Teaching Fractions: Is it Poetry or Mathematics?

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The teaching of fractions has been in crisis for over fifty years. Many theories have been advanced, and many improvements have been proposed.

Until recently, nobody seemed to notice that fractions have been taught as poetry rather than as mathematics. *This is almost a contradiction in terms.*

Mathematics from around **grade 5 onward** has to be precise and literal for the purpose of error-free communication, but the power of poetry is derived from ambiguity and evocativeness.

The need for precision in the mathematics of **grade 5 onward** is generally not appreciated.

Students in grades 5–7 must learn fractions, and the concept of a fraction is **abstract** compared with that of a whole number. We must give students *precise* guidance for them to navigate the abstractions.

Moreover, Grades 5–7 is roughly where the climb to algebra begins. Students must begin to learn to rein in the relatively *informal* mathematics education of the primary grades and learn to express themselves with precision. For example, they must learn to solve a quadratic equation *exactly.*

Let us see how the poetic approach to fractions affects student learning from **grade 5 onward.**

Consider **Hamlet**'s comment on Denmark after his father's death:

'Tis an unweeded garden
That grows to seed; things rank and gross in nature
Possess it merely.

Compare it with the **definition of** $\mathbf{\frac{3}{4}}$**:**

Take a pizza (or a fraction bar) and divide it into 4 equal parts. Take 3 parts.

Now Hamlet did not mean that Denmark *was* a garden, only that it was *like* a garden.

Likewise, $\frac{3}{4}$ is not meant to be *exactly* *"3 parts when a pizza is divided into 4 equal parts,"* only that it is **like** *"3 parts when a pizza is divided into 4 equal parts."*

Shakespeare could get away with the metaphor because patrons of the theater do not as a rule ask, if they were to fertilize Denmark, where to begin?

Unfortunately, we do ask how to divide 3/4 of a pizza by 7/5 of a pizza, **exactly**. **Very awkward.** In other words, we want the exact answer to $\frac{3}{4} \div \frac{7}{5}$.

Mathematics cannot be done by using analogies and metaphors. It requires knowing what every concept **is exactly**, not about what it **is like**. We don't ask: what is $\frac{3}{4} \div \frac{7}{5}$ **more or less like?**

In addition, if analogy is used, who decides which analogy is appropriate? Why are some reasonable analogies consider not good enough?

For example, why not teach the **addition of fractions** by analogy with the **addition of whole numbers** as "combining things"? That is a very good analogy, and yet it is not used.

The use of metaphors in poetry is essential because poetry exploits the reader's ability to make free associations, and each reader presumably has a wealth of past experiences to draw from. Ambiguity works to the poet's advantage.

But when we introduce children to fractions, we cannot appeal to their "wealth of experience in mathematics" because they barely know whole numbers. *We must be as precise as we can to provide the guidance they need*. All the more so because we do not follow a *vague* definition with *vague* questions, but instead demand precise answers from precise computations!

The VAGUE information we provide students does not support the PRECISE information we demand of them.

A friend of mine related to me her personal encounter with the teaching of fractions:

You're so right about the pizza model of fractions! Teresa has now learned about decimals (she had no idea they were fractions) so I asked her what was one tenth of her classroom pupils (they're 20, so no danger it would not be a whole number). She was totally surprised by my question and protested: "What a silly question, how am I suppose to cut my classmates in slices!"

Teresa is an articulate nine-year-old girl who could put her finger right on the button. However, millions of other children have probably rebelled against this kind of teaching without being able to vent their frustrations.

Another example that poetry does not need precision:

Shakespeare:

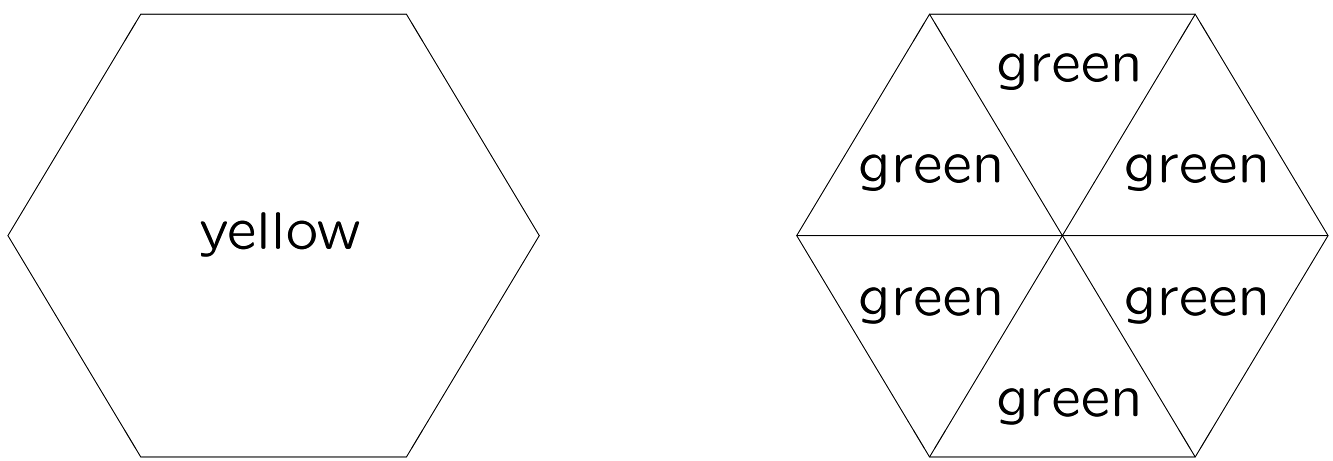
> Shall I compare thee to a summer's day?
> Thou art more lovely and more temperate.

If this were mathematics, you'd say this doesn't make any sense. Indeed, *what kind of summer's day?* Remember what Mark Twain once said: "The coldest winter I ever saw was the summer I spent in San Francisco."

However, anybody with any sense would know that Shakespeare had in mind a *hot* summer's day, and I am not going to tell you whether "hot" means 90° F or 100° F.

Now look at an analogous situation where a lack of precision leads to nonlearning.

In one of the case books, there is the case of a fifth grade teacher trying to teach *the fractional relationship between the different colored blocks* in pattern blocks. Recall that the hexagon is *yellow*, and that six *green* triangles make up a hexagon of the same size:



She wanted her students to write a fractional name for a green triangle. Her instruction was that the yellow hexagon equals 1, and that they should use the green triangles to make exactly the same size and shape as the yellow hexagon.

Since six green triangles make a yellow, she told them that they would use 6 as their denominator. She wrote "$\frac{}{6}$", and explained that if they had just one green triangle, then 1 is the numerator. She completed it to $\frac{1}{6}$.

Then she held up 2 green triangles and said, "If yellow equals 1, then how much is this?" The students first said, "Two." Upon being pushed, they said, "Two green triangles." The result did not vary with repetition. She was frustrated.

So you see that, whereas adults can take the intentional vagueness of Shakespeare's metaphor in stride, the children were unable to fill in the gap created by the teacher's lack of precision.

She did not have a clear conception of a fraction as a number, and therefore her students did not get it either. They failed to get the point that they had to use a *number* to describe the area of two green triangles in terms of the area of the hexagon (set to be 1).

They thought it was a game of of *counting the number of green triangles*. It did not help that she wrote down the denominator 6 ahead of time.

A poet can be effective in using familiar-sounding phrases without assigning them a precise meaning. A famous example is the following two lines of John Keats:

> "Beauty is truth, truth beauty," — that is all
> Ye know on earth, and all ye need to know.

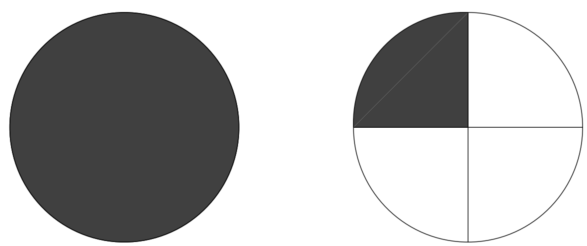
We all *think* we know what "truth" is and what "beauty" is, and who wouldn't want to have both? To have them mentioned in the same breath simply stirs the soul. We are thus predisposed to accept them without asking too many questions.

The fact is that for two hundred years, people have vigorously debated what Keats had in mind.

Now let us look at what happens in the teaching of fractions when we make students *feel* that they know something and yet not let them know exactly what it means.

The concept of a **mixed number** is usually brought up *right after fractions are introduced.*

Sometimes a mixed number such as $1\frac{1}{4}$ is mentioned in passing as **one and a quarter** pizzas, or sometimes it is simply said that **it is a mix of a whole number and a fraction** with no further explanation.



"One and a quarter pizzas" makes $1\frac{1}{4}$ sound entirely reasonable, so students are lulled into a sense of complacency with the concept. But how are pizzas going to help with

$$5\frac{2}{7} \; + \; 13\frac{3}{4} \quad \text{or} \quad 5\frac{2}{7} \; \times \; 13\frac{3}{4} \; \; ?$$

The confusion arises from the fact that a mixed number is a shorthand for the **addition** of the whole number and the fraction. The above is nothing but

$$(5+\frac{2}{7}) \; + \; (13+\frac{3}{4}) \quad \text{and} \quad (5+\frac{2}{7}) \; \times \; (13+\frac{3}{4})$$

Therefore mixed numbers should not be introduced until they can be precisely defined, after the addition of fractions has been introduced.

Here is another example of how the use of imprecise but familiar-sounding phrases hampers mathematics instruction. Consider the concept of "percent". The most common definition of **percent** is "out of 100".

The familiarity of the phrase "out of 100" is soothing, but does it suffice for instruction?

In another case book, there is a case of a sixth grade teacher trying to teach percent for "conceptual understanding" by the use of manipulatives and visual diagrams. He gave his class a diagram in which 6 out of 40 identical squares are shaded. The problem he gave them: What is the percent of the area that is shaded?

The kind of solutions he was looking for are all roughly of the level that

$$\frac{6}{40} = \frac{3}{20} = \frac{5 \times 3}{5 \times 20} = 15\%$$

However, even for solutions as simplistic as this, most of his students didn't get it. My guess is that, if they had been brought up on visual diagrams "out of 100", they might not have thought of *computations*.

Moreover, I believe that if we go strictly by the concept of "out of 100" and only work with manipulatives and visual diagrams, we cannot achieve much conceptual understanding of "percent".

Consider, for example, a similar problem: If 6 out of 41 identical squares are shaded, what is the percent of the area that is shaded?

A little reflection would reveal that, to the extent that 41 is relatively prime to 100, nothing as vague as "out of 100", manipulatives, or visual diagrams would help.

We must come to grips with what "percent" really means, precisely. For that, we need a correct definition.

Assume that the concept of division of fractions has been introduced. A division such as $\frac{2/5}{9/7}$ is called a **complex fraction**. In this case, we call $\frac{2}{5}$ the **numerator** of the complex fraction and $\frac{9}{7}$ its **denominator**.

Because a whole number is also a fraction, $\frac{22\frac{1}{2}}{100}$ is an example of a complex fraction. The common name for this complex fraction is $\mathbf{22\frac{1}{2}}$ **percent**, or $\mathbf{22\frac{1}{2}\%}$, as is well known.

More generally, a **percent** is a complex fraction whose denominator is 100. **A percent is a number.**

We also assume that multiplication of fractions has been defined: $\mathbf{\frac{5}{7} \times \frac{3}{4}}$ is by definition the fraction which is the totality of *5* parts when $\frac{3}{4}$ is divided into *7* equal parts.

In ordinary language, $\frac{5}{7} \times \frac{3}{4}$ is $\mathbf{\frac{5}{7}}$ **of** $\mathbf{\frac{3}{4}}$**.**

We can *prove* that $\frac{5}{7} \times \frac{3}{4} = \frac{5 \times 3}{7 \times 4}$. We call this the **product formula**. (More of this later.)

Everything on this page is also true for complex fractions, e.g., if $\frac{A}{B}$ and $\frac{C}{D}$ are complex fractions, then $\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}$.

Now, **strictly according to the definition**, what is $22\frac{1}{2}\%$ of 37? It is the totality of $22\frac{1}{2}$ parts when 37 is divided into 100 equal parts. This is exactly the usual meaning of "$22\frac{1}{2}\%$ of 37".

Moreover, according to the product formula, this percentage is:

$$22\frac{1}{2}\% \times 37 \;=\; \frac{22\frac{1}{2} \times 37}{100} \;=\; \frac{1665}{200} \;=\; 8\frac{13}{40}$$

We can now tackle the original problem: If 6 out of 41 identical squares are shaded, what is the percent of the area that is shaded?

The fraction of the area that is shaded is of course $\frac{6}{41}$. The problem asks that this fraction be expressed as **number** in the form of N% for some fraction N. Thus we simply write down:

$$\frac{6}{41} = \frac{N}{100},$$

so that $41N = 600$, and $N = 14\frac{26}{41}$.

This is NOT a rote skill. Without the concept of a complex fraction, percent could not be realized as a number and the equation $\frac{6}{41} = \frac{N}{100}$ would not even make sense.

The following problem shows the advantage of having a precise definition of percent: 17 is what percent of 36?

Let N be the fraction so that 17 is N% of 36. By the definition of percent, we can **directly translate the given data into symbolic language:**

$$17 \;=\; \frac{N}{100} \times 36\,,$$

so that $36N \;=\; 1700$. Thus $N = 47\frac{2}{9}$.

Notice that the solution is entirely straightforward.

Observe that understanding “percent” rests on a complete understanding of the multiplication of fractions. We want to revisit this concept.

Of the four arithmetic operations on fractions ($+$, $-$, $\times$, $\div$), multiplication is the most subtle, and is the cause of most misconceptions, mainly because the product formula $\frac{m}{n} \times \frac{k}{\ell} = \frac{mk}{n\ell}$ gives the misleading impression that multiplication is easy.

Education researchers on fractions are aware of the subtlety. They probably gave up on treating fraction multiplication as *mathematics* and decided to treat it *poetically*, in the sense of resorting to the use of allusive language.

Here is one approach to the multiplication of fractions:

We know that teachers and most other adults in our country have a limited understanding of the meaning of multiplication and division of fractions. . . . Teachers who are interested in changing this situation must first approach these topics themselves in ways that are very different from all their previous experiences with mathematics learning. They must completely reformulate their ideas about teaching the topics. . . .

The medium for this rethinking is language. How can we think about something for which we have no words? . . .

Now comes the mathematical discussion:

Multiplication of fractions is about finding multiplicative relationships between multiplicative structures. When students partition a continuous whole such as a circle, they actually find part of parts in the process. In order to create fourths, for example, a student first creates halves. The student then cuts the halves in half to create fourths. In so doing the student can verbalize that one-half of one-half is one-fourth.

This almost writes itself as free verse.

The ineffable meaning of
Multiplication of fractions,
Lies in multiplicative relationships
Between multiplicative structures.

Partition a continuous whole, a circle,
And thou shalt find part of parts.

To create fourths,
First create halves.
Then cut the halves in half,
To create fourths.

It thus comes to pass,
One-half of one-half is one-fourth.

To do mathematics, however, we need something more down-to-earth.

As before, we **define** $\mathbf{\frac{5}{7} \times \frac{3}{4}}$ to be the fraction which is the totality of **5** parts when $\frac{3}{4}$ is divided into **7** equal parts. We usually express this as **"**$\frac{5}{7} \times \frac{3}{4}$ **is $\mathbf{\frac{5}{7}}$ of $\mathbf{\frac{3}{4}}$."**

We want to prove the product formula:

$$\frac{5}{7} \times \frac{3}{4} = \frac{5 \times 3}{7 \times 4}$$

How to divide $\frac{3}{4}$ into 7 equal parts?

Let us first tackle an easier problem: How to divide $\frac{7}{4}$ into 7 equal parts? Answer: $\frac{1}{4}$.

Similarly, if we want to divide $\frac{35}{4}$ into 7 equal parts, one part would be $\frac{5}{4}$ because $35 = 7 \times 5$.

So if the numerator of $\frac{3}{4}$ were a multiple of 7 (but 3 isn't), one could easily divide $\frac{3}{4}$ into 7 equal parts.

However, even if $\frac{3}{4}$ doesn't have a numerator that is a multiple of 7, *we can force it to have this property* because the theorem on equivalent fractions allows us to change the fraction symbol to $\frac{7\times 3}{7\times 4}$.

Thus dividing $\frac{3}{4}$ into 7 equal parts is the same as dividing $\frac{7\times 3}{7\times 4}$ into 7 equal parts, and one part is therefore $\frac{3}{7\times 4}$.

If we take 5 such parts, we get:

$$\frac{3}{7\times 4}+\frac{3}{7\times 4}+\frac{3}{7\times 4}+\frac{3}{7\times 4}+\frac{3}{7\times 4} \;=\; \frac{5\times 3}{7\times 4}$$

Thus we have proved $\frac{5}{7}\times\frac{3}{4} \;=\; \frac{5\times 3}{7\times 4}$.

The more prosaic approach to multiplication of fractions is sufficient for the purpose of teaching fractions.

Analogies and metaphors have a place in mathematics. They can be very *helpful in the understanding of precise concepts and reasoning.* However, it is a mistake to allow them to replace precise concepts and reasoning.

Let us hope that fractions will be taught with less poetry, but with more emphasis on

> precise definitions, and
> precise reasoning.

Epilogue A treatment of fractions that conforms to the basic requirements of mathematics and is usable in elementary schools will be given in:

H. Wu, *Understanding Numbers in Elementary Schol Mathematics*, Amer. Math. Society, to appear in May, 2011.

In the meantime, a version that has been used for the professional development of middle school teachers can be found at:

http://math.berkeley.edu/~wu/Pre-Algebra.pdf