Key Mathematical Ideas in Grades 5-8*

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The curriculum of the middle grades revolves principally around the the following three topics: rational numbers, beginning algebra, and basic geometry. I will attempt to outline, with some details, what we want students to knows in each of these topics. At the end, I will make a few comments on how far we have to go before we can hope to implement these ideas.

Rational numbers The importance of rational numbers in the middle grades stems from the fact that what students learn here about this topic would have to serve them until at least the first two years of college. For the majority, much more is true because what they learn in grades 5-7 would be all they ever know about rational numbers for the rest of their lives. From this perspective, one can see all too clearly the difficulty with the teaching of rational numbers in the middle grades, and it is this. At this stage, students are not yet ready for the kind of mathematical sophistication that is needed for the complete understanding of the rational numbers,[1] and yet they must learn enough about this topic in order to function in the upper grades or in society. The tension between what is achievable with students at this level and what is mathematically correct underlies the notoriety of rational numbers in middle school mathematics. This

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[1] In the sense that rational numbers form the quotient field of the integers, and are constructed by taking equivalence classes of ordered pairs of integers.

tension may be the reason why basic questions such as what is a fraction, or why does negative times negative equal positive, are often left unanswered. I hasten to add that all such questions can be answered satisfactorily in a way that is grade-appropriate.

The subject of *fractions* (which is the term I will use for nonnegative rational numbers) is known to be a main source of mathphobia. If this is not reason enough for us to teach fractions better, let me cite another one: understanding fractions is the most critical step in the understanding of rational numbers because fractions are students' first serious excursion into abstraction. Whereas their intuition of whole numbers can be grounded on the counting of fingers, learning fractions requires first of all a mental substitute for their fingers. They need to be clearly told what a fraction is. A fraction has to be a *number*, and so the definition of a fraction as "parts-of-a-whole" simply doesn't cut it. Students have to be shown that fractions are the natural extension of whole numbers so that the arithmetic operations $+$, $-$, $\times$, and $\div$ on whole numbers can smoothly transition to those on fractions. The fact that there is such a smooth transition is certainly not common knowledge among teachers and students as of year 2005. See the discussion of the longitudinal coherence of the curriculum in Wu [2002]. Right now, most of our students are not even told what it means to multiply two fractions. The mournful refrain of the British educator Kathleen Hart says it all: "How can you multiply two pieces of pizza?" (Hart [2000]). Defining a fraction (or in fact any rational number) as a point on the number line obtained by a partitioning process would serve admirably to effect this transition. In case you are aghast at this suggestion, let me point out that mathematics education is in a state of flux and you are encouraged to come up with a better definition. But any (correct) definition is better than no definition at all, because mathematics cannot proceed without precise definitions.

Students must also know how to fluently execute the four arithmetic operations on fractions and, more importantly, know how to apply these operations to solve problems. Allowing for the use of a four-function calculator, a student with a minimal degree of computational fluency should see no difference, for example, between $\frac{2}{3}+\frac{4}{5}$ and $\frac{357}{68}+\frac{17}{29}$. She should also be able to compute the division

$$\frac{\frac{23}{15}}{\frac{28}{49}}$$

with no effort.

At the moment, the teaching of negative numbers is long on gimmickry and short on substance. For example, students should be shown that the validity of "negative $\times$ negative = positive" depends not on any cute looking patterns or seductive pseudoreasoning. Rather, it rests squarely on the fact that we want the distributive law to be true for rational numbers.

Algebra Students need to see introductory algebra as a natural extension of all they have learned about rational numbers (Wu [2001]; Section 1 of Wu [2005a]). In other words, introductory algebra is *generalized arithmetic.* This is one reason why we must teach rational numbers better. Students should be gradually but systematically acclimated to the use of symbols and to the concept of generality over a long period of time; symbols and generality go together. Like cramming for an exam, the current practice of stinting on the use of symbols before algebra and then suddenly throwing lots of symbols at students when they begin the study of algebra is just bad educational strategy. There are in fact plenty of opportunities for students to learn to use symbols in the process of learning fractions (cf. the thesis of Darley [2005]). Needless to say, such opportunity exists even before that. All you have to do is look at the Russian texts of grades 2 and 3 (Askey, Milgram, and Wu [2005]). Currently, there is an effort to put "algebraic thinking" into all the grades. If I understand this term correctly, it means looking for patterns and working with manipulatives and technology. The intention is laudable, but this kind of algebraic thinking is not enough to promote the learning of algebra from a mathematical perspective; it must go further in the direction of making use of symbols and computing with them whenever it is natural to do so. I suggest, for example, that the next time you teach the primary grades, instead of writing $15 + \underline{\quad\quad} = 22$, try instead, "find a number x so that $15 + x = 22$". When teaching the addition of fractions, tell students that the formula $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ is valid for all fractions $\frac{a}{b}$ and $\frac{c}{d}$, and point out that this is an *identity* in whole numbers a, b, c, and d ($bd \neq 0$). This is the kind of algebraic thinking students need.

Students' discomfort with the use symbols contributes to their inability to solve word problems. The critical process of transcribing verbal information into the symbolic one becomes an insurmountable hurdle. Only extensive practice with the use of symbols can cure this problem, and there is no better way for this practice to take root than to spread it through all the grades. There should also be a concerted effort at the beginning of algebra to give students plenty of practice to perfect this skill of transcription (cf. Section 2 in Wu [2005a]).

Students need to be totally at ease in moving between the geometric data of a straight line and the algebraic data of a linear equation. This cannot happen if they are never taught similar triangles before embarking on the study of linear equations and their graphs, and have never been exposed to the explanation of why the equation of a line is linear and why the graph of a linear equation is a line. Currently, our curriculum, be it reform or traditional or anything in between, does not allow students to learn about similar triangles before taking algebra or during algebra. A high school teacher once objected to a question proposed for a High School Exit Exam which asked for the equation of the line passing through two given points. She said it it would be too hard for her students, and a second teacher concurred. When asked what would make it easier, they said "give nine or ten points instead", because then students could do guess-and-check better. So long as we continue to keep the interplay between the algebra and geometry of a straight line a mystery, such anecdotes will continue to ring down the ages. Compare the discussion in Wu [2005b].

The high point in the study of quadratic equations of one variable is the quadratic formula, but what makes the formula possible is the technique of completing the square. This is a basic technique in mathematics, and students need to be fluent with it. They should also be shown its literal pictorial meaning (thanks to the Babylonian of thirty-eight centuries ago) as *completing a square-with-a-corner-missing to the whole square*. Like the multiplication table, the quadratic formula should be committed to memory.

It is not sufficiently emphasized that the quadratic formula makes all the exercises in factoring trinomials trivial. While there are good mathematical reasons why factoring trinomials with integer coefficients by mental math is a useful skill to have, it is nonetheless true that the present orgy on factoring trinomials in many classrooms should be toned down.

Geometry Geometry in K-12 mathematics is the quantitative study of the space around us. Geometry in the middle grades is mainly concerned with two main topics: mensuration formulas for length, area, and volume, and exploration of the concepts of congruence and similarity. The concept of congruence underlies the mensuration formulas, but unhappily, this fact has been kept from middle school students for far too long.

There are two major problems concerning the teaching of the well-known mensuration formulas, e.g., areas of triangles and circles and volumes of rectangular prisms. First, there is insufficient attention given to a definition of length, area, or volume, and to the similarity between these definitions. Second, not enough emphasis is given to the reasoning that leads to these mensuration formulas. Note that because we are here talking about the middle grades, a completely correct definition of length, etc., is not at issue here. Nevertheless, students need definitions that are *essentially* correct. To find ways to put forward essentially correct definitions that are usable for school mathematics ought to be the basic obligation of mathematics education, but I am rather under the impression that this basic obligation is not being met very well. In fact, the overall absence of definitions in school mathematics is a scandal, and there is no time to waste in putting this scandal behind us.

This may be the place for me to reiterate the importance of definitions in mathematics. Unless I am completely off base, and I am not, this importance is news to most of our teachers. I already touched on the need of a precise definition for fractions earlier. But in geometry, definitions are especially critical because at least the formal reasoning (in contrast with intuitive arguments) about a geometric configuration has to be conducted entirely on the basis of these definitions. Any lack of precision in the definitions would therefore result in the loss of information about the original configuration. In the case of formulas about length, area, and volume, for example, it is not possible to prove these formulas without precise definitions of length, area and volume. One can hardly over-emphasize this message.

Right now, many of our teachers have never been shown why definitions are important, because *our pre-service professional development is in general that defective*. School textbooks pay lip service to the need of definitions by attempting to give some. Unfortunately, these so-called definitions are usually not correct and, even when they are correct, they are not put to use so that they may as well not have been given. For example, it is routinely asserted that the solution of a pair of simultaneous linear equations in two variables is the point of intersection of the graphs of the equations in question. This is actually a theorem, but since the *graph* of an equation and the *solution* of an equation are concepts used *informally* all the way through, this theorem has no hope of ever being proved. In any case, this proof is not to be found in most of these texts.

In the summers of 2003-04, I happened to have taught teachers in both California and Australia, and it occurred to me to ask them if they knew the difference between a definition and a theorem. The answer was 100% negative in both cases. What have we done to our teachers?

There is no better illustration of the need of definitions than the case of "congruence" and "similarlity". The usual definition of *congruence* is *same size and same shape*, and that of *similarity* is *same shape but not necessarily the same size*. These sound very attractive until we try to use them explain *in what way* a photograph of a person is similar to the same photograph shrunk to half its size. Of course it is impossible.

In the middle grades, it is eminently possible to teach congruence and similarity in the plane correctly *and* effectively. We begin with the concepts of *rotation, translation,* and *reflection.* These used to be difficult concepts to teach, but with the availability of transparencies and overhead projectors, students can get to know them via hands-on activities. A *congruence* is then defined to be a composition of rotations, translations and reflections. Such a definition is correct, and grade-level appropriate.

Next, dilation. For simplicity, we define it using coordinates (but this is not necessary). A *dilation* with *center* at the origin O and *scale factor* r ($r \neq 0$) is a transformation of the plane that sends a point (a, b) to the point (ra, rb). So for instance, $(0, 1)$ goes to $(0, r)$, i.e., it changes the distance of every point from O by a factor of r. This concept is a Godsend in the teaching of mathematics because I do not believe there is another opportunity quite like this for the teacher to both astound the students and teach substantive mathematics at the same time. For example, ask students how to shrink a wiggly curve to half the size, and most of them wouldn't know where to begin. Now you just pick a random point and use that as your center O, and start shrinking a few well-chosen points on the curve to half the distance (relative to O) to get the rough contour of a new curve. By increasing the number of points, students gets to see the emergence of the shrunken curve. They usually find this demonstration truly impressive. Once they buy into this concept of dilation, they are ready for the definition two figures to be *similar* if one figure is congruent to a dilated version of the other. Incidentally, this definition of similarity puts in evidence the dependence of the concept of similarity on the concept of congruence. One should not, therefore, try to introduce similariy ahead of congruence, as it is sometimes done.[2]

Certainly, this definition of similarity has much greater impact, and infinitely more mathematical substance than "same shape but not necessarily the same size".

There are at least three reasons why the teaching of geometry in the middle grades must improve. The first has already been mentioned in the discussion of algebra: without a thorough grounding in similar triangles, the teaching of the graphs of linear equations can only proceed by rote. A second one is that we cannot explain to students the meaning of length, area, etc., if we do not have a *mathematical definition* of congruence in the first place. Indeed, the measurements of length, area, etc., must satisfy the basic requirement that congruent sets have equal measurements (of length, area, or volume,

[2]The reason this is done is based on the best of intentions: similarity is more common in real world situations than congruence, so why not teach similarity first? But as we have so often observed, good intention is not enough in mathematics education.

whichever is applicable). Finally, without a precise definition of similarity, your favorite test question of what happens to the area of a polygon when each side is expanded by a factor of $\frac{3}{2}$ does not even make sense. Unhappily, for a very long time now, our students have been forced to answer questions about things that do not make sense to them because these things have never been *properly explained* to them. It is sobering to realize that mathematics education has sunk this low.

Let us bring closure to this discussion of what students in the middle grades ought to know by putting it in the context of the available textbooks. Are there any textbooks that come close to helping us realize the preceding vision? To my knowledge, the answer is sadly an emphastic *No*. Out there there are textbooks of all sizes and styles: traditional, reform, New Math, New New Math, etc. None measures up, and upon closer examination, they fail in different ways. Here failure refers not to defective pedagogical conceptions but to defective presentations of the substance of mathematics. One can get an idea of this failure from the ample references to some of these mathematical defects in the preceding discussion. As we know, the Math War was precipitated by fights in school districts over textbook adoptions, and each side of this war wanted to claim that nothing but the textbooks it favored would do. While each side had some cause, ultimately, one must say that such fierce loyalty to any of these flawed materials is uncalled for in the face of their mathematical flaws.

I believe it is time for us all to step back and take stock of the cold reality: for decades, we have failed *collectively* to produce a reasonable textbook series for our children,[3] and it is time for us to atone for our sin. Fighting is not the correct method of atonement. What is needed is a joint effort, by both the mathematics and education communities, to create some usable textbooks. Neither side can do it alone. At the moment, the need of this constructive effort is unfortunately drowned out by shrill rhetoric from both sides. Let us do better in the future.

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[3]And of course we have also failed to teach our teachers adequately, but this failing is beyond the limited scope of the present article.

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