

(♢) if $x = \frac{8}{7}$, then $4x - 1 = 7 - 3x$.

Some would consider the distinction between (†) and (♢) to be the worst kind of pedantic hair-splitting. After all, once we know $x = \frac{8}{7}$, isn't the verification of $4x-1 = 7-3x$ automatic? Yes and no. It is trivial to verify that $4(\frac{8}{7}) - 1 = 7 - 3(\frac{8}{7})$, but to confuse these two statements would be to commit one of the cardinal sins in mathematics. One cannot afford, under *any* circumstance, to conflate a theorem with its converse. If school mathematics education is to realize its potential, then it must try to instill in students the ability to think clearly and logically. Therefore learning algebra should include knowing what it means to solve an equation, and the difference between obtaining a solution (statement ($\Diamond$)) and proving the uniqueness of a solution (statement ($\dagger$)). In a well taught algebra class, students should be made aware that the usual symbol manipulations which lead to $x = \frac{8}{7}$ need an extra step, a simple step to be sure, to complete the solving of this linear equation. Of course, they should also be taught that the whole solution method depends only on routine applications of arithmetic operations, and that none of the "balancing" arguments associated with operations on both sides of an equation that sometimes creep into textbooks or classroom instructions is necessary.

Finally, one message comes out of the preceding discussion loud and clear. It is that arithmetic is the foundation of algebra. Without a totally fluent command of arithmetic operations, it is impossible to access the most basic part of algebra such as solving a linear equation.

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